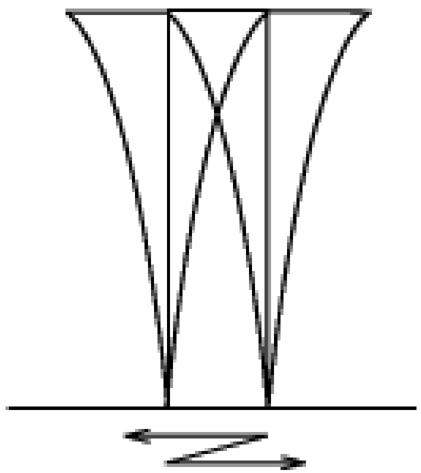
Fundamentals of Dynamics (1)

Department of Mechanical and Control Engineering

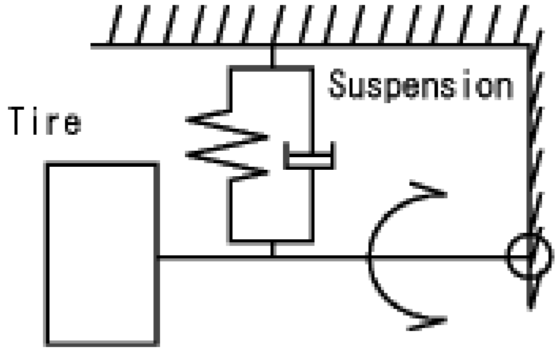
Hiroshi Yamaura

One-degree-of-freedom Vibration System (1)

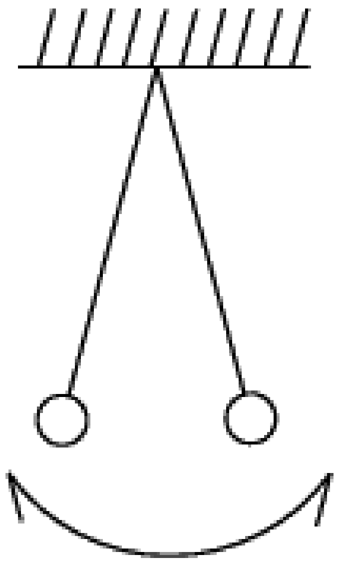
Actual vibration system (1)



(a) Tall Building



(b) Tire and suspension



(c) Simple pendulum

Fig.1 Actual vibration systems

Actual vibration system (2)

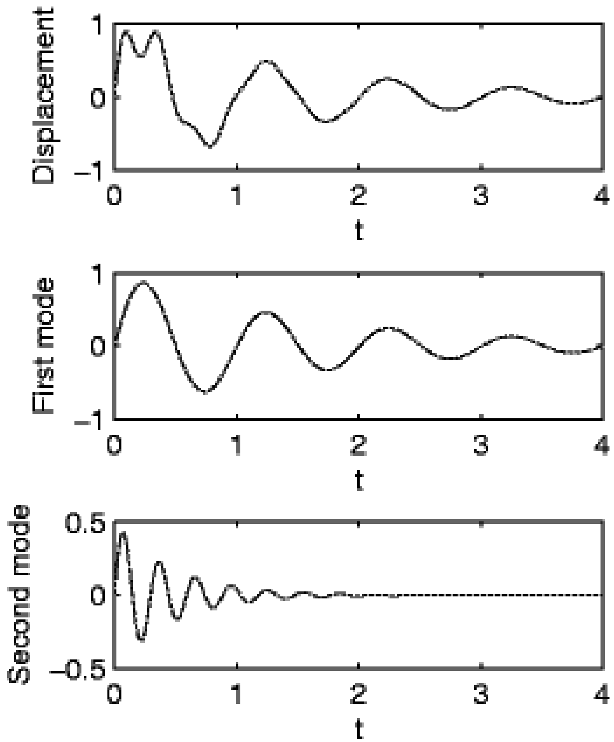


Fig.2 Example of free vibration

In many cases, the first mode is dominant in the free vibration.

Most vibration systems can be modeled as one degree-of-freedom vibration system.

