Fundamentals of Dynamics (12)

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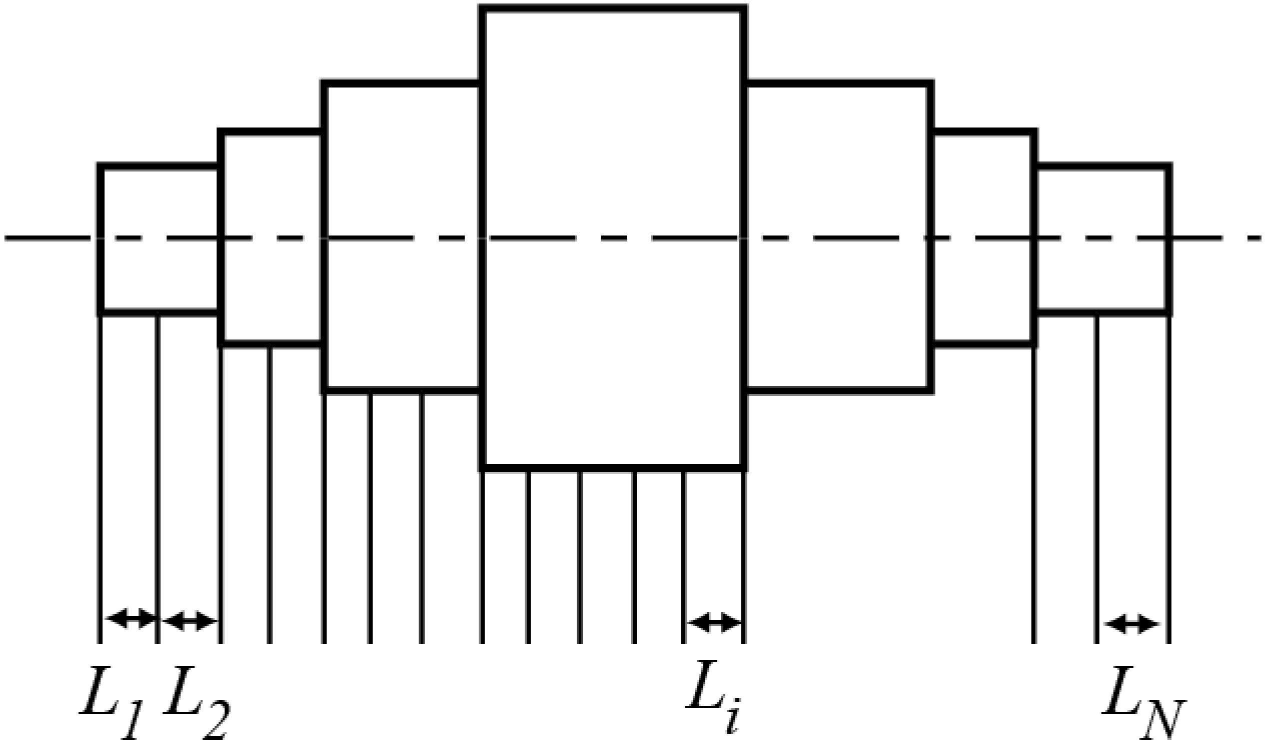
Multi-degree-of-freedom Vibration System (1)

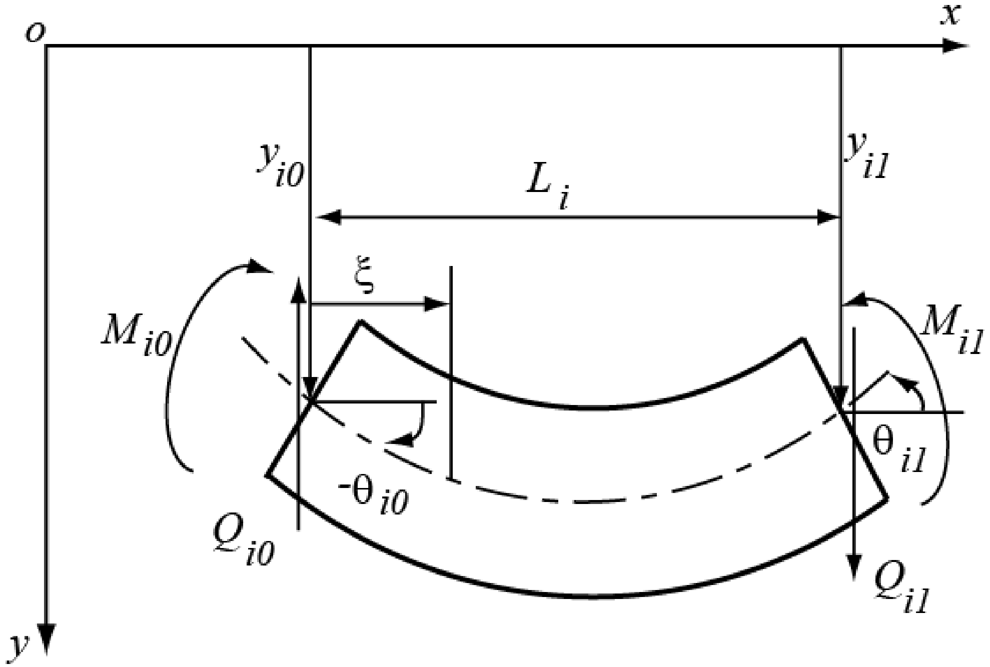
Finite Element Method

Bending Vibration of a Beam (1)



