REPEATED GAMES AND FOLK THEOREM

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Abstract. This article illustrates the vast variety of equilibria that can be achieved in infinitely repeated games.

1. Repeated games

Repeated games simplest multistage games. Intuitively, repeated games are the repeated plays of stage games. However, there is a little complication regarding the information sets, representing the fact that the players know the plays of the previous stage games.

Definition 1.1 (Repeated game)**.** Repeated game is a structure $\Gamma =< \mathscr{I}, \mathcal{T}, G, \hat{\pi} >$ where:

- $\mathscr{I} = \{1, ..., I\}$ is a finite set of players,
- $\mathcal{T} = \{0, ..., T\} \quad (T = 1, ..., +\infty)$ is a set of periods,
- $G =< \mathscr{I}, A, u >$ is a stage game where:
  - $A = (A_i)_{i \in \mathscr{I}}$ is an action profile set
  - $u = (u_i)_{i \in \mathscr{I}} : A \to \Re^{\mathscr{I}}$ is a stage payoff profile

  and
- $\hat{\pi} : A^{\mathcal{T}} \to \Re^{\mathscr{I}}$ is a payoff profile.

It is useful to define the set of possible *histories* $H = \cup_{t \in \mathcal{T}} H^t$, where $H^t = A^t$ is a set of feasible histories up to period t for $\forall t \in \mathcal{T}$, $h^t = (a_9, ..., a_t) \in H^t$ representing the realized choices of actions up to period t. $[H^t | t \in \mathcal{T}]$ forms a partition on H. $H^0 = A^0$ is a singleton set with $h^0 = \emptyset \in H^0$ called an empty history which corresponds to the root of the game. Each history $h \in H$ corresponds to an information set. $h \in A^{\mathcal{T}}$ is called a terminal history. For each history $h \in H$, subhistory h^t up to period t can be defined. For the case of infinite games, a payoff profile is defined on each stage and the payoffs of the repeated game are associated with the stage payoffs by

$$\forall h \in A^{\mathcal{T}} : \hat{\pi}(h^{\mathcal{T}}) = \lim_{t \to \infty} \hat{\pi}(h^t)$$

where $h^t (t = 0, ...)$ are subhistories of h. The existence of the limit is called *continuity* condition.

Definition 1.2 (Discounted payoff)**.** Discounted payoffs of a repeated game $\Gamma(\delta)$ $=< \mathscr{I}, \mathcal{T}, G, \hat{\pi} >$ with a discount factor $\delta = (\delta_i)_{i \in \mathscr{I}}$ satisfying $\forall i \in \mathscr{I} : \delta_i \in [0, 1)$ is defined by:

$$\hat{\pi}_i(a^{\mathcal{T}}) = \sum_{s=0}^{T} \delta_i^s u_i(a^s)$$

Any discounted payoff profile function is continuous.

Definition 1.3 (Mixed (behavior) strategy)**.** Let $\mathcal{A}$ be the space of probability distributions on A_i. Player i's mixed strategy $\sigma_i : H \to \mathcal{A}_i$ determines an action according to the past histories.

Note that the players cannot observe the randomizing probabilities of other players.

Definition 1.4 (Normal form representation of a repeated game)**.** Normal form representation of a repeated game $\Gamma =< \mathscr{I}, \mathcal{T}, G, \hat{\pi} >$ is given by $\Gamma =< \mathscr{I}, \Sigma, \pi >$ where:

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- $\Sigma = A^H$ is a set of pure strategies, and
- $\pi : \Sigma \to \Re^{\mathscr{I}}$ is a payoff profile function defined by $\pi(s) = \hat{\pi}(h^{\mathcal{T}}(s))$ for $\forall s \in \Sigma$.

When players choose actions according to $s \in \Sigma$, for all $t \in \mathcal{T}$, a^t and h^t are recursively determined by:
(i) $h^0(s) = h^0 = \emptyset$ and
(ii) $a^t(s) = s(h^t(s))$ and $h^{t+1}(s) = h^t(s) \times a^t(s)$,
which is why $h^t(s)$ is well-defined.

All proper subgames denoted $\Gamma(h)$ begin after some history h and are repeated games with the same discount factor, except that the payoffs are given by the affine transformation of the original game in the case of infinite repeated games with the same discount factor.

Proposition 1.5 (Subgame)**.** Take any period t history h^t of repeated game $\Gamma =< \mathscr{I}, \mathcal{T}, G, \hat{\pi} >$. The payoffs of the subgame $\Gamma(h^t)$ are given by:

$$\pi_i(\sigma) = (\sum_{s=0}^{t} \delta_i^s u_i(a^s)) + \delta_i^t \sum_{s=t}^{T} \delta_i^{s-t} u_i(a^s)$$

$\sum_{s=0}^{T} \delta_i^s u_i(a^{s+t})$ is called a (renormalized) continuation payoff.

The structure of continuation payoffs to the payoffs of repeated games with discount factors is the core of various folk theorems.

Definition 1.6 (Subgame-perfect equilibrium)**.** Let $s_{i|h}$ be the strategy on the subgame $\Gamma(h)$ induced by $s^i \in \Sigma_i$ if for all histories h' of $\Gamma(h)$, $s_{i|h}(h') = s_i(h \times h')$.