Analytical model

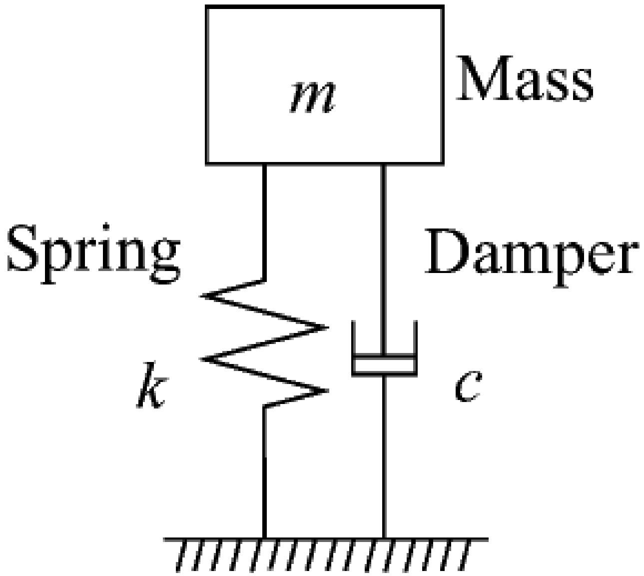


Fig.3 Analytical model of one-degree-of-freedom vibration system

SI Unit

$m:\mathrm{kg}$

$k:\mathrm{N/m}$ Stiffness constant

$c:\mathrm{Ns/m}$ Damping constant

Deriving the equation of motion (1)

- Coordinate system
  - The origin of the displacement should be placed on <u>the equilibrium point</u>.
  - The direction of the force should agree with that of the displacement. Both of the Case 1 and 2 in Fig.4 are acceptable.

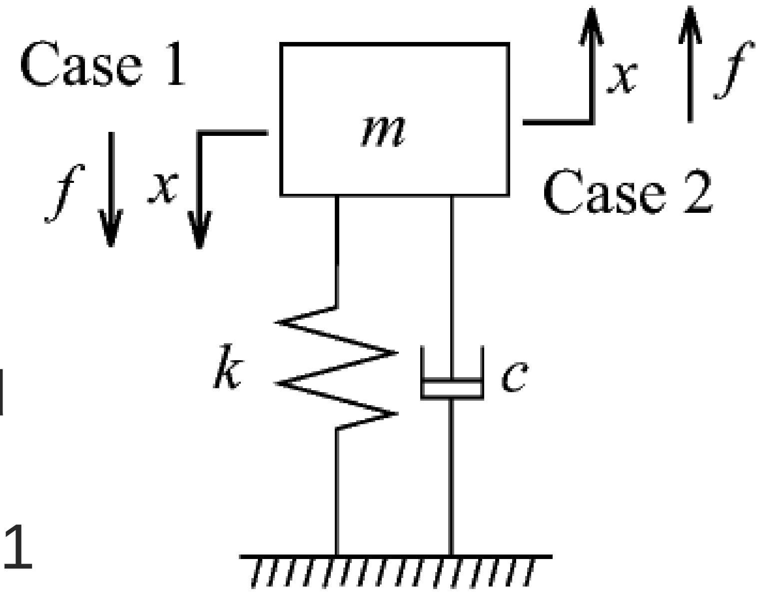


Fig.4 Coordinate system

Deriving the equation of motion (2)

- Reaction Force
  - Reaction force of the damper is proportional to the velocity of the mass and its sign is minus.
  - Reaction force of the spring is proportional to the displacement of the mass and its sign is minus.

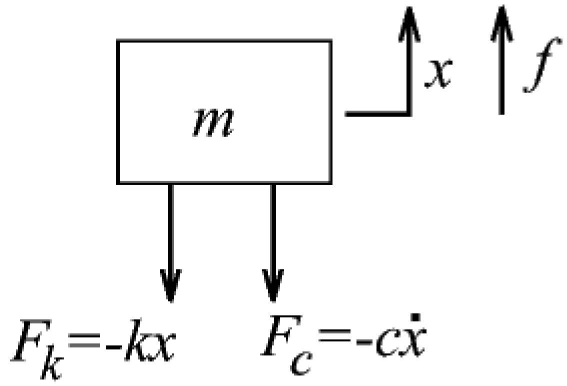


Fig.5 Free body and acting force

Deriving the equation of motion (3)

- d'Alembert's principle (1)

The sum of the differences between the generalized forces acting on a system and the time derivative of the generalized momentum of the system itself along an infinitesimal displacement compatible with the constraints of the system (a virtual displacement), is zero.