Beam and finite elements

Bending Vibration of a Beam (2)



Equation of Motion

$$\rho_i A_i \frac{\partial^2 y}{\partial t^2} + E_i I_i \frac{\partial^4 y}{\partial x^4} = 0$$

Coordinates and acting forces on a finite element

Finite Element Method (1)

Static Deformation

$$E_i I_i \frac{\partial^4 y}{\partial x^4} = 0 \longrightarrow$$

$y(x)$ is a third order polynomial function

⇒Four boundary conditions can determine the function

$$y(\xi) = N_1(\xi) y_{i0} + N_2(\xi)\theta_{i0} + N_3(\xi) y_{i1} + N_4(\xi)\theta_{i1}$$

$N_j(\xi)$: Shape function

Finite Element Method (2)

$y(\xi)$ is a approximate solution for the equation of motion

$$\rho_i A_i \frac{\partial^2 y}{\partial t^2} + E_i I_i \frac{\partial^4 y}{\partial x^4} \neq 0$$

$$\int_0^{L_i} N_k(\xi) \left(\rho_i A_i \frac{\partial^2 y}{\partial t^2} + E_i I_i \frac{\partial^4 y}{\partial x^4} \right) d\xi = 0$$

$$(k = 1, \cdots, 4)$$

Finite Element Method (3)

$$y(\xi) = \mathbf{N}^T \mathbf{u}_i$$

$$\mathbf{N} = \{N_1(\xi)\ N_2(\xi)\ N_3(\xi)\ N_4(\xi)\}^T$$

$$\mathbf{u}_i = \{y_{i0}\ \theta_{i0}\ y_{i1}\ \theta_{i1}\}^T$$

Finite Element Method (4)

$$\rho_i A_i \int_0^{L_i} N_k(\xi) \left(\mathbf{N}^T \frac{d^2 \mathbf{u}_i}{dt^2} \right) d\xi$$

$$= \rho_i A_i \left[\int_0^{L_i} N_k(\xi) N_1(\xi) d\xi \;\; \int_0^{L_i} N_k(\xi) N_2(\xi) d\xi \;\; \int_0^{L_i} N_k(\xi) N_3(\xi) d\xi \;\; \int_0^{L_i} N_k(\xi) N_4(\xi) d\xi \right] \ddot{\mathbf{u}}_i$$

$$E_i I_i \int_0^{L_i} N_k(\xi) \left(\frac{d^4 \mathbf{N}^T}{d\xi^4} \mathbf{u}_i \right) d\xi$$

$$= E_i I_i \left[\int_0^{L_i} N_k'(\xi) N_1'(\xi) d\xi \;\; \int_0^{L_i} N_k'(\xi) N_2'(\xi) d\xi \;\; \int_0^{L_i} N_k'(\xi) N_3'(\xi) d\xi \;\; \int_0^{L_i} N_k'(\xi) N_4'(\xi) d\xi \right] \mathbf{u}_i$$

$$- Q_{i1} N_k(L_i) + Q_{i0} N_k(0) + M_{i1} N_k'(L_i) - M_{i0} N_k'(0)$$

Finite Element Method (5)

$$\mathbf{M}_i \ddot{\mathbf{u}}_i + \mathbf{K}_i \mathbf{u}_i = \mathbf{f}_i$$

$$\mathbf{M}_i = \frac{\rho_i A_i L_i}{420} \begin{bmatrix} 156 & 22L_i & 54 & -13L_i \\ & 4L_i^2 & 13L_i & -3L_i^2 \\ & & 156 & -22L_i \\ \text{Sym.} & & & 4L_i^2 \end{bmatrix}$$

$$\mathbf{K}_i = \frac{E_i I_i}{L_i^3} \begin{bmatrix} 12 & 6L_i & -12 & 6L_i \\ & 4L_i^2 & -6L_i & 2L_i^2 \\ & & 12 & -6L_i \\ \text{Sym.} & & & 4L_i^2 \end{bmatrix} \qquad \mathbf{f}_i = \begin{Bmatrix} -Q_{i0} \\ M_{i0} \\ Q_{i1} \\ -M_{i1} \end{Bmatrix}$$

Finite Element Method (6)

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}$$

$$\mathbf{M} \in R^{(2N+2)\times(2N+2)}$$

$$\mathbf{K} \in R^{(2N+2)\times(2N+2)}$$

$$\mathbf{u} = \{y_{10}\ \theta_{10}\ y_{20}\ \theta_{20} \cdots y_{N0}\ \theta_{N0}\ y_{N1}\ \theta_{N1}\}^T$$

$$\mathbf{f}_i = \{-Q_{i0}\ M_{i0}\ 0\ 0 \cdots 0\ 0\ Q_{N1}\ -M_{N1}\}^T$$

Four from 2n+6 unknown parameters are determined from four boundary conditions.

Finite Element Method (7)

$$\boxed{\mathbf{M}_r \ddot{\mathbf{u}}_r + \mathbf{K}_r \mathbf{u}_r = \mathbf{0}}$$

Natural frequencies are calculated by solving a generalized eigenvalue problem.

The response is also calculated numerically.