ÉCOLE POLYTECHNIQUE

## Today's Plan

# Linear viscoelasticity

# Recall : linear elasticity

- Hooke's law : Stress and strain are linearly proportionnal

- 3D

$$\boxed{\sigma = C : \varepsilon}$$

elastic tensor $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

$$\boxed{\varepsilon = S : \sigma}$$

compliance tensor

where $C = S^{-1}$

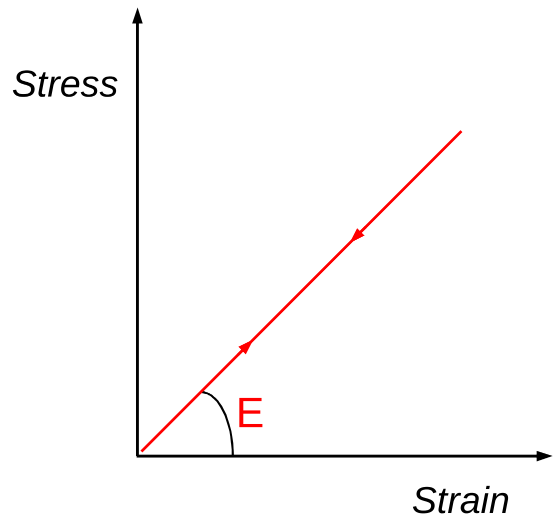
- Unidimensional relationship :

$$\boxed{\sigma = E\varepsilon} \quad \text{or} \quad \boxed{\varepsilon = J\sigma}$$

elastic modulus

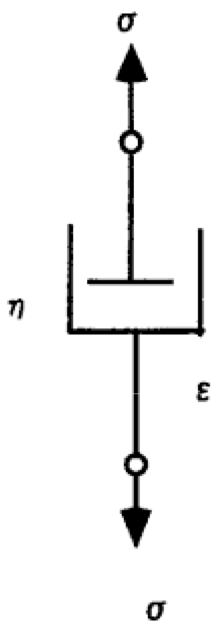
compliance modulus

where $E = J^{-1}$



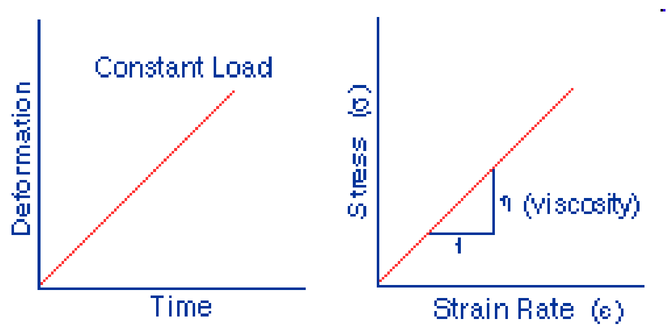

*Stress-strain curve in uniaxial test*

# Recall : viscous liquid




$$\sigma = \eta \frac{d\varepsilon}{dt}$$

viscosity

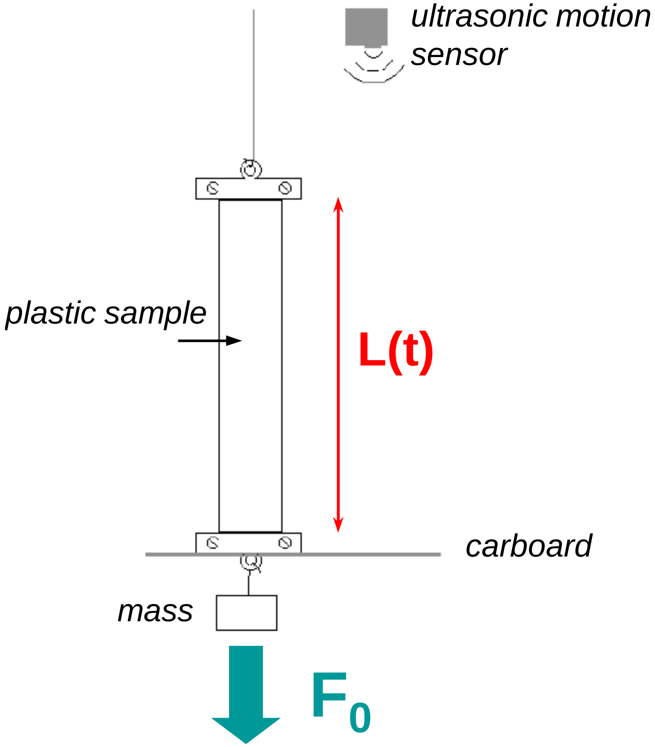



rate-dependent response

Viscoelastic materials display the characteristics of both elastic solids and viscous fluids. These 2 ideal behaviours are modeled by the spring and the dashpot.

