Fundamentals of Dynamics (2)

Department of Mechanical and Control Engineering

Hiroshi Yamaura

One-degree-of-freedom Vibration System (2)

Complex exciting force and response (1)

Damped one degree-of-freedom system

$$\boxed{m\ddot{x} + c\dot{x} + kx = f}$$

Exciting force　　　　Response

$$f_c = F\cos\omega t \Rightarrow x(t) = A\cos(\omega t - \phi)$$

$$f_s = F\sin\omega t \Rightarrow x(t) = A\sin(\omega t - \phi)$$

Complex exciting force and response (2)

Complex exciting force

$$f(t) = F\cos\omega t + iF\sin\omega t$$
$$= F\exp(i\omega t)$$

Response

$$x(t) = A\cos(\omega t - \phi) + iA\sin(\omega t - \phi)$$
$$= A\exp(-i\phi)\exp(i\omega t)$$
$$= X(i\omega)\exp(i\omega t)$$

Complex exciting force and response (3)

If once $X(i\omega)$ is calculated,

$$x_c(t) = \mathrm{Re}\{X(i\omega)\exp(i\omega t)\}$$

$$x_s(t) = \mathrm{Im}\{X(i\omega)\exp(i\omega t)\}$$

and

$$A(\omega) = | X(i\omega) |$$

$$\phi(\omega) = -\angle X(i\omega)$$

Frequency response function for force excitation (1)

Damped one degree-of-freedom system

$$\boxed{m\ddot{x} + c\dot{x} + kx = f}$$

Assume $x(t) = X(i\omega)\exp(i\omega t)$

and $f(t) = F\exp(i\omega t)$

then

$$\boxed{\frac{X(i\omega)}{F} = \frac{1}{(k - m\omega^2) + ic\omega}}$$

Frequency response function for force excitation (2)

Define $$X_{st} = \frac{F}{k}$$

Dynamic amplitude ratio
$$\boxed{\begin{aligned}\frac{X(i\omega)}{X_{st}} &= \frac{k}{(k - m\omega^2) + ic\omega} \\ &= \frac{1}{(1-\Omega^2) + i2\zeta\Omega}\end{aligned}}$$

where $$\Omega = \frac{\omega}{\omega_n}$$

Frequency response function for force excitation (3)

