

12. Resonator

Operation principles and designs are addressed in several types of resonators. Also, the class includes an important parameter, Q factor.

12.1 Transmission line resonator

12.2 Ring resonator

12.3 Application to a wavelength meter

12.4 Q factor

Resonant condition : transmission line

Current flowing in a series connection of Z_1 and Z_2 :

$$I = \frac{E}{Z_1 + Z_2}$$

$$Z_1 + Z_2 = 0 \quad \longrightarrow \quad I = \infty \quad (\text{resonance})$$

Circuit can be divided into two at an arbitrary position.

Example : transmission line terminated with short circuit:

$$\text{input impedance (length } l) \quad Z_{in} = jR_c \tan \beta l$$

$$R_c (\tan \beta l_1 + \tan \beta l_2) = 0$$

$$\tan \beta(l_1 + l_2) = \frac{\tan \beta l_1 + \tan \beta l_2}{1 - \tan \beta l_1 \tan \beta l_2} = 0$$

$$\beta(l_1 + l_2) = m\pi$$

$$\therefore l_1 + l_2 = m \frac{\lambda}{2} \quad (m = 1, 2, \dots)$$

Resonance : voltage and current distribution

$$V(y) = V_i e^{j\beta y} + V_r e^{-j\beta y}$$

$$I(y) = \frac{1}{R_c} (V_i e^{j\beta y} - V_r e^{-j\beta y})$$

$$V=0 \text{ at } y=0 \quad \longrightarrow \quad V_i = -V_r$$

$$\therefore V(y) = V_i (e^{j\beta y} - e^{-j\beta y}) = 2jV_i \sin \beta y$$

$$I(y) = \frac{V_i}{R_c} 2 \cos \beta y$$



$$\therefore v(y,t) = \sqrt{2} \operatorname{Re}[V(y)e^{j\omega t}] = 2\sqrt{2}V_i \sin \beta y \cos(\omega t + \frac{\pi}{2})$$

$$i(y,t) = 2\sqrt{2} \frac{V_i}{R_c} \cos \beta y \cos \omega t$$

- (a) The phases of fields (voltage and current) are not a function of position (constant in the cavity).
- (b) 90-deg phase difference exists between a magnetic and an electric field.

Q factor

Quality factor of resonator:

$$Q = 2\pi f_0 \frac{L}{R} = \frac{\frac{1}{2} 2\pi f_0 L |I_0|^2}{\frac{1}{2} R |I_0|^2} = \omega_0 \frac{\text{Maximum energy stored in an inductance}}{\text{Energy dissipated in one cycle}}$$

if conductor loss is dominant:

$$Q = \omega_0 \frac{\frac{\mu}{2} \iiint |H|^2 dv}{\frac{R_s}{2} \iint |\mathbf{i}|^2 dS}$$

$$R_s = \frac{1}{\sigma \delta} \quad (\delta : \text{skin depth}) \text{ is a surface resistance.}$$

$$|\mathbf{i}| = |\mathbf{H}_t| \quad (\mathbf{H}_t : \text{a magnetic field component tangential to a conductor surface})$$

$$\text{Transmission line : } Q = \frac{\omega_0 L}{R + G \frac{L}{C}} = \frac{\omega_0}{\frac{R}{L} + \frac{G}{C}} = \frac{\omega_0}{\frac{2\alpha}{\sqrt{LC}}} = \frac{\beta_0}{2\alpha}$$

External Q

total Quality factor :

$$Q_t = \frac{\omega_0 L}{R + R_{ext}}$$

$$\frac{1}{Q_t} = \frac{R}{\omega_0 L} + \frac{R_{ext}}{\omega_0 L} = \frac{1}{Q_0} + \frac{1}{Q_{ext}}$$

$$Q_{ext} \equiv \frac{\omega_0 L}{R_{ext}} > 0 : \text{external } Q$$

$$\frac{1}{Q_t} > \frac{1}{Q_0} \quad Q_0: \text{unloaded } Q$$

13.Multi/Demultiplexer

13.1 Equivalent circuit of a resonator

13.2 Band-pass filter

13.3 Band-stop (band-reject) filter

13.4 Demultiplexer (mainly used in lightwave circuit)

Arrayed waveguide grating (AWG)