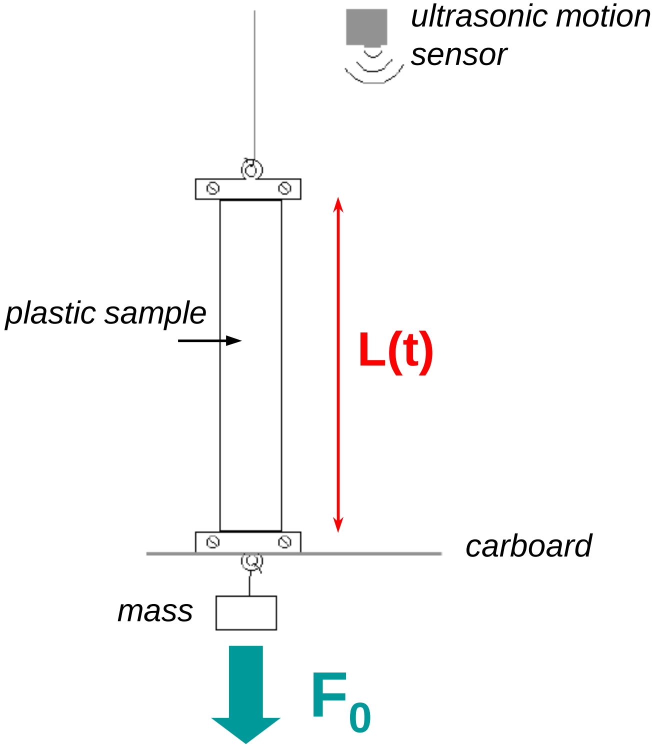
# I. Some observations

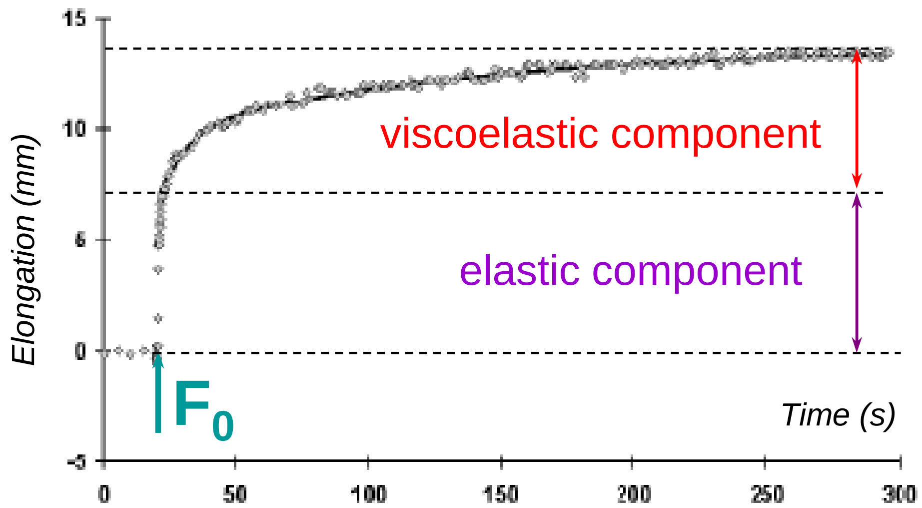
Why plastic bags give way when you are halfway home ?


[Vilela et al., Eur. J. Phys, 1999]

# Experiments



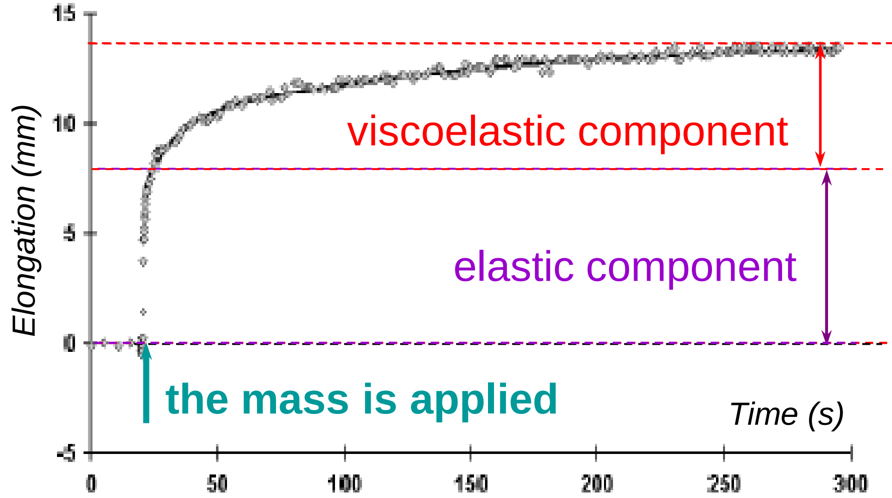

*Deformation produced by the addition of a mass of 0.5 kg to the sample as a function of time.*

➢viscoelastic deformation is time dependant. The stress gives rise to a strain that approaches its equilibrium value slowly.