$$Gain(\mathrm{dB}) = 20\log_{10}\left|\frac{X(i\omega)}{X_{st}}\right|$$

$$= 20\log_{10}\left|\frac{1}{(1-\Omega^2) + i2\zeta\Omega}\right|$$

$$Phase(\mathrm{deg}) = \tan^{-1}\left(\frac{-2\zeta\Omega}{1-\Omega^2}\right)$$

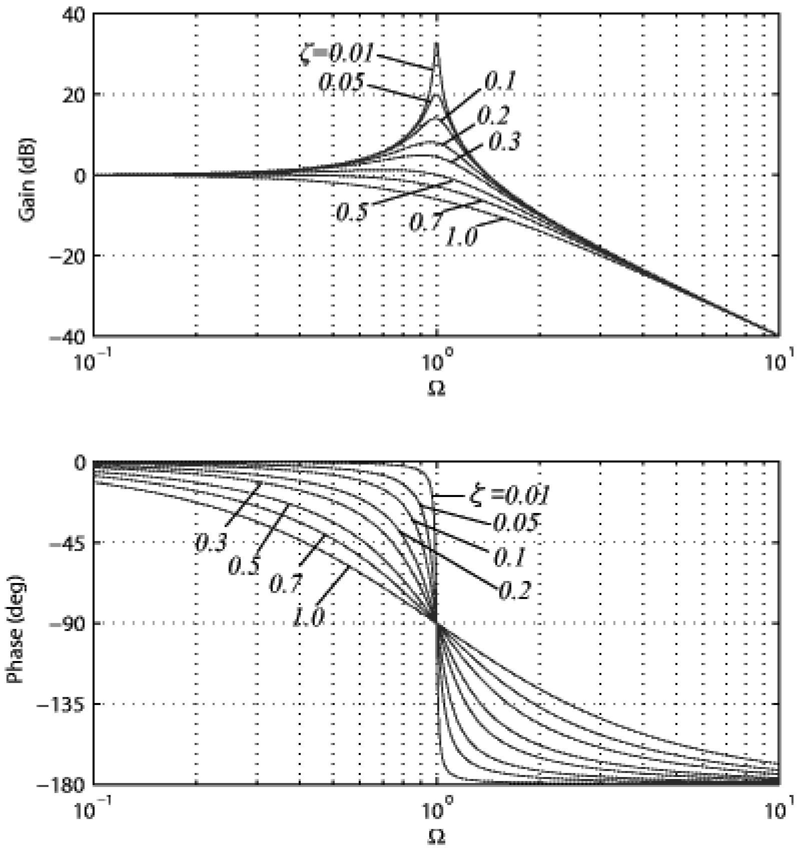

Fig.1 Frequency response function of the dynamic amplitude ratio

Frequency response function for base excitation (1)

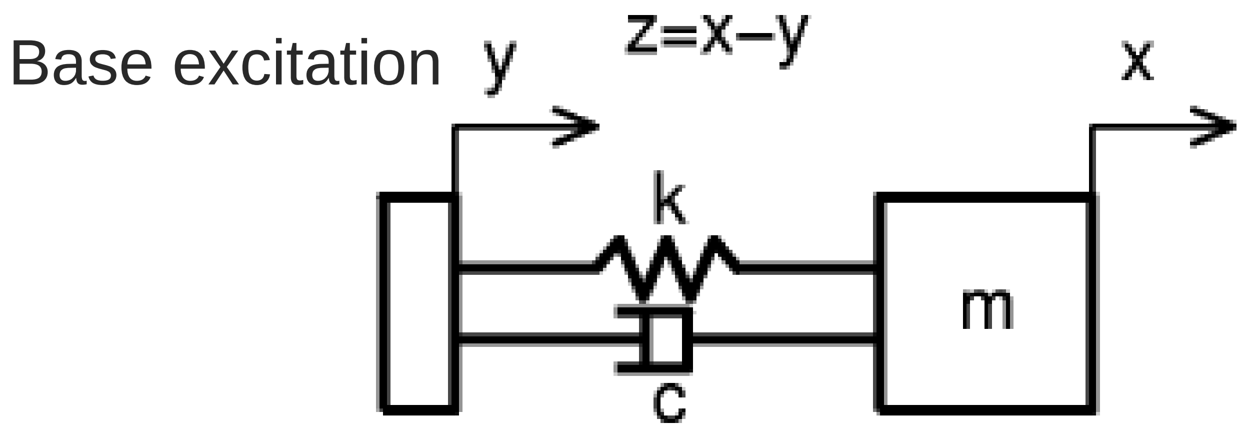

Fig.2 One degree-of-freedom vibration system with base excitation

Absolute displacement *x* $\boxed{m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky}$

Relative displacement *z* $\boxed{m\ddot{z} + c\dot{z} + kz = -m\ddot{y}}$

Frequency response function for base excitation (2)

Absolute displacement *x*

$$\boxed{\begin{aligned}\frac{X(i\omega)}{Y} &= \frac{k+ic\omega}{(k-m\omega^2)+ic\omega}\\ &= \frac{1+i2\zeta\Omega}{(1-\Omega^2)+i2\zeta\Omega}\end{aligned}}$$

Frequency response function for base excitation (3)