Mach-Zehnder interferometer

Equivalent circuit of resonator

short-open transmission line

$$Y_{in} = \frac{1}{jR_c} \cot \beta l \quad \longrightarrow \quad \text{resonance at } \beta l = \frac{\pi}{2}$$

open short

$$Y_{in} = \frac{1}{jR_c} \cot \beta l = -j \frac{1}{R_c} \cot(\omega \sqrt{\epsilon \mu} l) = -j \frac{1}{R_c} \cot(\omega_r \sqrt{\epsilon \mu} l + \Delta \omega \sqrt{\epsilon \mu} l)$$

$$= -j \frac{1}{R_c} \cot\left(\frac{\pi}{2} + \Delta \omega \sqrt{\epsilon \mu} l\right) \approx j \frac{1}{R_c} \Delta \omega \sqrt{\epsilon \mu} l = j \frac{1}{R_c} \frac{\pi}{2} \frac{\Delta \omega}{\omega_r}$$

$$L_e \parallel C_e \quad \longrightarrow \quad Y_e = j\omega C_e + \frac{1}{j\omega L_e} = j(\omega_r + \Delta \omega)C_e + \frac{1}{j(\omega_r + \Delta \omega)L_e}$$

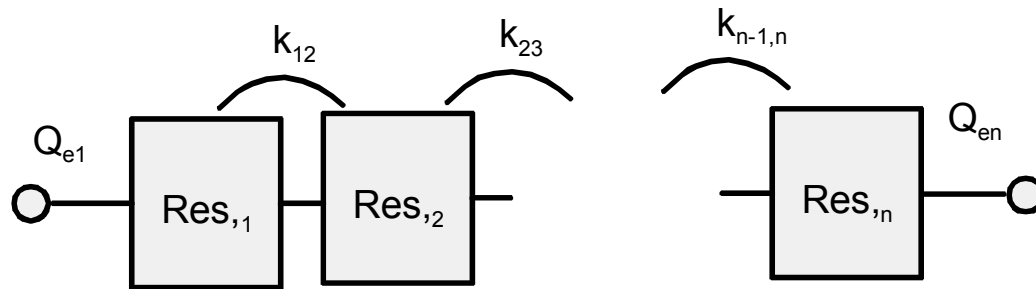
$$\approx j(\omega_r + \Delta \omega)C_e + \frac{1}{j\omega_r L_e} \left(1 - \frac{\Delta \omega}{\omega_r}\right) = j\Delta \omega C_e + j \frac{1}{\omega_r L_e} \frac{\Delta \omega}{\omega_r}$$

$$= j\Delta \omega C_e \left(1 + \frac{1}{\omega_r^2 L_e C_e}\right) = j2\Delta \omega C_e$$

$$Y_{in} = Y_e \quad \longrightarrow \quad \boxed{C_e = \frac{\pi}{4\omega_r R_c}, \quad L_e = \frac{4R_c}{\pi\omega_r}}$$

Band-pass filter

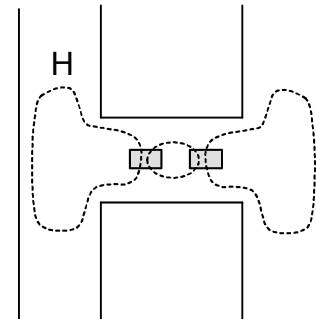
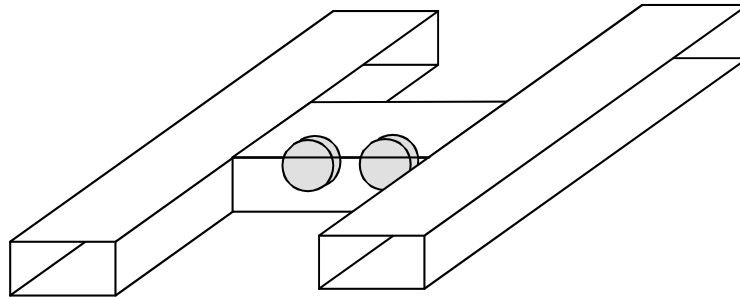
Line - resonator -line