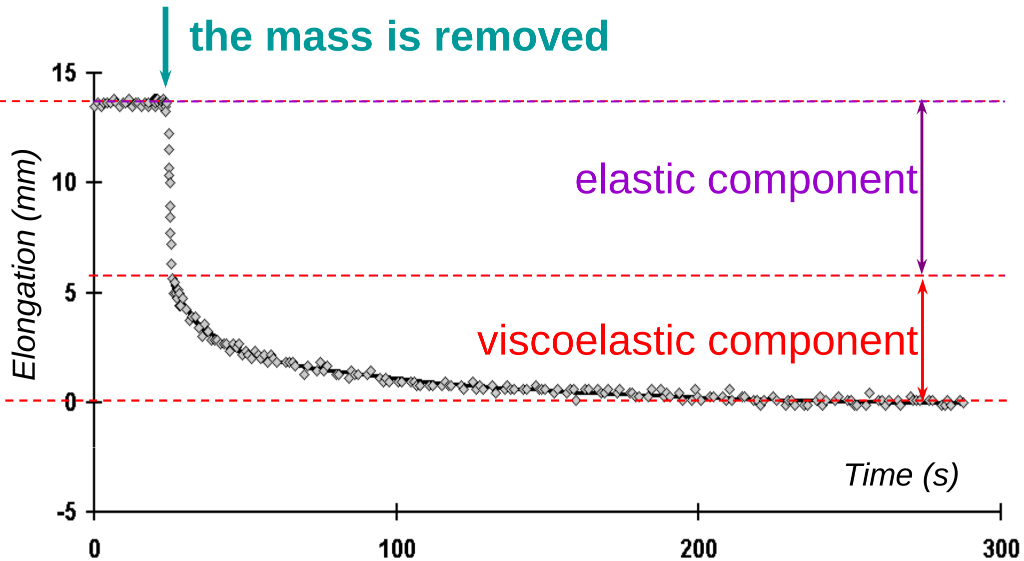


[Vilela et al., Eur. J. Phys, 1999]

# Recovery of deformation




*Deformation produced by applying a mass of 0.5 kg to the sample as a function of time.*

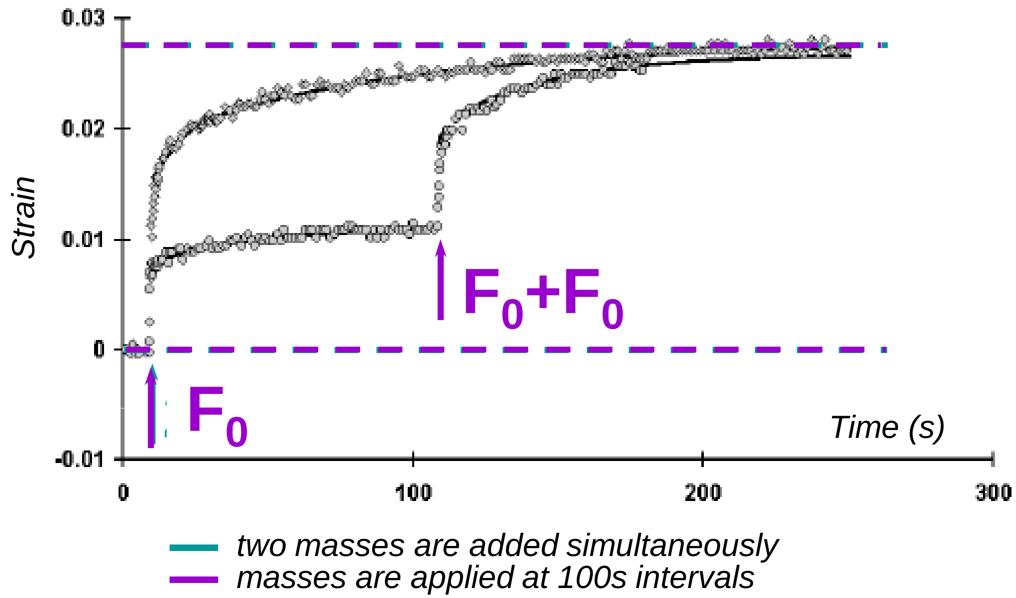



*Recovery of the sample when the mass is removed.*

- Elastic deformation is instantaneously recoverable

- Viscoelastic deformation is slowly recoverable
  - linear viscoelastic deformation is totally recovered
  - nonlinear viscoelastic deformation is partially recovered