$s^* \in \Sigma$ is a subgame-perfect equilibrium of game Γ iff for all subgames $\Gamma(h)$, $s^*_{|h}$ is a Nash equilibrium.

We most often focus on the simplest stationary equilibria.

Definition 1.7 (Stationary Equilibrium)**.** An equilibrium $s^* \in \Sigma$ is stationary if there exists a repetition of actions $h(s^*) = (a^0, ..., a^K, a^0, ..., a^K, ...)$ with the repetition periods K on the equilibrium path such that there exists a bijection $\phi(h^t(s^*))$ for any period t satisfying

$$\pi(s_{|h}) = \pi(\phi)$$

2. Folk theorems

Definition 2.1 (Minmax Payoff)**.** Let $\underline{v}_i = \min_{a_{-i}} \max_{a_i} u_i(a)$ be the minmax payoff of player i. Let m^i_{-i} be the solution of the above equality. m^i_{-i} is called minmax profile against player i.

Definition 2.2 (Individual rationality)**.** Payoff profile $u \in \Re^{\mathscr{I}}$ is individually rational if $u \geq \underline{v}$ and is strictly individually rational if $u \gg \underline{v}$. Denote the set of feasible and individually rational payoff vectors

$$V = \{u \in \mathcal{R}^{\mathscr{I}} | \exists a \in A : u = u(a) \wedge u \gg \underline{v}\}$$

Theorem 2.3 (Folk theorem)**.** *For any feasible individually rational payoff vector* $\forall v \in V$, *there esists a* $\underline{\delta}_i < 1$ *such that for* $\forall \delta_i \in [\underline{\delta}_i, 1)$ *(that is players are patient enough), there exists a Nash equilibrium* $s^* \in \Sigma$ *of* $\Gamma(\delta)$ *with payoffs* u *(that is for all period* t, $u = u(a^t(s^*))$ *).*

Proof. Trigger strategies with maxmin strategies m^i_{-i} taken by the other players $-i$ for punishment are sufficient as a threat to keep each player i from deviation.

□

Theorem 2.4 (Folk theorem (subgame-perfect with stage Nash threat))**.** *Let* $s^G \in A$ *be a stage game Nash equilibrium. For* $u \in V$ *satisfying* $u \gg u(s^G)$, *there esists a* $\underline{\delta} < 1$ *such that for* $\forall \delta \in [\underline{\delta}, 1)$ *(that is players are patient enough), there exists a subgame-perfect equilibrium* $s^* \in \Sigma$ *of* $\Gamma(\delta)$ *with payoffs* u *(that is for all period* t, $u = u(a^t(s^*))$ *).*

Proof.

$$\underline{\delta}_i = \frac{max_{b_i} f_i(b_i, a_{-i}) - f_i(a)}{max_{b_i} f_i(b_i, a_{-i}) - f_i(e)}, \qquad \forall i = 1, ..., n$$

□

The theorem is most often applied to prisoners' dilemma games. Notice that the form of possible cooperation in repeated prisoners' dilemma games is not at all restricted to (cooperation, cooperation). It is possible for instance in two-player prisoner's dilemma game that one player gives and the other player takes in each period.

Theorem 2.5 (Perfect folk theorem (Fudenberg and Maskin, 1986))**.** *Assume sufficiently patient players with full-dimentional payoffs V. For any feasible individually rational payoff vector $u \in V$, there esists a $\underline{\delta} < 1$ such that for $\forall \delta \in [\underline{\delta}, 1)$ (that is players are patient enough), there exists a subgame-perfect equilibrium $s^* \in \Sigma$ of $\Gamma(\delta)$ with payoffs u (that is for all period t, $u = u(a^t(s^*))$).*

Proof. (Sketch) If this theorem is correct (and indeed it is correct), if some player i deviates from an equilibrium giving payoff profile u, the other players may punish player i by moving onto another subgame equilibrium giving payoff profile u^1 satisfying $\underline{v} \ll u^1 \ll u$. Likewise, the same procedure can be repeated if another player deviates from u^k by moving to u^{k+1}. Such a strategy is possible because of the fact that in infinitely repeated games, the payoff space is virtually continuous by 'public randomizing' even in stage games with discrete payoff space. □

Fudenberg and Maskin propose an ultimate punishment procedure in which the other players in $-i$ specially target the deviator i to punish with minmax strategy m^i_{-i}. After the punishment period follows the reward period that compensates for the act of punishing actions.

Public randomization can be achieved through coordination over multiple periods.

Example 2.6 (Coordination)**.** Consider the repetition of the battle of sexes game. The players

Table 1. Battle of sexes

1 \2	A	B
A	$0, 0$	a, b
B	b, a	$0, 0$

where $a > b > 0$

1 and 2 can coordinate to achieve $((a+b)/2, (a+b)/2)$ in average for each stage by coordinating to choose (A, B) and (B, A) in turn in each two successive periods. This has the same effect as choosing correlated equilbria in the stage game. The threat can be constructed by moving from the coordinated action to the independent mixed strategy equilibrium.

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