Error Analysis

For the solution of the elliptic equation, we can consider the steady state of the parabolic equation.

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} - \rho$$

Discretized form:

$$\frac{\partial \vec{\phi}}{\partial t} = A\vec{\phi} - \vec{\rho}$$

Error Analysis

where

$\vec{\phi} = \{\phi_1, \phi_2, \cdots, \phi_{N-1}, \phi_N\}$ $\quad \vec{\rho} = \{\rho_1, \rho_2, \cdots, \rho_{N-1}, \rho_N\}$

$$A = \frac{1}{\Delta x^2}\begin{pmatrix} 1 & & & & & & \\ 1 & -2 & 1 & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & 1 & -2 & 1 & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & 1 & -2 & 1 \\ & & & & & & 1 \end{pmatrix}$$

Error Analysis

We can change the matrix A to a diagonal one by means of the eigen value λ_m and the eigen vector $\vec{X}_m (m=1, N)$.

$$X^{-1}AX = \Lambda$$

where $X = \{X_1, X_2, \cdots, X_N\}$ is the eigen matrix and $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_N)$ is the diagonal matrix composed of the eigen value.

Error Analysis

Multiplying X^{-1} to the both sides,

$$X^{-1}\frac{\partial \vec{\phi}}{\partial t} = X^{-1}X\Lambda X^{-1}\vec{\phi} - X^{-1}\vec{\rho}$$
$$= \Lambda X^{-1}\vec{\phi} - X^{-1}\vec{\rho}$$

We have $\dfrac{\partial \vec{\varphi}}{\partial t} = \Lambda\vec{\varphi} - \vec{g}$

where $\vec{\varphi} = X^{-1}\vec{\phi}, \quad \vec{g} = X^{-1}\vec{\rho}$

Linear Algebra

Error Analysis

The given parabolic equation is decomposed to N-independent equations in the eigen-vector system. Each component is written by

$$\frac{\partial \varphi_m}{\partial t} = \lambda_m \varphi_m - g_m \qquad m = 1, \cdots, N$$

The solution is

$$\varphi_m = c_m e^{\lambda_m t} + \frac{g_m}{\lambda_m}$$

Linear Algebra

Error Analysis

The solution for the original variable ϕ is

$$\phi(t) = X\vec{\varphi} = \sum_{m=1}^{N} c_m e^{\lambda_m t} \vec{X}_m + X\Lambda^{-1} X^{-1} \vec{\rho}$$

$$= \sum_{m=1}^{N} c_m e^{\lambda_m t} \vec{X}_m + A^{-1} \vec{\rho}$$

- The second term is the solution of the Poisson equation. The first term is the temporal change of the parabolic equation.
- When the decay of the first term is fast, the solution reaches the steady state rapidly.

Linear Algebra

Error Analysis

The first term expresses the error of the numerical solution for the Poisson equation.

The eigen values and the corresponding eigen vectors:

$$\lambda_m = -2 + 2\cos(\Delta x \cdot m\pi)$$

$$X_{j,m} = \sin(j\Delta x \cdot m\pi)$$

In the case of $N = 7$,

$$\lambda_1 = -0.15244$$
$$\lambda_2 = -0.58579$$
$$\cdots$$
$$\lambda_7 = -3.84776$$

Linear Algebra

Error Analysis

- The solutions are expressed by the summation of the wave determined by the mesh size.
- All the eigen value are negative. For the larger wave number, the solution decreases faster.

small wave number → slow converging

MULTI-GRID Method

By preparing different coarse grids, the iteration process moves from a fine grid to a coarse grid. We try to decrease the error as fast as possible.

In a coarse grid, the wave number is thought to be large, and the correction for the error can be distributed to the long-distance grid.

Iteration Method

- Initial several iterations can decrease the residual error rapidly, but the convergence is not effective after that.
- The error is effectively decreased when the wave number is comparable to the grid size.
- It requires many iterations to decrease the error of long wave length.

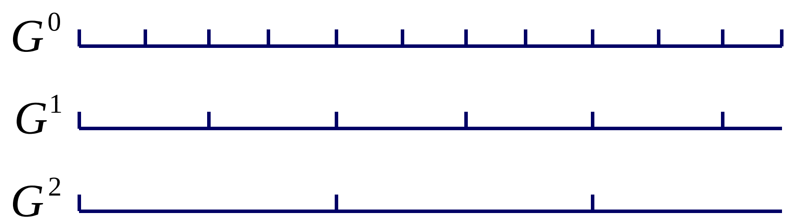
MG Process

- Approximate solution is obtained by a conventional iterative method.
- Correction value to decrease the error is estimated on the coarse grid.
- Nesting of different coarse grid iteration.

The frequency components of the error is decreased effectively by using the suitable coarse grid.

Application to Poisson Eq.

- Hierarchy coarse grid G^k : the superscript k indicates the fineness of the gird.
- The grid distance of G^k is Δx^k and $\Delta x^k = 2\Delta x^{k-1}$



(1) Discretization of Poisson to be solved on the grid G^k

$$L^k F^k = S^k$$

L^k : Operator on the grid k

F^k : Exact Solution

S^k : Source term (const.)

Starting from a proper initial value f_0^k, the approximation value f_1^k is obtained by n-th iteration of SOR method

$$f_1^k = SOR\left(L^k, S^k, f_0^k, n\right)$$

Residual : $R^k = S^k - L^k f_1^k$

Correction : $v^k = F^k - f_1^k$

Equation for correction value :

$$\begin{aligned} R^k &= S^k - L^k f_1^k \\ &= L^k F^k - L^k f_1^k = L^k\left(F^k - f_1^k\right) \end{aligned}$$

$$\boxed{\therefore \quad L^k v^k = R^k}$$

The correction value is obtained by the above equation on the coarse grid in order to decrease the components of the long wavelength.

Restriction interpolation to the coarse grid G^{k+1}

$$R^{k+1} = I_k^{k+1} R^k$$

Solving the correction on the coarse grid

$$L^{k+1} v^{k+1} = R^{k+1}$$

The correction on the fine grid G^k is obtained by the prolongation interpolation.

$$v^k = I_{k+1}^k v^{k+1}$$

Correcting the approximation by using the above correction value. $f_2^k = f_1^k + v^k$

SOR operation again:

$$f_3^k = SOR\left(L^k, S^k, f_2^k, n_2\right)$$

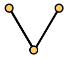
Repeating

↓

2－Grid Cycle

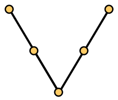
2-grid Cycle

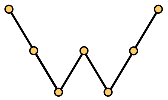
Source Code

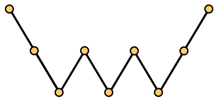


3-grid Cycle

Source Code

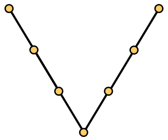


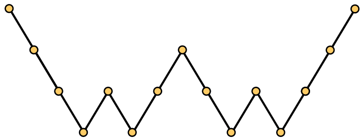




4-grid Cycle

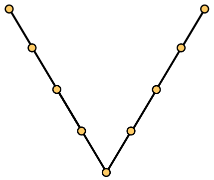
Source Code





5-grid Cycle

Source Code



cf : non-MG

Source Code