## Nonlinearity and Boltzmann's superposition principle



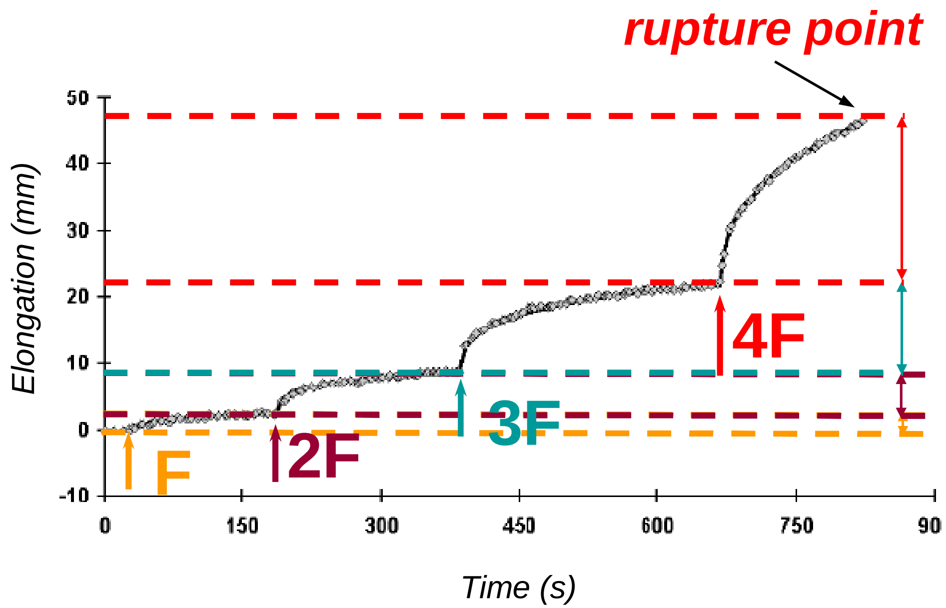

- Viscoelastic deformation doesn't depend linearly of the applied stress.

- Boltzmann's superposition principle :
  Asymptotic viscoelastic deformation is independant of the loading path.

Why plastic bags give way when you are halfway home ?

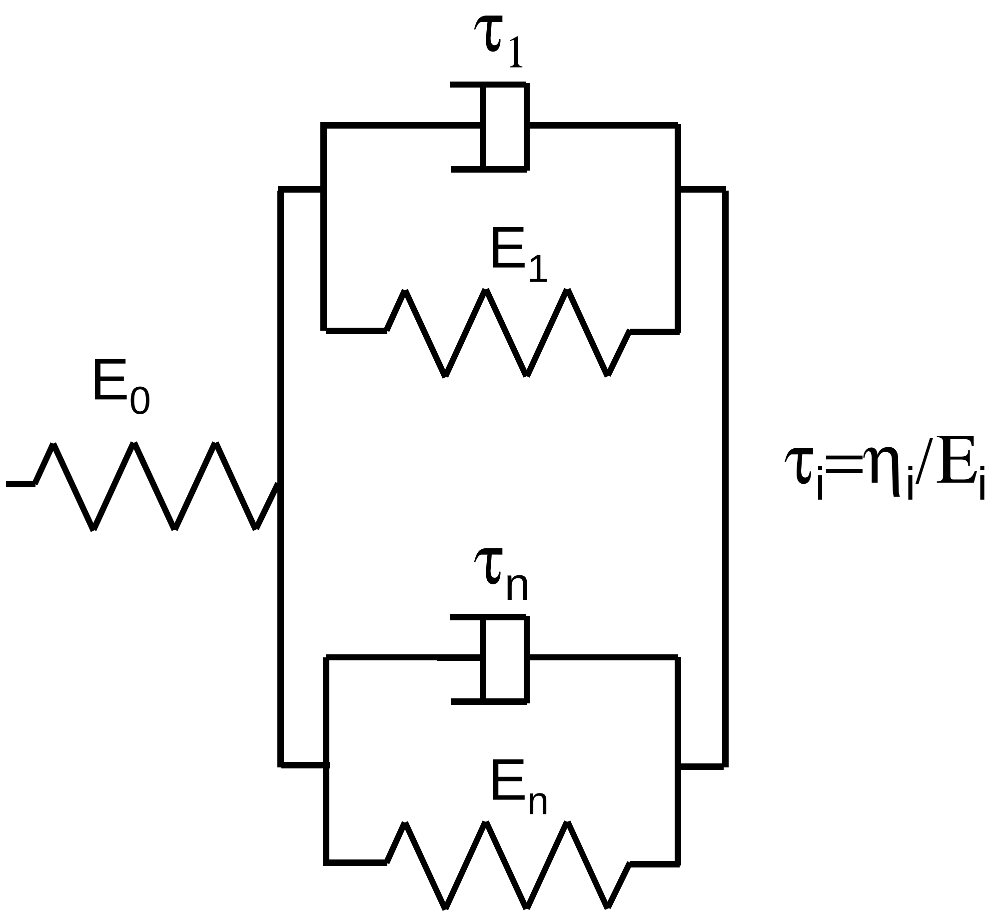
F

because of its viscoelastic origine



# Application

- Fit the data with such a model.



# Introduction

- In usual materials, viscoelastic deformation is negligle. In polymeric materials however (as in metals at very high temperatures or bitumen materials..), viscoelastic phenomena is predominent.

- Why study viscoelastic deformation?

  $\rightarrow$ Provide information about the microstructure (disposition and interaction of the macromolecules in both their short range and their long range interrelations)

- We aim at

  - determine the viscoelastic functions describing the material responses under various loadings (creep and relaxation functions, dynamic moduli..)