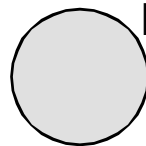
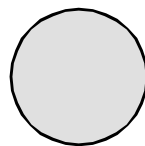
electrical length
between resonators

$d=0$: double size
($2R, 2L$), invariant Q

$d=\pi/4$: transformer
improved Q



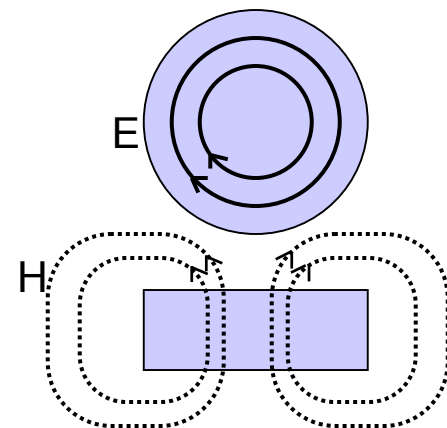
Micro strip line



Dielectric
resonator

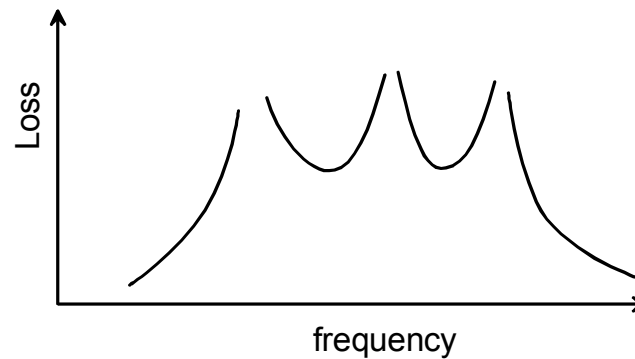
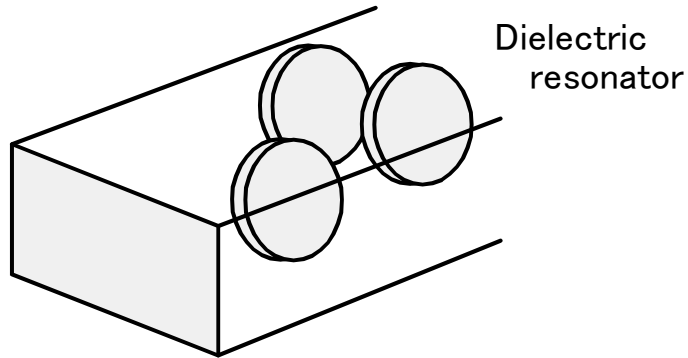


$TE_{01\delta}$ mode



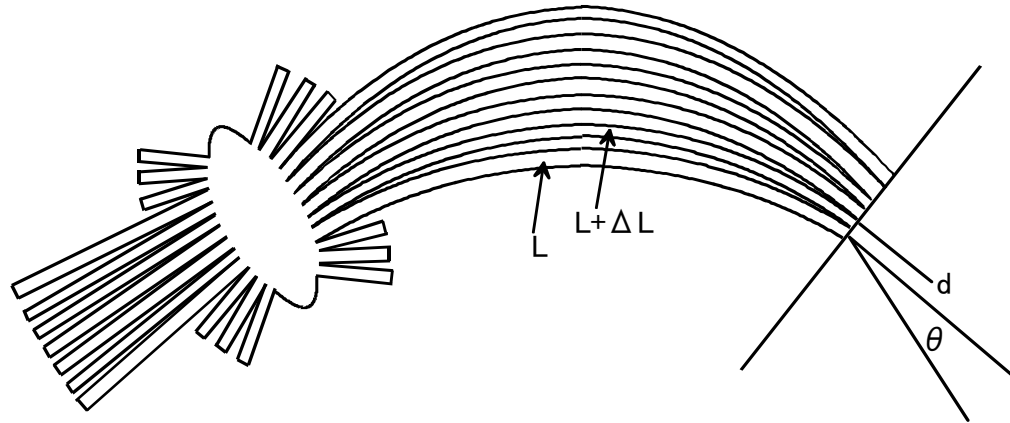
Band-stop filter

at resonance condition : Resonator prohibits transmission.