Deriving the equation of motion (4)

- d'Alembert's principle (2)

$$\left\{ F - \frac{d(mv)}{dt} \right\} \delta x = 0$$

where $F = -c\dot{x} - kx + f$

Thus, the equation of motion is represented as the following.

$$\boxed{m\ddot{x} + c\dot{x} + kx = f}$$

Natural angular frequency, natural frequency and natural period (1)

Equation of Motion of an undamped and free vibration system

$$m\ddot{x} + kx = 0$$

The characteristic equation

$$m\lambda^2 + k = 0 \quad \Rightarrow \quad \lambda = \pm i\,\omega_n, \quad \omega_n = \sqrt{\frac{k}{m}}$$

Natural angular frequency, natural frequency and natural period (2)

General solution

$$x(t) = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t}$$

$$= D_1 \cos \omega_{\mathrm{n}} t + D_2 \sin \omega_{\mathrm{n}} t$$

$$= A \cos (\omega_{\mathrm{n}} t - \phi)$$

Natural angular frequency, natural frequency and natural period (3)

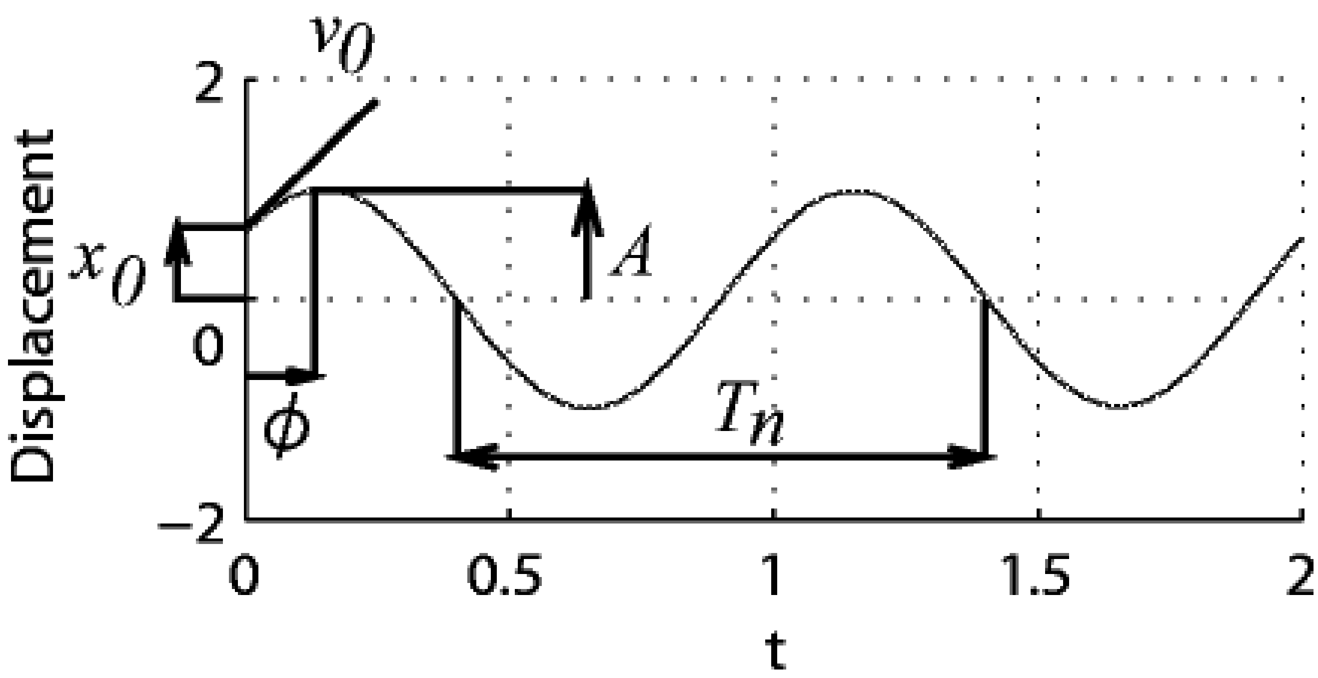


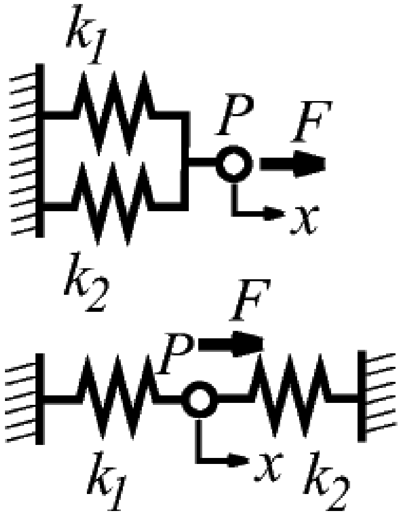
Fig.6 Example of initial value response

Natural angular frequency, natural frequency and natural period (4)