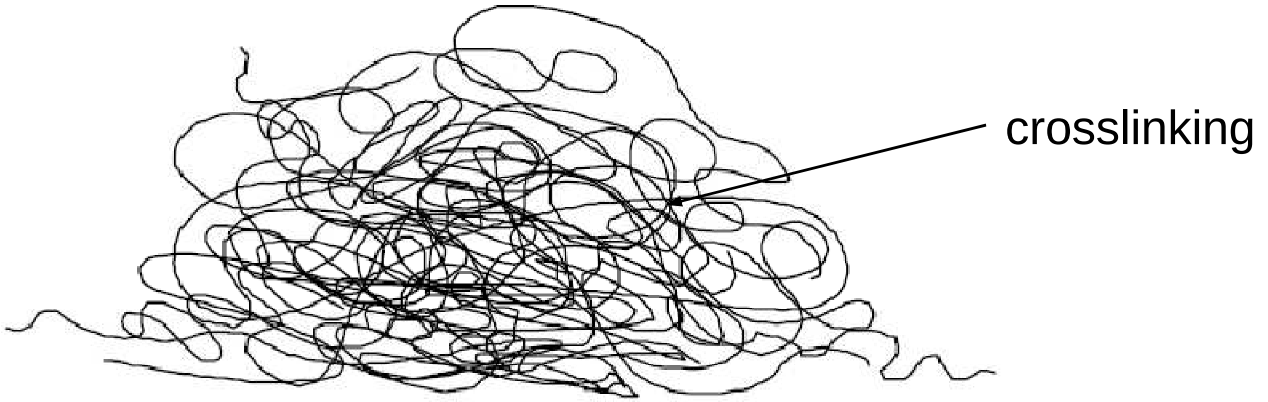
  - review the effect of polymer structure on these viscoelastic functions

  - study simple rheological models accounting for linear viscoelasticity

  - present usual mechanical tests

  - present the time-temperature equivalency and it's limitations

## Microscopic origin of viscoelasticity ?




Materials Polymer molecules are usually arranged in the form of random coils, analogous to cooked spaghetti.

Elastic deflection is due to the ability of each chain to stretch.
Viscous flow is caused by the sliding of the molecules over one another.

# Framework

- Small Perturbation Hypothesis (SPH)
  small deformations $_{\sup}|\varepsilon|<<1$
  infinitesimal strain tensor : $\varepsilon_{ij}=\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right]$

- Linear viscoelasticity

  if $\varepsilon=f(\sigma)$

  then $f(m\sigma 1+n\sigma 2)=mf(\sigma 1)+nf(\sigma 2)$

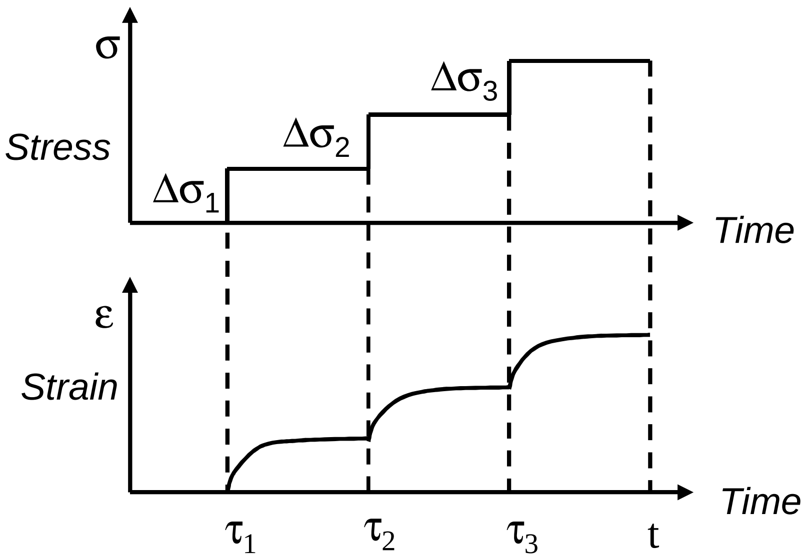
  (it means that viscoelastic functions are independant of level of loading)

- isothermal
- isotropic and non-ageing materials
- uniaxial frame

# II. Boltzmann's superposition principle

Hypothesis of linearity  effect (∑cause) = ∑effect (cause)

creep




compliance or creep function at time (t - τ)

$$\varepsilon(t) = \Delta\sigma_1 J(t-\tau_1) + \Delta\sigma_2 J(t-\tau_2) + \Delta\sigma_3 J(t-\tau_3)$$

# For any loading's history

## Hereditary constitutive relationships

1D frame

- For *n* steps $\varepsilon(t) = \sum_{i=1}^{n} \Delta\sigma_i J(t - \tau_i)$
- For *n* very slight stress steps: $\varepsilon(t) = \int_{-\infty}^{t} J(t-\tau) d\sigma(\tau)$

*J(t-τ)* is the creep function at time (t - τ)

- rewritten as : $\boxed{\varepsilon(t) = J_u \sigma + \int_{-\infty}^{t} J(t-\tau) \frac{d\sigma(\tau)}{d\tau} d\tau}$

$J_u$ is the unrelaxed compliance

- We use the convolution product $\varepsilon = J \otimes \frac{D\sigma}{Dt}$

# For any loading's history

## Hereditary constitutive relationships

For a relaxation type test: $$\sigma(t)=\int_{-\infty}^{t}E(t-\tau)d\varepsilon(\tau)$$

*E(t-τ)* is the relaxation modulus at time (t - τ)

- rewritten as : $$\boxed{\sigma(t)=E_r\varepsilon+\int_{-\infty}^{t}E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}dt}$$

*Er* is the relaxed modulus

- We use the convolution product $\sigma=E\otimes\frac{D\varepsilon}{Dt}$

The creep function (or the relaxation function) rules entirely the material's response.