Demultiplexer : AWG

Arrayed waveguide grating (AWG)



path difference ΔL  $(n_s d \sin \theta + n_c \Delta L)k_0$

in-phase condition

$$(n_s d \sin \theta + n_c \Delta L)k_0 = 2m\pi$$

$$n_s d \sin \theta + n_c \Delta L = m\lambda$$

$$\therefore \sin \theta = \frac{m\lambda - n_c \Delta L}{n_s d}$$

Lightwave of wavelength λ focuses at angle θ .

AWG wavelength sensitivity

$$n_s d \sin \theta_1 + n_c \Delta L = m \lambda_1$$

$$n_s d \sin \theta_2 + n_c \Delta L = m \lambda_2$$

$$m(\lambda_1 - \lambda_2) = m \Delta \lambda = n_s d (\sin \theta_1 - \sin \theta_2)$$

$$= n_s d (\sin \theta_1 - \sin \theta_1 \cos \Delta \theta - \cos \theta_1 \sin \Delta \theta)$$

$$\approx -n_s d \Delta \theta \cos \theta$$

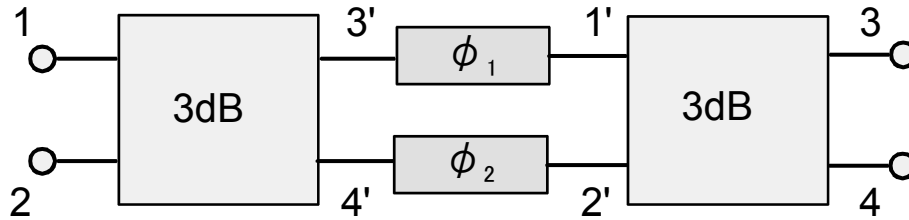
$$\therefore \Delta \lambda = \frac{-n_s d \cos \theta}{m} \Delta \theta$$

angular displacement -(lens with focal distance f) --> spatial displacement

$$\Delta x = f \Delta \theta = \frac{-mf}{n_s d \cos \theta} \Delta \lambda$$

$$\therefore \frac{\Delta x}{\Delta \lambda} = \frac{-mf}{n_s d \cos \theta}$$

Mach-Zehnder interferometer



3dB directional coupler

$$\begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & -j\frac{1}{\sqrt{2}} \\ 0 & 0 & -j\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -j\frac{1}{\sqrt{2}} & 0 & 0 \\ -j\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

Output

$$\begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}(e^{j\phi_1} - e^{j\phi_2}) \\ -\frac{j}{2}(e^{j\phi_1} + e^{j\phi_2}) \end{bmatrix}$$

output from cross port

$$\varphi_2 - \varphi_1 = 2n\pi \rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ -je^{j\varphi_1} \end{bmatrix}$$

output from bar port

$$\varphi_2 - \varphi_1 = (2n-1)\pi \rightarrow \begin{bmatrix} 0 \\ 0 \\ e^{j\varphi_1} \\ 0 \end{bmatrix}$$

Mach-Zehnder interferometer

When the phase difference is

(a) controlled by an external field:

$$\varphi_2 - \varphi_1 = \begin{cases} 2n\pi \\ (2n-1)\pi \end{cases} \quad \longrightarrow \text{switch / modulator}$$

(b) dependent on frequency :

$$\varphi_2(f) - \varphi_1(f) = \begin{cases} 2n\pi \\ (2n-1)\pi \end{cases} \quad \longrightarrow \text{demultiplexer}$$

(c) dependent on propagation direction :

$$\varphi_2 - \varphi_1|_{\pm} = \begin{cases} 2n\pi \\ (2n-1)\pi \end{cases} \quad \longrightarrow \text{circulator}$$