Fundamentals of Dynamics (4)

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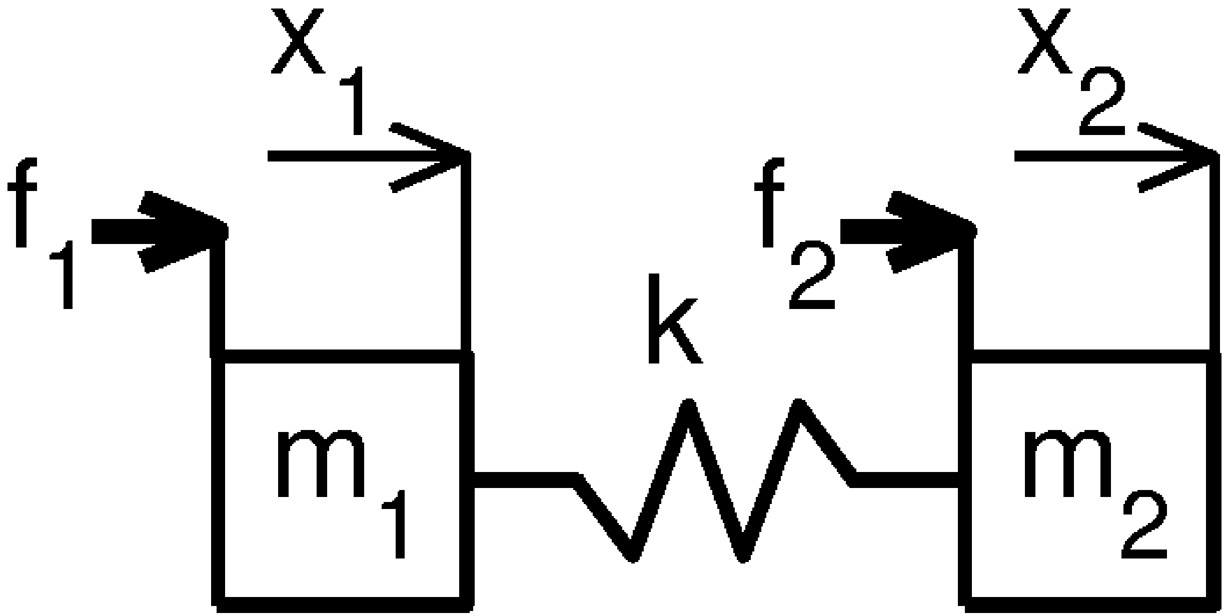
Two-degree-of-freedom Positioning Mechanism (1)

Analytical Model



Analytical model of a two-degree-of-freedom positioning mechanism

Equation of Motion

$$m_1\ddot{x}_1 + k(x_1 - x_2) = f_1$$

$$m_2\ddot{x}_2 + k(x_2 - x_1) = f_2$$

These equations can be written as $\boxed{\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}}$

where $\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$

$$\mathbf{K} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}, \quad \mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

Similar expression with One-degree-of-freedom vibration system

Modal Analysis (1)

No exciting force = Free vibration = Initial value response

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$$

General eigen value problem

$$(\mathbf{K} - \lambda \mathbf{M})\mathbf{v} = \mathbf{0}$$

Solutions

$$\lambda_1 = 0,\ \lambda_2 = \frac{k(m_1 + m_2)}{m_1 m_2}$$

and

$$\mathbf{v}_1 = \begin{Bmatrix} \alpha \\ \alpha \end{Bmatrix},\ \mathbf{v}_2 = \begin{Bmatrix} \beta \\ -\frac{m_1}{m_2}\beta \end{Bmatrix}$$

The value of α and β are determined later.

Modal Analysis (2)

Orthoganatity of eigen vectors

$$\mathbf{v}_i^T \mathbf{M} \mathbf{v}_j = \begin{cases} 0 & i \neq j \\ \overline{m}_i & i = j \end{cases}, \quad \mathbf{v}_i^T \mathbf{K} \mathbf{v}_j = \begin{cases} 0 & i \neq j \\ \overline{k}_i & i = j \end{cases}$$

$\overline{m}_i$: Modal mass, $\overline{k}_i$: Modal stiffness $\longrightarrow$ $\boxed{\lambda_i = \dfrac{\overline{k}_i}{\overline{m}_i}}$

If we define a modal matrix $\mathbf{V} \equiv \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$,

then

$$\mathbf{V}^T \mathbf{M} \mathbf{V} = \begin{bmatrix} \overline{m}_1 & 0 \\ 0 & \overline{m}_2 \end{bmatrix}, \quad \mathbf{V}^T \mathbf{K} \mathbf{V} = \begin{bmatrix} \overline{k}_1 & 0 \\ 0 & \overline{k}_2 \end{bmatrix}.$$

Modal Analysis (3)

Determination of α and β

(1) $\alpha = \beta = 1$

(2) $\left\| \mathbf{v}_i \right\| = 1$

(3) $\overline{m}_i = 1 \ (\overline{k}_i = \lambda_i)$

Modal Analysis (4)

(3) $\bar{m}_i = 1 \ \ (\bar{k}_i = \lambda_i)$

$$\alpha = \frac{1}{\sqrt{m_1 + m_2}}, \quad \beta = \sqrt{\frac{m_2}{m_1(m_1 + m_2)}}$$

$$\mathbf{v}_1 = \begin{Bmatrix} \dfrac{1}{\sqrt{m_1 + m_2}} \\ \dfrac{1}{\sqrt{m_1 + m_2}} \end{Bmatrix}, \ \mathbf{v}_2 = \begin{Bmatrix} \sqrt{\dfrac{m_2}{m_1(m_1 + m_2)}} \\ -\sqrt{\dfrac{m_1}{m_2(m_1 + m_2)}} \end{Bmatrix}$$