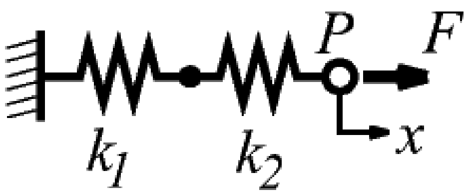
Table 2 Important parameters

Notation	Unit	Meaning	
ω_n	rad/s	Natural angular frequency	$=\sqrt{\dfrac{k}{m}}$
f_n	Hz (=1/s)	Natural frequency	$=\dfrac{\omega_n}{2\pi}$
T_n	s	Natural period	$=\dfrac{1}{f_n}=\dfrac{2\pi}{\omega_n}$

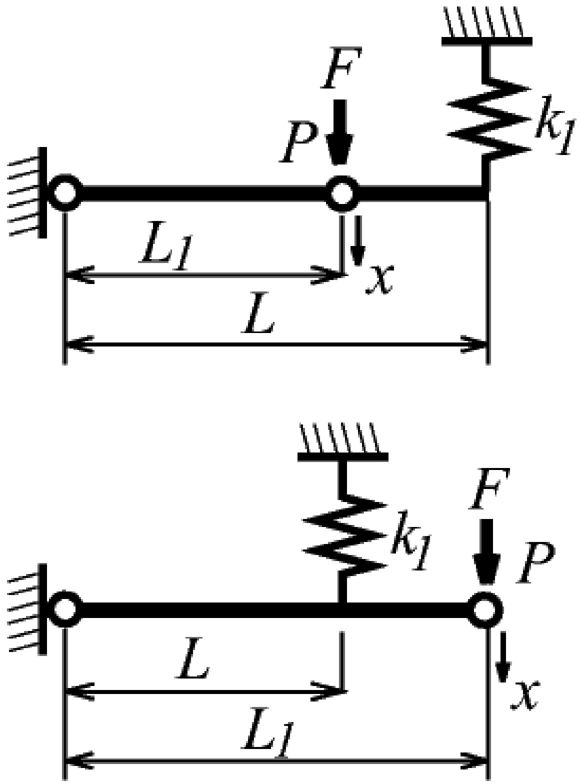
Equivalent stiffness (1)



(a) Parallel springs

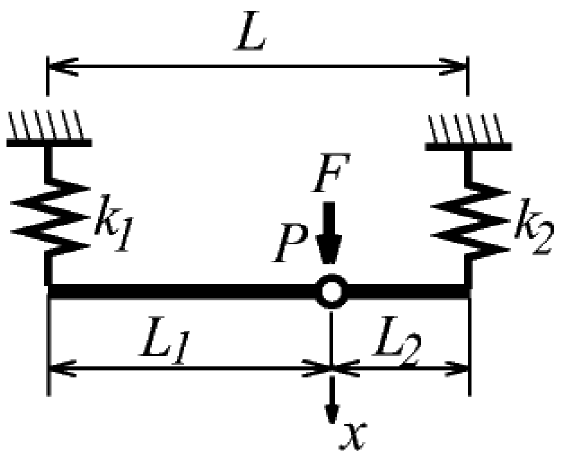


(b) Serial springs

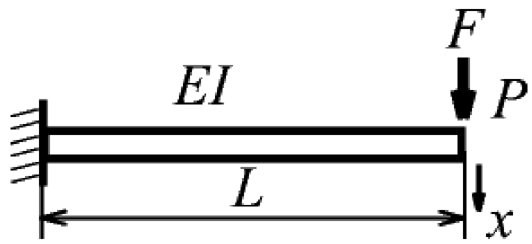


(c) Lever spring mechanism

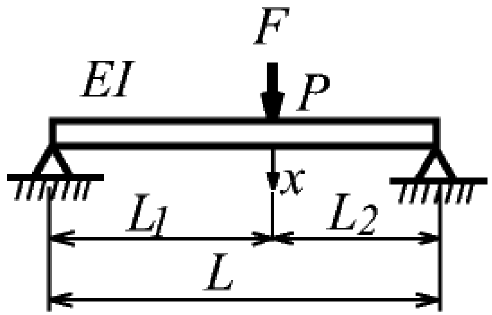
Equivalent stiffness (2)