Absolute displacement *x*

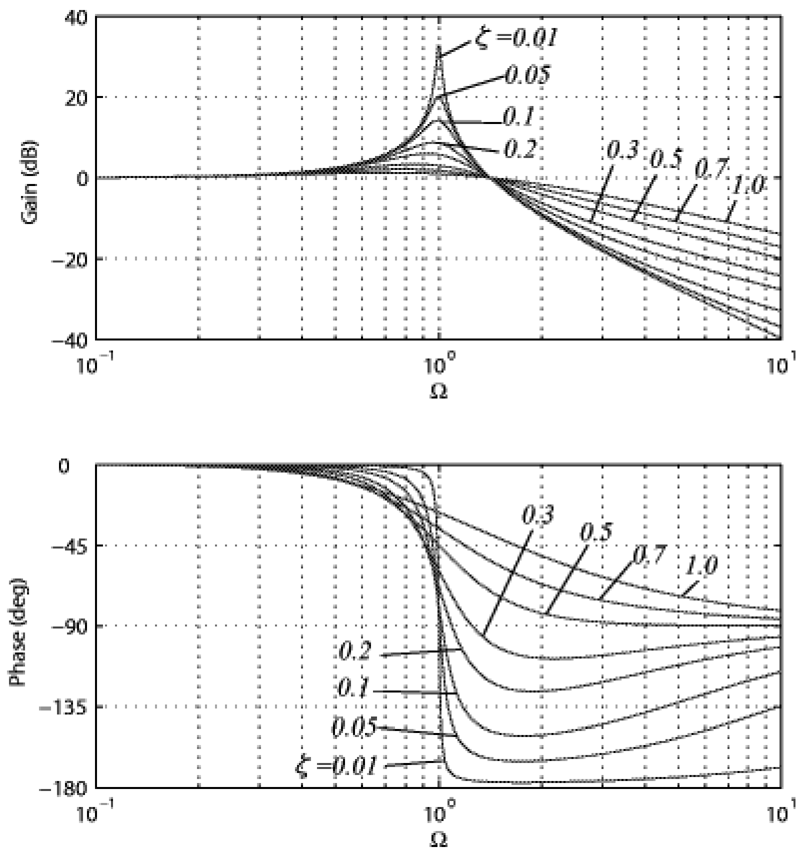

Fig.3 Frequency response function of $\frac{X(i\omega)}{Y}$

Frequency response function for base excitation (4)

Relative displacement *z*

$$\frac{Z(i\omega)}{Y} = \frac{m\omega^2}{(k - m\omega^2) + ic\omega}$$

$$= \frac{\Omega^2}{(1 - \Omega^2) + i2\zeta\Omega}$$

Frequency response function for base excitation (5)

Relative displacement *z*

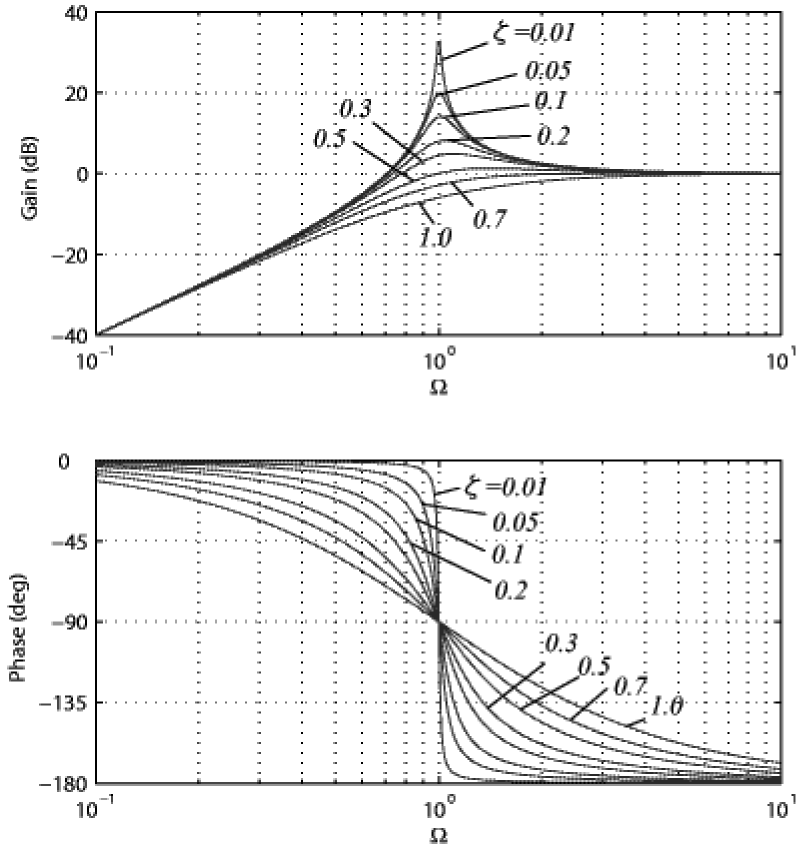

Fig.4 Frequency response function of $\frac{Z(i\omega)}{Y}$

Impulse response function (1)