## Relationship between creep and stress relaxation

Consider a stress program in which $$\frac{d\sigma(\tau)}{d\tau}=\frac{dG(\tau)}{d\tau}$$

$$\varepsilon(t)=\int_0^t J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau$$

$$\varepsilon(t)=\int_0^t J(t-\tau)\frac{dG(\tau)}{d\tau}d\tau=cst$$

normalize : $$\int_0^t J(t-\tau)\frac{dG(\tau)}{d\tau}d\tau=1$$

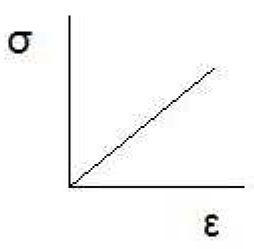
hence : $$\boxed{\int_0^t G(\tau)J(t-\tau)d\tau=t.}$$

# III. Usual mechanical tests

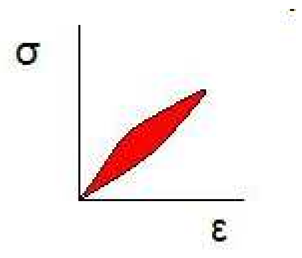
Viscoelastic deformation is time (and temperature) dependant. Hence, we characterise the influence of strain rate, frequency (and temperature).
We do :

① Quasi-static tests at various strain rate

*An hysteresis is seen in the stress-strain curve.*

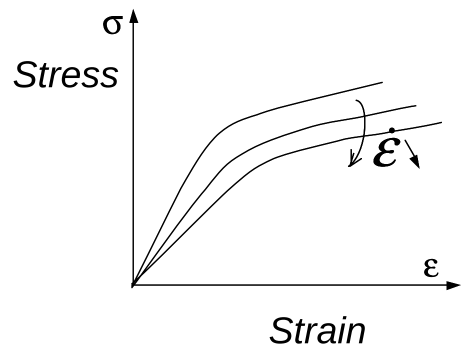



*elastic solid*



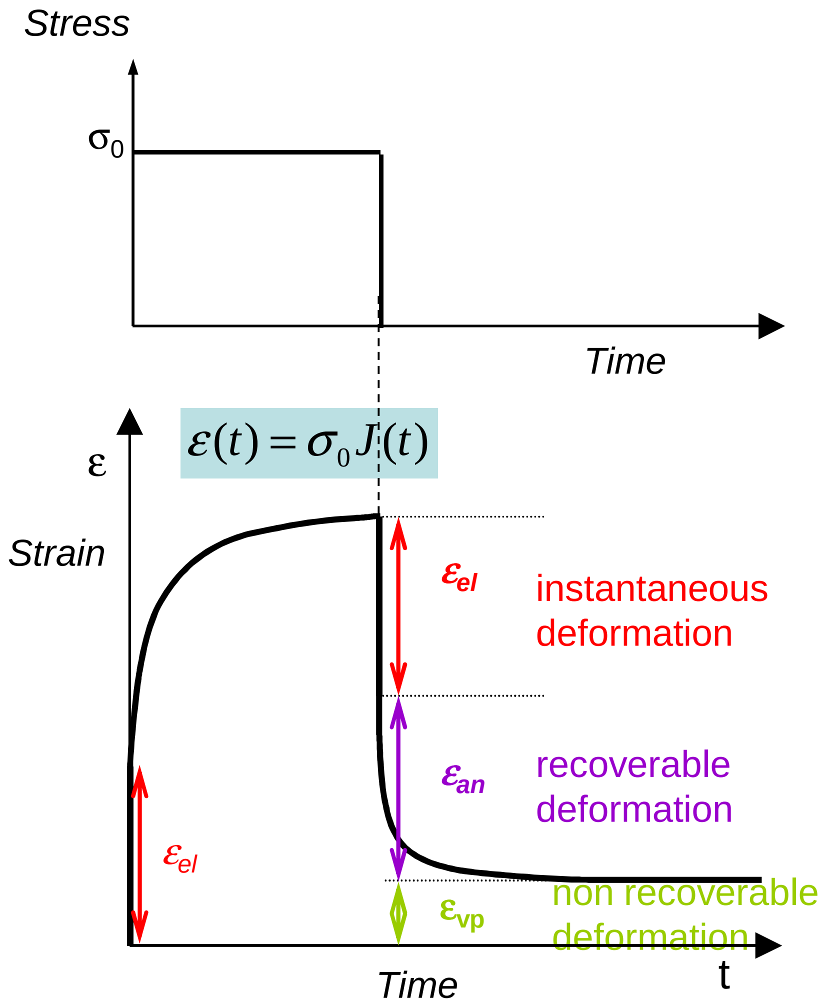

*viscoelastic solid*

*The stress-strain curve is strain rate dependant.*




The slower you pull, the more deformation can occur

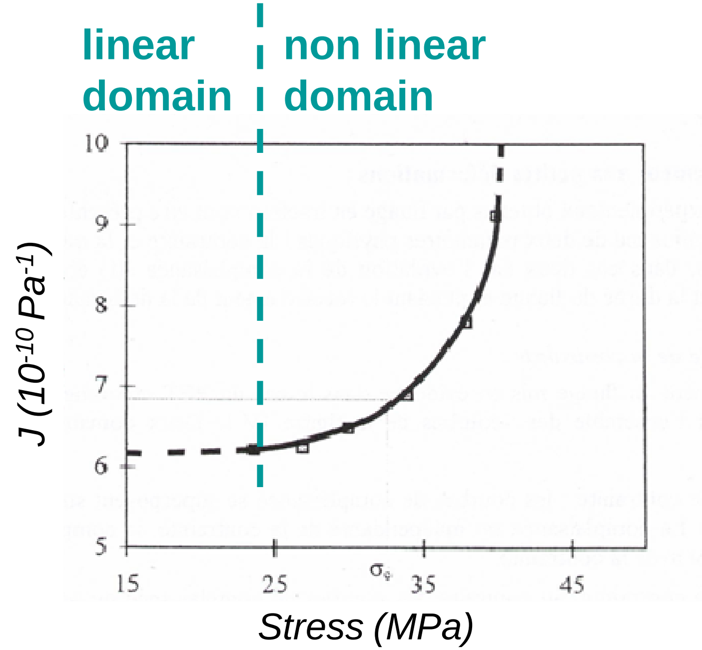
② Creep test : apply a stress and then follow strain vs. time