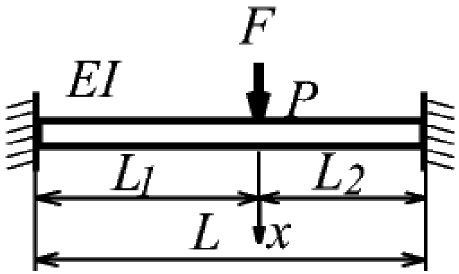
(d) Distributed spring mechanism



(e) Cantilever



(f) Beam with simple supported ends



(g) Beam with fixed ends

Rayliegh's method (1)

Undamped one degree-of-freedom system

$$m\ddot{x} + kx = 0$$ Conservative System

Mechanical Energy $$E = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

The mechanical energy of a conservative system is invariant during motion.

$\Rightarrow$ $\boxed{E = Const.}$ $\Rightarrow$ $\boxed{T_{\max} = U_{\max}}$

Rayliegh's method (2)

(1) Assume the shape of the vibration that is often called the mode shape.

(2) Calculate the maximum value of the kinetic energy and that of the potential energy of the vibration system.

(3) From the equivalence of the maximum value of the kinetic energy and that of the potential energy, the natural angular frequency is calculated.

Rayliegh's method (3)

Example 1：Linear Spring

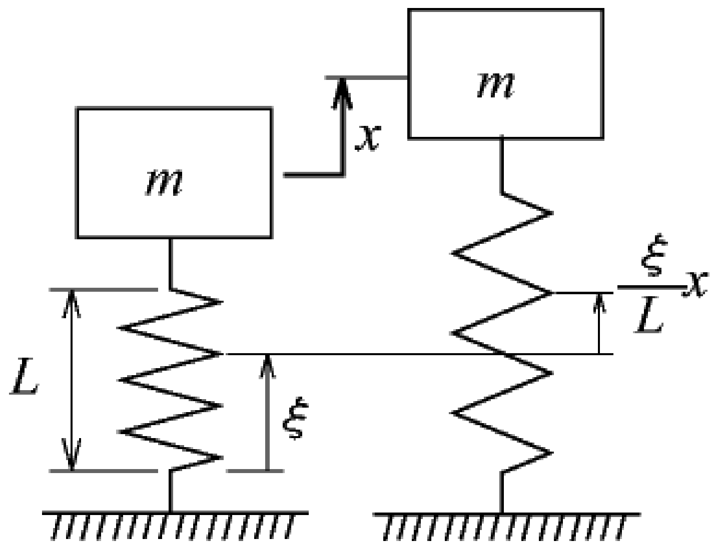


Fig.1 Effect of the mass of the spring on the natural angular frequency

Rayliegh's method (4)

Example 2：Mass-Cantilever System

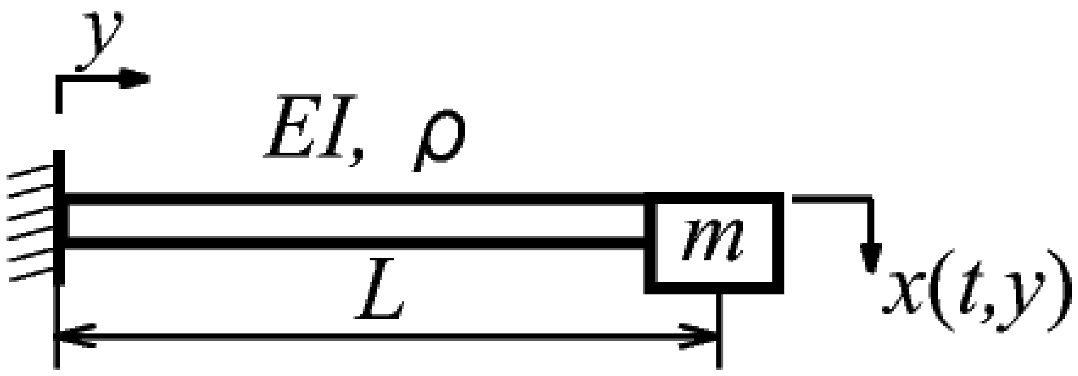


Fig.2 Effect of the mass of the beam on the natural angular frequency

Analysis of a damped one degree-of-freedom vibration system (1)

