8. Coupled Mode Theory

8.1 Coupled mode equation

unperturbed (uncoupled) eigen mode

$$\nabla \times \tilde{\mathbf{H}}_i = j\omega\varepsilon_0 N_i^2 \tilde{\mathbf{E}}_i$$

$$\nabla \times \tilde{\mathbf{E}}_i = -j\omega\mu_0 \tilde{\mathbf{H}}_i$$

$$\begin{cases} \tilde{\mathbf{E}}_i = \mathbf{E}_i e^{j(\omega t - \beta_i z)} \\ \tilde{\mathbf{H}}_i = \mathbf{H}_i e^{j(\omega t - \beta_i z)} \end{cases} \qquad (i = 1,2)$$

Perturbed (coupled) mode is expressed by superposition of unperturbed modes.

$$\begin{cases} \tilde{\mathbf{E}} = A(z)\tilde{\mathbf{E}}_1 + B(z)\tilde{\mathbf{E}}_2 \\ \tilde{\mathbf{H}} = A(z)\tilde{\mathbf{H}}_1 + B(z)\tilde{\mathbf{H}}_2 \end{cases}$$

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu_0 \tilde{\mathbf{H}} \qquad \frac{dA}{dz}\mathbf{u_z} \times \tilde{\mathbf{E}}_1 + \frac{dB}{dz}\mathbf{u_z} \times \tilde{\mathbf{E}}_2 = 0 \qquad \text{(a)}$$

$$\nabla \times \tilde{\mathbf{H}} = j\omega\varepsilon_0 N^2 \tilde{\mathbf{E}} \qquad \frac{dA}{dz}\mathbf{u_z} \times \tilde{\mathbf{H}}_1 + \frac{dB}{dz}\mathbf{u_z} \times \tilde{\mathbf{H}}_2 - j\omega\varepsilon_0 (N^2 - N_1^2) A\tilde{\mathbf{E}}_1 - j\omega\varepsilon_0 (N^2 - N_2^2) B\tilde{\mathbf{E}}_2 = 0$$

(b)

Coupled mode equation

$$\iint\left(\tilde{E}_1^{\ *}\cdot(b)-\tilde{H}_1^{\ *}\cdot(a)\right)dS=0$$

$$\iint\left(\tilde{E}_2^{\ *}\cdot(b)-\tilde{H}_2^{\ *}\cdot(a)\right)dS=0$$

$$\Rightarrow$$

$$\frac{dA}{dz}+C_{12}\frac{dB}{dz}e^{-j(\beta_2-\beta_1)z}+j\kappa_{11}A+j\kappa_{12}Be^{-j(\beta_2-\beta_1)z}=0$$

$$\frac{dB}{dz}+C_{21}\frac{dA}{dz}e^{j(\beta_2-\beta_1)z}+j\kappa_{21}Ae^{j(\beta_2-\beta_1)z}+j\kappa_{22}B=0$$

$$C_{ij}=\frac{\iint\mathbf{u_z}\cdot(\mathbf{E}_i^{\ *}\times\mathbf{H}_j+\mathbf{E}_j\times\mathbf{H}_i^{\ *})dS}{\iint\mathbf{u_z}\cdot(\mathbf{E}_i^{\ *}\times\mathbf{H}_i+\mathbf{E}_i\times\mathbf{H}_i^{\ *})dS}$$

$$\kappa_{ij}=\frac{\omega\varepsilon_0\iint(\mathrm{N}^2-\mathrm{N}_\mathrm{j}^{\ 2})\mathbf{E}_i^{\ *}\cdot\mathbf{E}_j dS}{\iint\mathbf{u_z}\cdot(\mathbf{E}_i^{\ *}\times\mathbf{H}_i+\mathbf{E}_i\times\mathbf{H}_i^{\ *})dS}\qquad(i,j=1,2)$$

normalization condition:

$$\iint u_z\cdot(E_i^{\ *}\times H_i+E_i\times H_i^{\ *})dS=4P_i=1$$

$$\Rightarrow\begin{cases}C_{12}^{\ *}=C_{21}\\ \kappa_{11}=\kappa_{11}^{\ *},\quad\kappa_{22}=\kappa_{22}^{\ *}\end{cases}$$

Coupled mode equation

power carried by electromagnetic wave

$$P = \frac{1}{2}\iint (\tilde{E} \times \tilde{H}^*) \cdot u_z dS$$

$$= \frac{1}{4}(|A|^2 + |B|^2 + AB^* C_{12}{}^* e^{j2\delta z} + A^* B C_{12}\ e^{-j2\delta z})$$

$$\delta = \frac{\beta_2 - \beta_1}{2}$$

power conservation condition： $\frac{dP}{dz} = 0$

$$\frac{dA}{dz}(A^* + B^* C_{12}{}^* e^{j2\delta z}) + \frac{dA^*}{dz}(A + BC_{12} e^{-j2\delta z})$$

$$+ \frac{dB}{dz}(B^* + A^* C_{12} e^{-j2\delta z}) + \frac{dB^*}{dz}(B + AC_{12}{}^* e^{j2\delta z})$$