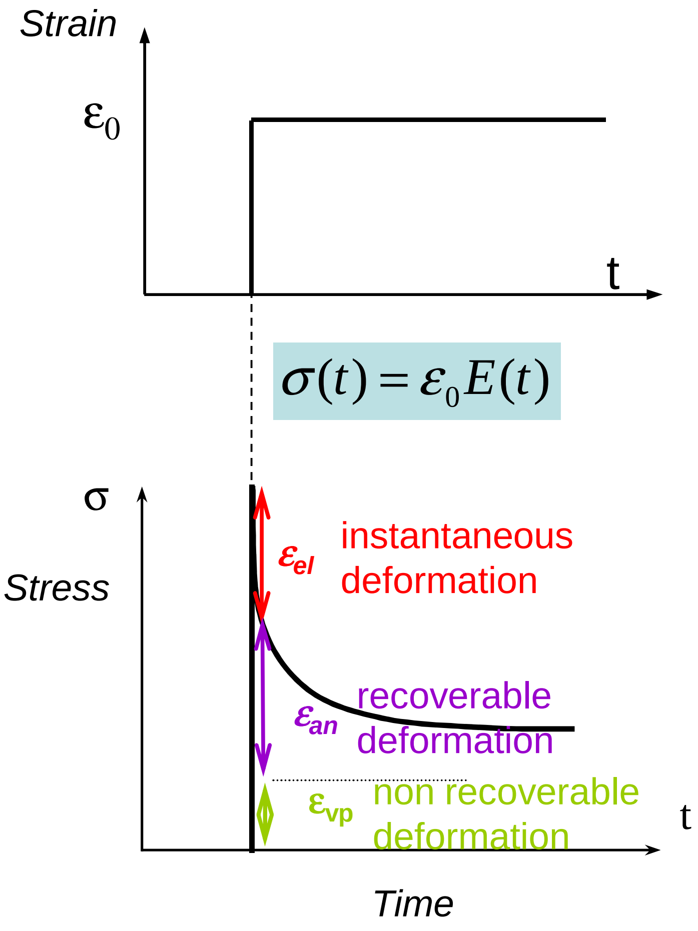

**Features of creep response**

- Instantaneous elastic response
- Retardation
- Equilibrium compliance

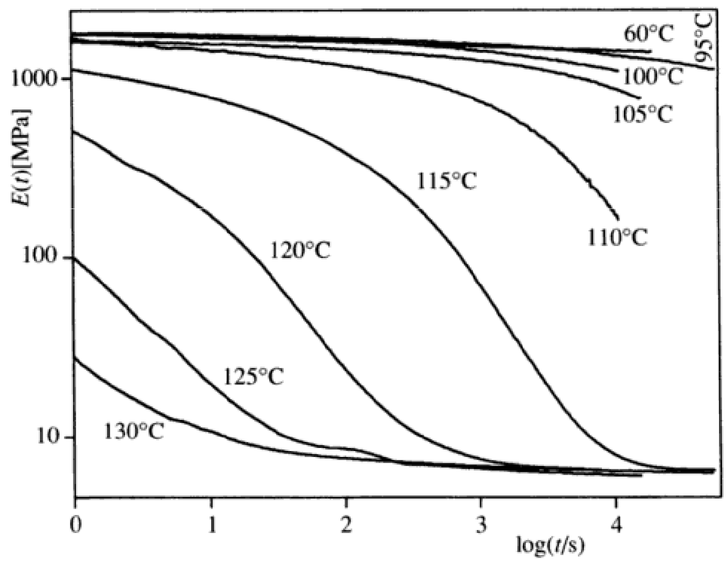
*Ex : Creep function of amorphous PET*



③ Stress relaxation : apply a sudden length change and then watch the stress decay.



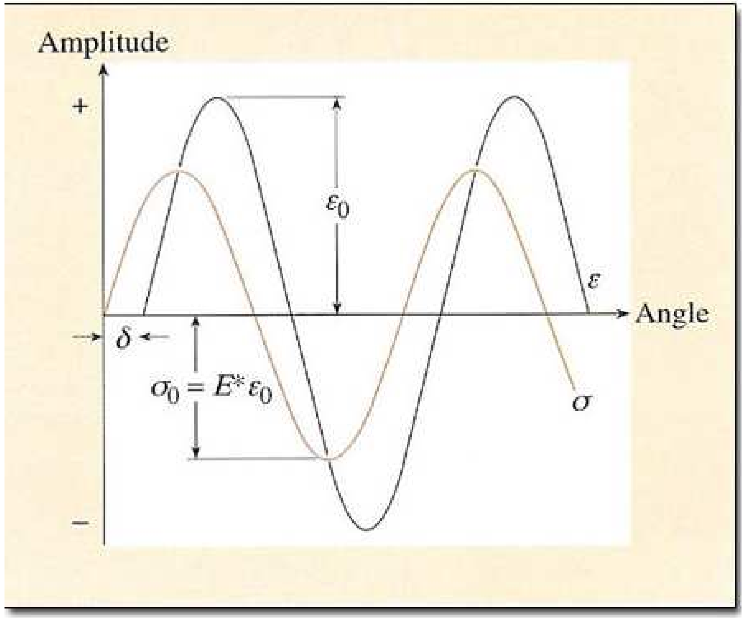

*Ex: Modulus of relaxation $E=\sigma(t)/\varepsilon_o$ for PMMA, at $\varepsilon_o$= 0.5%*



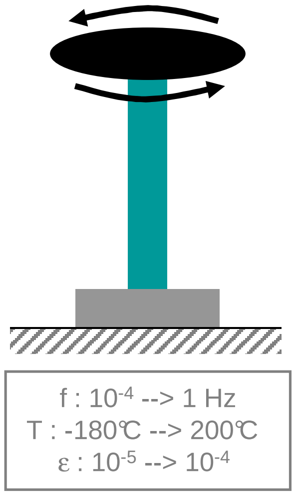

[Kausch, Heymans, Plummer, Decroly, Traité des matériaux vol 14, 2001]

④ Dynamic measurements : apply an oscillating strain (or stress) at a fixed frequency and measure the amplitude and phase of the response

DMA (Dynamic Mechanical Analysis )




*Example : torsion test*




➢ Input