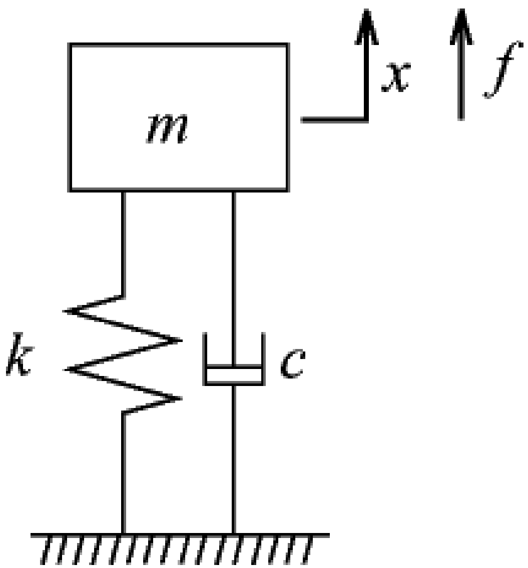

Fig.1 Damped one degree-of-freedom vibrationsystem with force excitation

Equation of motion

$$\boxed{m\ddot{x} + c\dot{x} + kx = f}$$

Unit impulse exciting force

$$f(t) = \delta(t)$$

Dirac's delta function

Impulse response function (2)

Table 1 Change of the states of the system with the unit impulse

Time	Momentum	Velocity	Displacement
$t=0$	$m\dot{x}=0$	$\dot{x}=0$	$x=0$
$t=\epsilon$	$\mathrm{m}\,\dot{x}=1$	$\dot{x}=1/m$	$x=0$

$$x(t) = e^{-\zeta\omega t}\frac{1}{m\omega_d}\sin\omega_d t \equiv h(t)$$

$h(t)$: Impulse response function

Transient response (1-1)

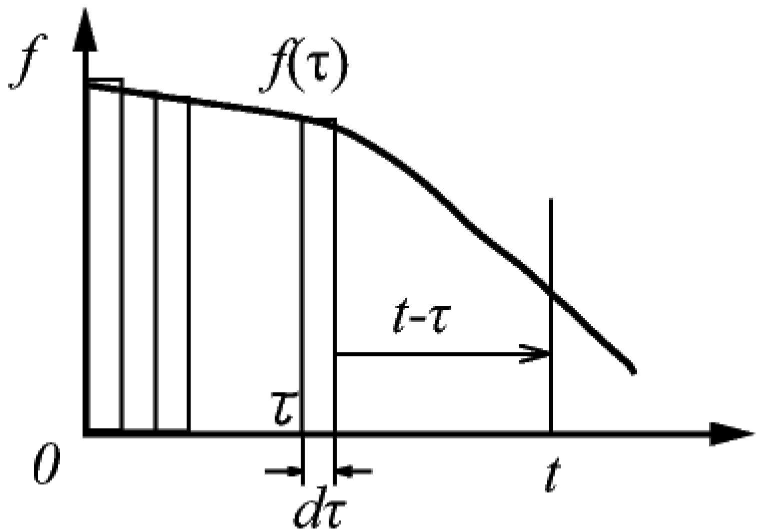

Fig.3 Decomposition of the exciting force into impulses

Transient response

$$x(t) = \int_0^t h(t-\tau) f(\tau) d\tau$$

Transient response (1-2)