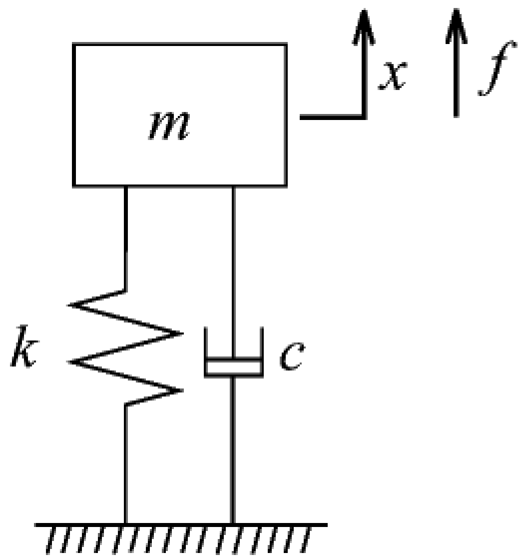


Fig.3 Damped one-degree-of-freedom vibration system

$$\boxed{m\ddot{x} + c\dot{x} + kx = f}$$

Analysis of a damped one degree-of-freedom vibration system (2)

Free Vibration $$\boxed{m\ddot{x} + c\dot{x} + kx = 0}$$

Free Vibration Response $$\boxed{x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}}$$

where

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$\zeta = \frac{c}{2\sqrt{mk}}$$ Damping Ratio

Damping ratio and initial value response (1)

(a) Unstable

$$x(t) = \frac{x_0\lambda_2 - v_0}{\lambda_2 - \lambda_1} e^{\lambda_1 t} + \frac{x_0\lambda_1 - v_0}{\lambda_1 - \lambda_2} e^{\lambda_2 t}$$

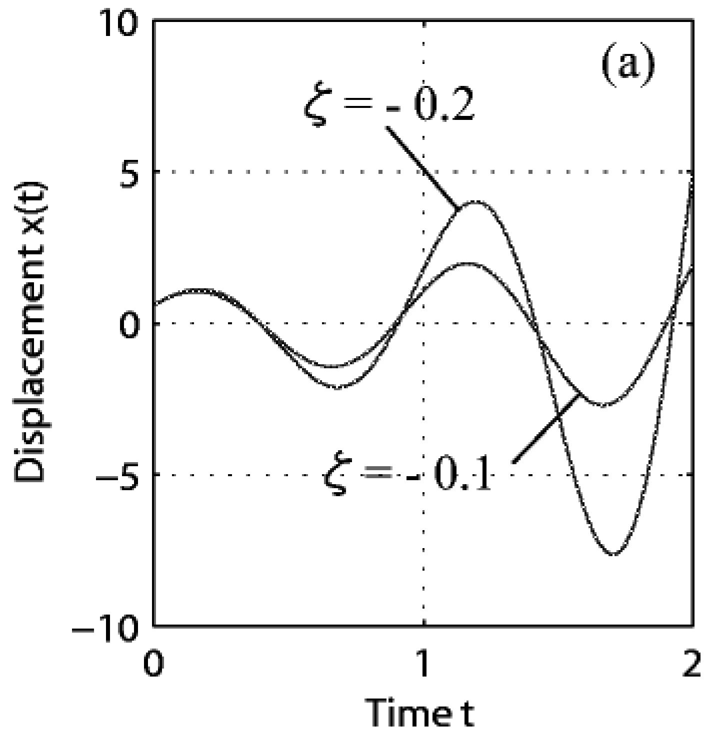


Fig.4 Initial value response
$(x_0 = 0.6, v_0 = 0.8\omega_n, \omega_n = 2\pi)$

Damping ratio and initial value response (2)

(b) Under damping

$$x(t) = A_0 e^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

$$A_0 = \sqrt{x_0^2 + \left(\frac{x_0\zeta\omega_n + v_0}{\omega_d}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{x_0\zeta\omega_n + v_0}{x_0\omega_d}\right)$$