$$\sigma = \sigma_o \sin(\omega t)$$

$\sigma_0$ : stress amplitude

➢ Response

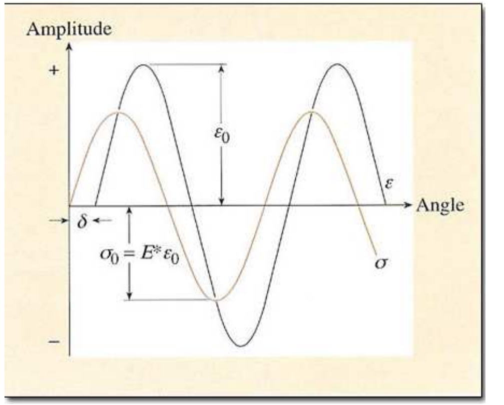
$$\varepsilon = \varepsilon_o \sin(\omega t - \delta)$$

$\varepsilon_0$ : strain amplitude

where 0°< δ <90°

δ : phase

## … Dynamic measurements




- Dynamic modulus

$$E^*(i\varpi) = \frac{\sigma(i\omega)}{\varepsilon(i\varpi)} = E'(\varpi) + iE''(\varpi)$$

- Conservation modulus (stored elastic energy)

$$E' = \frac{\sigma_0}{\varepsilon_0}\cos\delta$$ *in phase with the strain*

- Lost modulus (dissipated energy)

$$E'' = \frac{\sigma_0}{\varepsilon_0}\sin\delta$$ *out of phase with the strain*

- Internal friction (damping factor, loss tangent)

$$\tan\delta = E'/E''$$

**Thank you for your attention**

**Lecture# 7 will be given on June 5th**