Example

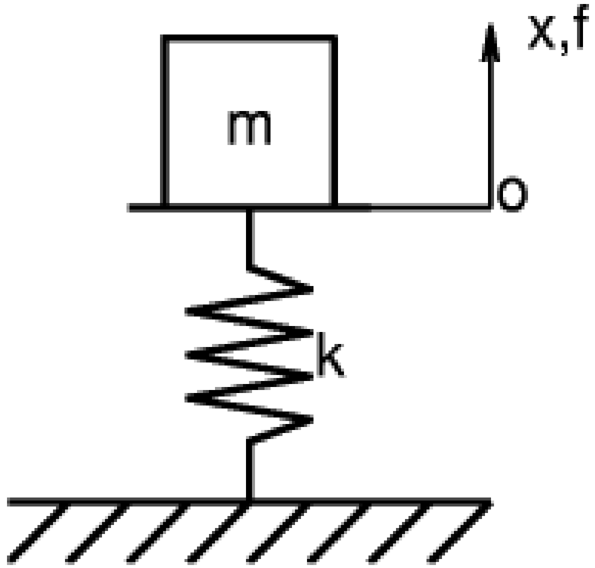

Fig.4 Undamped one degree-of-freedom vibration system

Impulse response function

$$h(t) = \frac{1}{m\omega_n}\sin\omega_n t$$

Transient response (1-3)

Example

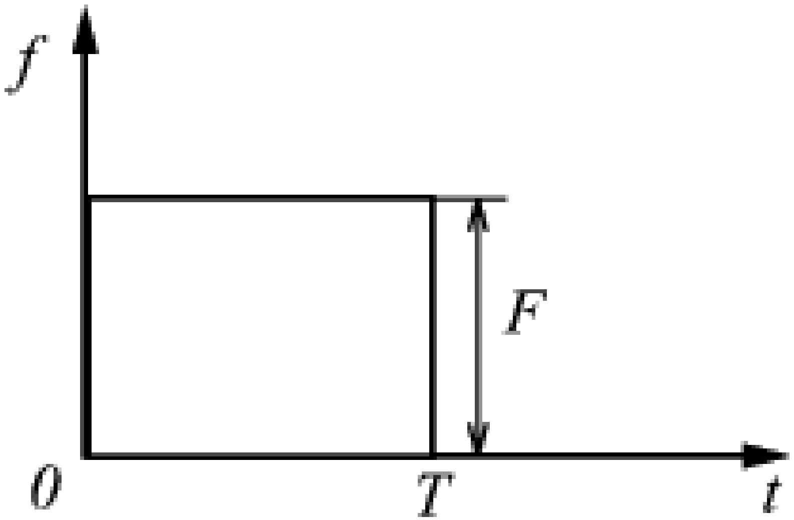

Fig.5 Exciting force

Exciting force

$$f(t) = \begin{cases} 0 & (t < 0) \\ F & (0 \le t \le T) \\ 0 & (t > T) \end{cases}$$

Transient response (1-4)

Example

$$x(t)=\begin{cases} 0 & (t<0) \\ \int_0^t Fh(t-\tau)dt & (0\le t\le T) \\ \int_0^T Fh(t-\tau)d\tau & (t>T) \end{cases}$$

Transient response (2-1)

Define the frequency response function $H(i\ \omega)$

$$H(i\omega) = \frac{X(i\omega)}{F}$$

Assume a complex exciting force

$$f(t) = F\exp(i\omega t)$$

Response against the complex exciting force

$$x(t) = FH(i\omega)\exp(i\omega t)$$

Transient response (2-2)

Inverse Fourier transformation of the exciting force

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)\exp(i\omega t)d\omega$$

Response against the exciting force

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)H(i\omega)\exp(i\omega t)d\omega$$

Fourier transformation of the response

$$X(i\omega) = F(i\omega)H(i\omega)$$

Transient response (2-3)

Necessary condition of the Laplace transformation is not strict compared to that of the Fourier transformation.

$$x(t) = \frac{1}{2\pi}\int_{\sigma - i\infty}^{\sigma + i\infty} F(s)H(s)\exp(st)ds$$

where

$$F(s) = \int_0^\infty f(t)\exp(-st)dt$$

$$H(s) = \int_0^\infty h(t)\exp(-st)dt = H(i\omega)\,|_{i\omega = s}$$

Transient response (2-4)

Table 2 Laplace transformation

Time function	Laplace transformation
$\delta(t)$	1
$u(t)$ (Unit function)	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$f(t-T)$	$F(s)e^{-sT}$
$\frac{d}{dt}f(t)$	$sF(s) - f(0)$
$\frac{d^2}{dt^2}f(t)$	$s^2F(s) - sf(0) - \frac{d}{dt}f(0)$

Transient response (2-5)