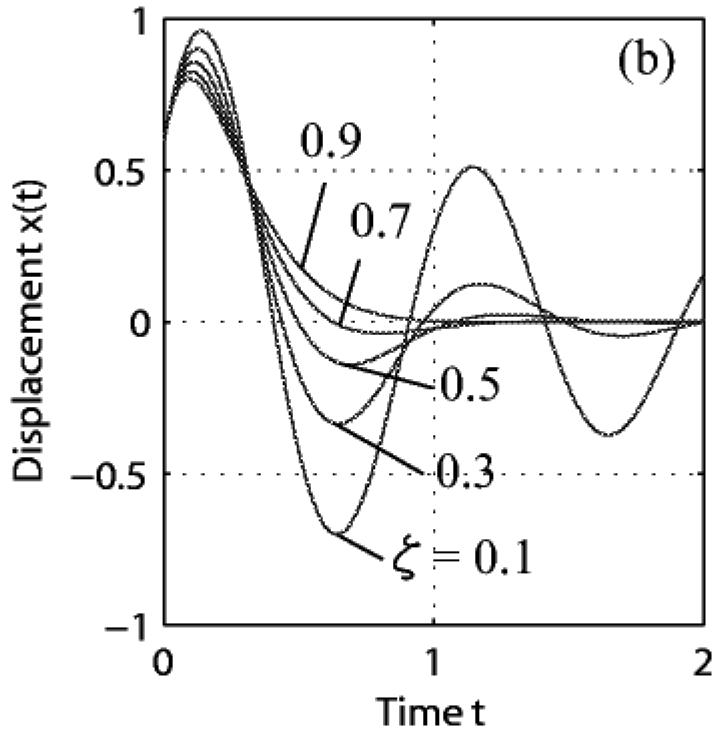


Fig.4 Initial value response
$(x_0 = 0.6, v_0 = 0.8\omega_n, \omega_n = 2\pi)$

Damping ratio and initial value response (3)

(c) Critical damping

$$x(t) = x_0 e^{\lambda_1 t} + (v_0 - x_0\lambda_1) t e^{\lambda_1 t}$$

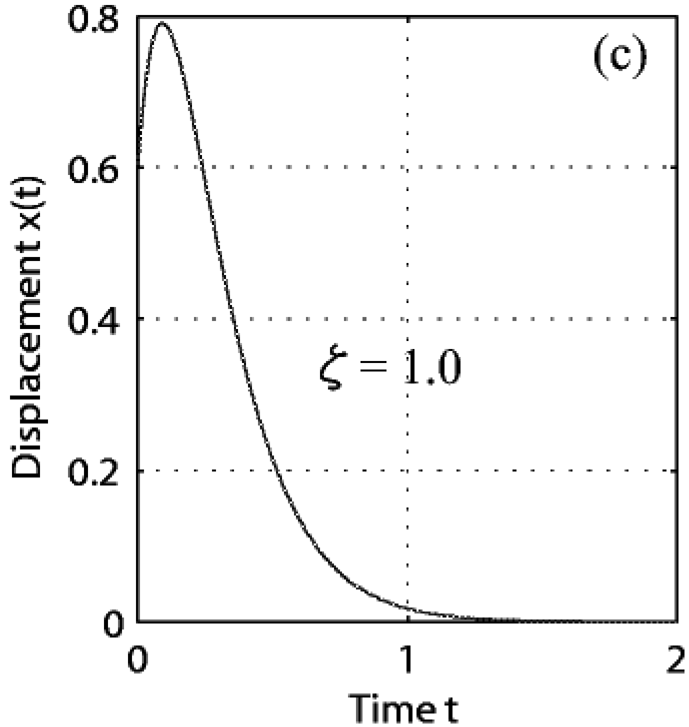


Fig.4 Initial value response
$(x_0 = 0.6, v_0 = 0.8\omega_n, \omega_n = 2\pi)$

Damping ratio and initial value response (4)

(d) Over damping

$$x(t) = \frac{x_0\lambda_2 - v_0}{\lambda_2 - \lambda_1} e^{\lambda_1 t} + \frac{x_0\lambda_1 - v_0}{\lambda_1 - \lambda_2} e^{\lambda_2 t}$$

