# 2.Transmission Line

## 2-1.Transmission-line equations

$$V(z+\Delta z) = V(z) + \frac{dV(z)}{dz}\Delta z$$

$$I(z+\Delta z) = I(z) + \frac{dI(z)}{dz}\Delta z$$

$$V(z) = (Z_d \Delta z) I(z) + V(z+\Delta z) \qquad (Z_d = R + j\omega L)$$

$$\frac{dV(z)}{dz} = -Z_d I(z)$$

$$\frac{dI(z)}{dz} = -Y_d V(z) \qquad (Y_d = G + j\omega C)$$

These equations result in the following differential equation (transmission line equation).

$$\frac{d^2V(z)}{dz^2} = Z_d Y_d V(z)$$

$$\therefore V(z) = V_1 e^{-\gamma z} + V_2 e^{+\gamma z}$$

$$I(z) = \frac{\gamma}{Z_d}(V_1 e^{-\gamma z} - V_2 e^{+\gamma z}) = \frac{1}{Z_c}(V_1 e^{-\gamma z} - V_2 e^{+\gamma z})$$

# Propagation constant & characteristic impedance

propagation constant：$\gamma = \sqrt{Z_d Y_d} = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta$

characteristic impedance：$\dfrac{Z_d}{\gamma} = \sqrt{\dfrac{Z_d}{Y_d}} = \sqrt{\dfrac{R + j\omega L}{G + j\omega C}} = Z_c$

in the case of lossless transmission line;

$$\gamma = \sqrt{j\omega L j\omega C} = j\omega\sqrt{LC} = j\beta$$

$$Z_c = \sqrt{\frac{j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}} = R_c$$

# 2-2.Reflection coefficient and input impedance

Use a coordinate y directed from load to source, then

$$V(y) = V_i e^{\gamma y} + V_r e^{-\gamma y}$$

$$I(y) = \frac{1}{Z_c}(V_i e^{\gamma y} - V_r e^{-\gamma y}) = I_i e^{\gamma y} + I_r e^{-\gamma y}$$

load impedance：

$$V(0) = V_i + V_r$$

$$I(0) = \frac{1}{Z_c}(V_i - V_r) = I_i + I_r$$

$$Z_L = \frac{V(0)}{I(0)} = Z_c \frac{V_i + V_r}{V_i - V_r}$$

the reflection coefficient：

$$\frac{V_r}{V_i} = \frac{Z_L - Z_c}{Z_L + Z_c} = S_v(0)$$ ⇨ matching condition：$Z_L = Z_c$

$$S(y) = \frac{V_r e^{-\gamma y}}{V_i e^{\gamma y}} = S(0) e^{-2\gamma y}$$

# Input impedance

The impedance observed at an arbitrary position is given by

$$Z(y) = \frac{V(y)}{I(y)} = Z_c \frac{V_i e^{\gamma y} + V_r e^{-\gamma y}}{V_i e^{\gamma y} - V_r e^{-\gamma y}} = Z_c \frac{1 + S(y)}{1 - S(y)} = Z_c \frac{Z_L + Z_c \tanh \gamma y}{Z_c + Z_L \tanh \gamma y}$$

in case of a loss-less transmission line：

$$Z(y) = R_c \frac{Z_L + jR_c \tan \beta y}{R_c + jZ_L \tan \beta y}$$

normalized representation：

$$z_L = \frac{Z_L}{Z_c}$$

$$S(0) = \frac{z_L - 1}{z_L + 1}$$

$$z(y) = \frac{z_L + \tanh \gamma y}{1 + z_L \tanh \gamma y}$$

# 2-3. Standing wave

Consider a loss-less transmission line.

$$V(y) = V_i e^{j\beta y}(1 + S(0)e^{-j2\beta y})$$

$$I(y) = \frac{V_i e^{j\beta y}}{R_c}(1 - S(0)e^{-j2\beta y})$$

the ratio of max. and min. voltage along the line : standing wave ratio

$$\rho = \frac{|V(y)|_{\max}}{|V(y)|_{\min}} = \frac{1 + |S(0)|}{1 - |S(0)|}$$

When $|V(y)|$ takes a max. value,

$$Z_{in} = R_{\max} = \frac{|V|_{\max}}{|I|_{\min}} = R_c \frac{1+|S(0)|}{1-|S(0)|} = R_c \rho \quad \text{(resistance)}$$

When $|V(y)|$ takes a min. value,

$$Z_{in} = R_{\min} = \frac{|V|_{\min}}{|I|_{\max}} = R_c \frac{1}{\rho} \quad \text{(resistance)}$$

# 2-4. λ/4 transformer

Consider a loss-less transmission line.

$$z(l) = \frac{z_L + j\tan\beta l}{1 + jz_L \tan\beta l}$$

$$z(l+\frac{\lambda}{4}) = \frac{z_L + j\tan\beta(l+\frac{\lambda}{4})}{1 + jz_L \tan\beta(l+\frac{\lambda}{4})} = \frac{jz_L \tan\beta l + 1}{j\tan\beta l + z_L} = \frac{1}{z(l)}$$

$$z(l+\frac{\lambda}{4})z(l) = 1$$

$$\therefore\ Z(l+\frac{\lambda}{4})Z(l) = R_c^2$$

λ/4 segment of transmission line $R_x$ is connected to $R_L$

$$Z_{in}R_L = R_x^2$$

No reflection occurs, if $Z_{in} = R_c$

$$\Rightarrow\ R_x = \sqrt{R_c R_L}$$

# 2-5. Smith chart

Map the reflection coefficient on a complex plane, and show the correspondence to an impedance.

$$S = \frac{z-1}{z+1} = \frac{(r-1)+jx}{(r+1)+jx} = U + jV$$

(i)map $r$ on a complex plane

$$\left(U - \frac{r}{r+1}\right)^2 + V^2 = \left(\frac{1}{r+1}\right)^2$$

a set of circles centered at $(r/(r+1),0)$ with a radius of $1/(r+1)$

(ii)map $x$ on a complex plane

$$(U-1)^2 + \left(V - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$$

a set of circles centered at $(1,1/x)$ a radius of $1/x$

Plot $z_L = r + jx$, then you will find $S(0)$.

Rotate $S(0)$ by an angle of $-2\beta y = -2\pi(2y/\lambda)$,
then you will find the impedance measured at this position $r' + jx'$.