Example 1 Unit impulse response function

$$m\ddot{x} + c\dot{x} + kx = f \quad \text{and} \quad f(t) = \delta(t)$$

$$\Rightarrow \quad ms^2X(s) + csX(s) + kX(s) = 1$$

$$\Rightarrow \quad X(s) = \frac{1}{ms^2 + cs + k} \qquad \text{Transfer function}$$

$$= \frac{1}{m}\frac{i}{2\omega_d}\left(\frac{-1}{s-\lambda_1} + \frac{1}{s-\lambda_2}\right)$$

where $\lambda_{1,2} = -\zeta\omega_n \pm i\omega_d$

Transient response (2-6)

Example 1　Unit impulse response function

$$x(t) = \frac{1}{m}\frac{i}{2\omega_d}\left(-e^{\lambda_1 t} + e^{\lambda_2 t}\right)$$

$$= e^{-\zeta\omega_n t}\frac{1}{m\omega_d}\sin\omega_d t \equiv h(t)$$

Transient response (2-7)

Example 2

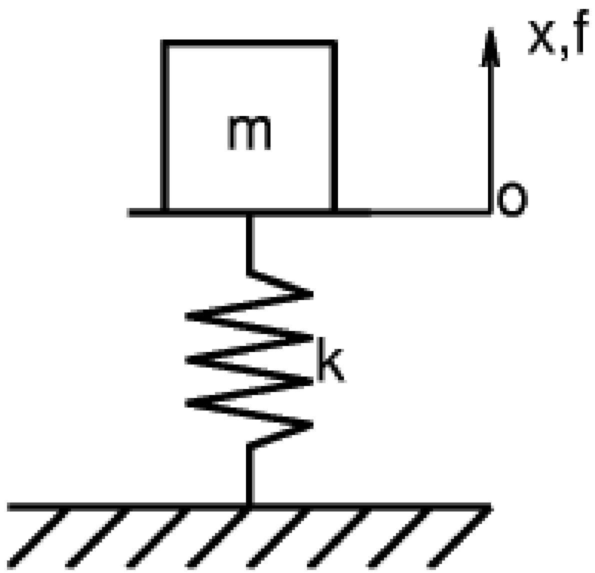

Fig.4 Undamped one degree-of-freedom vibration system

Transfer function

$$H(s) = \frac{1}{m}\frac{1}{s^2 + \omega_n^2}$$

Transient response (2-8)

Example 2

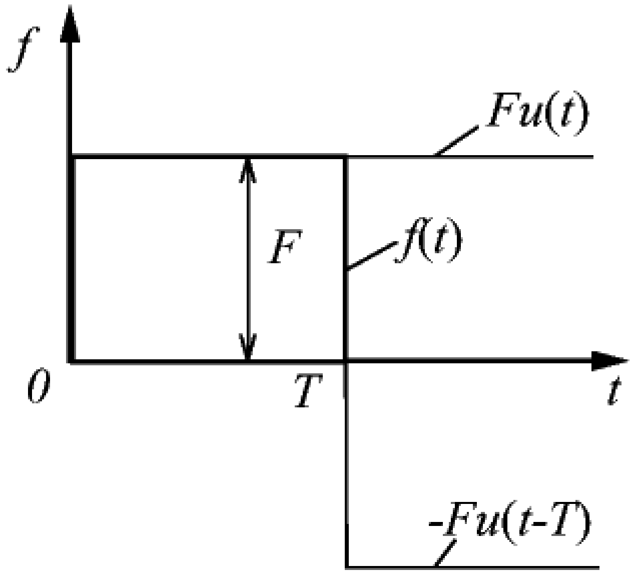

Fig.6 Decomposition of the exciting force into unit step functions

Exciting force

$$f(t)=\begin{cases} 0 & \\ Fu(t) & (0\le t\le T) \\ Fu(t)-Fu(t-T) & (t>T) \end{cases}$$

Transient response (2-9)

Example 2

$$x(t) = \begin{cases} 0 & 0 \\ L^{-1}\left[H(s)\dfrac{F}{s}\right] & (0 \le t \le T) \\ L^{-1}\left[H(s)\dfrac{F}{s}\left(1 - e^{-sT}\right)\right] & (t > T) \end{cases}$$

$L^{-1}[\ ]$: Inverse Laplace transformation