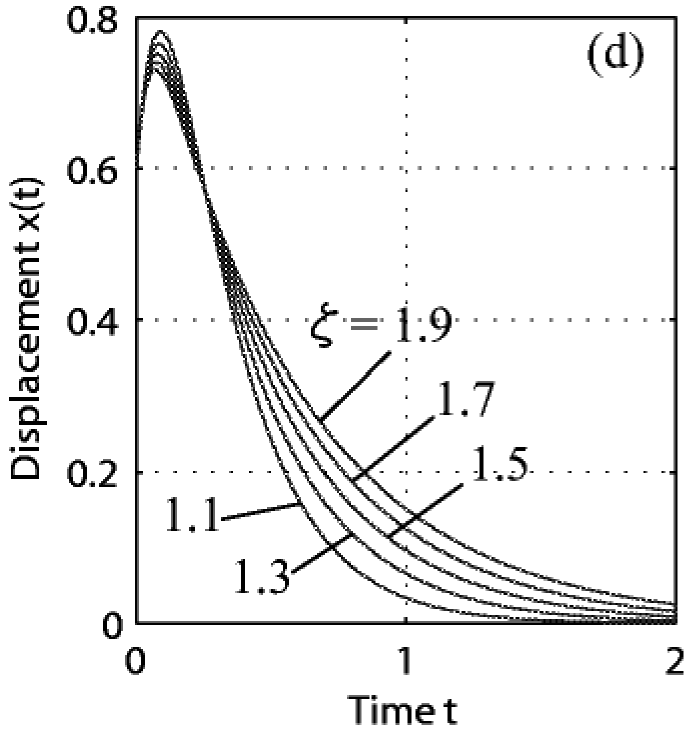


Fig.4 Initial value response
$(x_0 = 0.6, v_0 = 0.8\omega_n, \omega_n = 2\pi)$

Identification of system parameter (1)

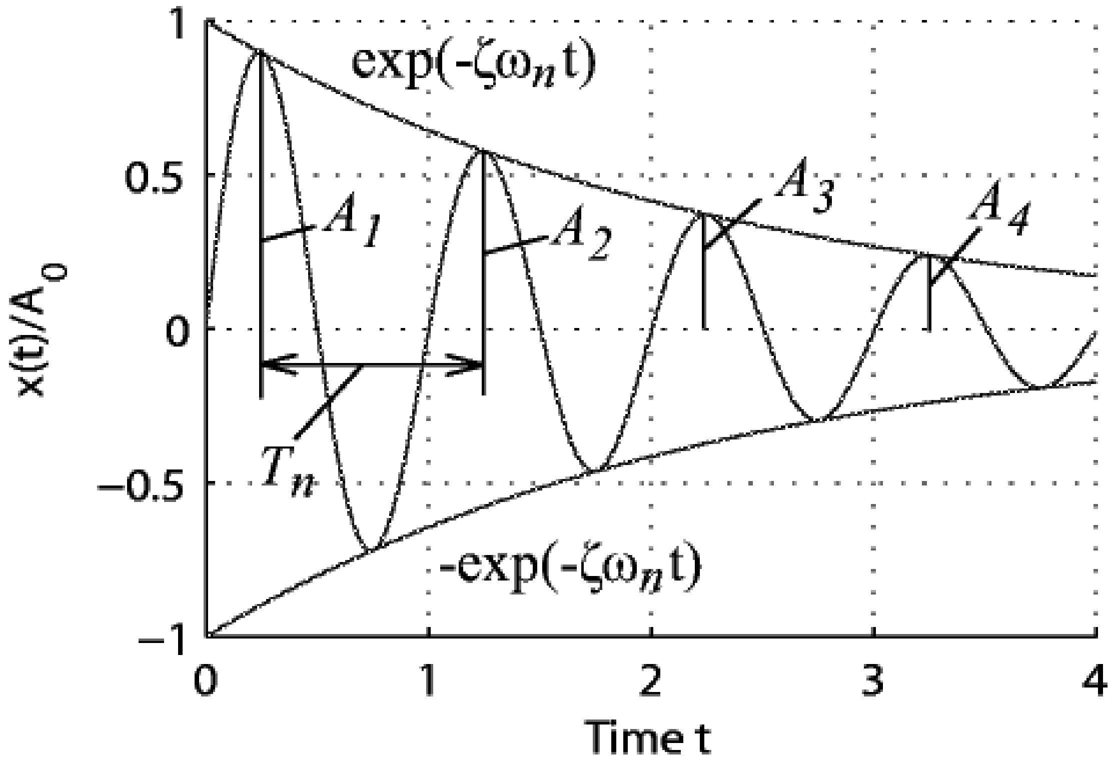


Fig.5 Free vibration of an actual vibration system to be modeled

Identification of system parameter (2)

Natural period T_n

Logarithmic damping ratio $$\delta = \frac{1}{N}\sum_{i=1}^{N}\ln\left(\frac{A_i}{A_{i+1}}\right)$$

Damping ratio $$\zeta = \frac{\delta}{2\pi\sqrt{1+\left(\frac{\delta}{2\pi}\right)^2}}$$

Identification of system parameter (4)

Case 1) *m* can be measured $$k = m\omega_n^{\ 2} \quad c = 2\sqrt{mk}\,\zeta$$

Case 2) *k* can be measured $$m = \frac{k}{\omega_n^{\ 2}} \quad c = 2\sqrt{mk}\,\zeta$$