$$+ j2\delta(AB^* C_{12}{}^* e^{j2\delta z} - A^* B C_{12} e^{-j2\delta z}) = 0$$

$$(-\kappa_{21} + \kappa_{12}{}^* + 2\delta C_{12}{}^*) AB^* e^{j2\delta z} + (-\kappa_{12} + \kappa_{21}{}^* - 2\delta C_{12}) A^* B e^{-j2\delta z} = 0$$

$$-\kappa_{21} + \kappa_{12}{}^* + 2\delta C_{12}{}^* = 0$$

$$\therefore \kappa_{21} = \kappa_{12}{}^* + 2\delta C_{12}{}^*$$

Coupled mode equation

in case of symmetric coupled waveguide: $\delta = 0$

in case of weakly coupled waveguides : $C_{12} \approx 0$ ⇒ $\kappa_{21} = \kappa_{12}{}^*$

$$\kappa_a = \frac{\kappa_{12} - \kappa_{22} C_{12}}{1 - |C_{12}|^2}, \quad \kappa_b = \frac{\kappa_{21} - \kappa_{11} C_{21}}{1 - |C_{12}|^2}$$

$$\alpha_a = \frac{-\kappa_{11} + \kappa_{21} C_{12}}{1 - |C_{12}|^2}, \quad \alpha_b = \frac{-\kappa_{22} + \kappa_{12} C_{21}}{1 - |C_{12}|^2}$$

$$\frac{dA}{dz} = -j\kappa_a B(z) e^{-j2\delta z} + j\alpha_a A(z)$$

$$\frac{dB}{dz} = -j\kappa_b A(z) e^{j2\delta z} + j\alpha_b B(z)$$

$C_{12} = 0 \quad \kappa_{11} = \kappa_{22} = 0$ ⇒ $\kappa_a = \kappa_{12} \quad \kappa_b = \kappa_{21} \quad \alpha_a = \alpha_b = 0$

$$\frac{dA}{dz} = -j\kappa_{12} B(z) e^{-j(\beta_2 - \beta_1)z}$$

$$\frac{dB}{dz} = -j\kappa_{21} A(z) e^{j(\beta_2 - \beta_1)z} \qquad \text{(coupled mode equation)}$$

8.2 Coupling coefficients

Consider a TE mode propagating in aslab waveguide extending y-z plane.

$$E = (0, E_y, 0)$$

$$H = (H_x, 0, H_z)$$

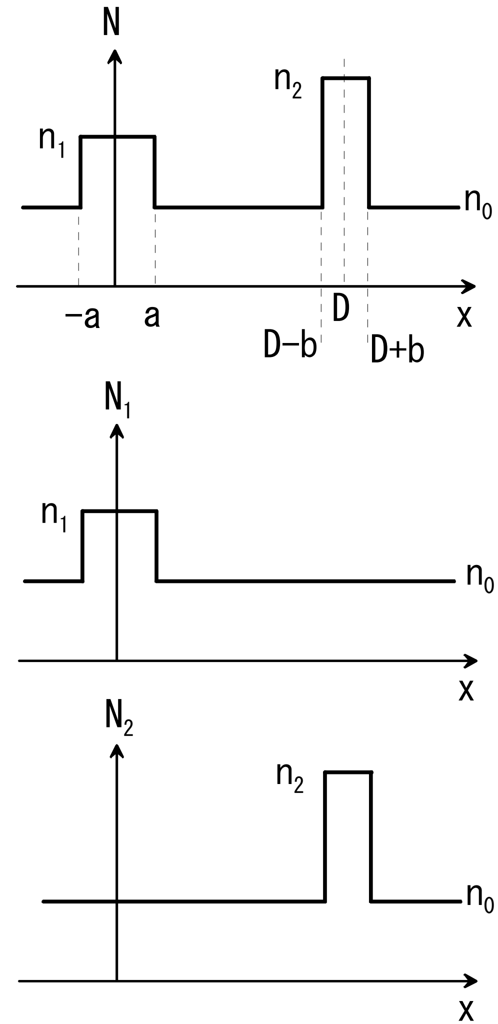
$$H_x = -\frac{\beta}{\omega\mu_0} E_y$$

$$(\mathbf{E}_1^* \times \mathbf{H}_1 + \mathbf{H}_1 \times \mathbf{E}_1^*) \cdot \mathbf{u}_z$$

$$= 2\,\mathrm{Re}[\mathbf{E}_1^* \times \mathbf{H}_1 \cdot \mathbf{u}_z] = \frac{2\beta}{\omega\mu_0} \left| E_{1y} \right|^2$$

normalization condition:

$$\iint (\mathbf{E}_1^* \times \mathbf{H}_1 + \mathbf{H}_1 \times \mathbf{E}_1^*) \cdot \mathbf{u}_z = 4P_1 = 1$$



Coupling coefficients

coupling coefficient:

$$C_{12} = \int (\mathbf{E}_1^* \times \mathbf{H}_2 + \mathbf{E}_2 \times \mathbf{H}_1^*) \cdot \mathbf{u}_z dx$$
$$= \int (-E_{1y}{}^* H_{2x} - E_{2y} H_{1x}{}^*) dx$$
$$= \frac{\beta_1 + \beta_2}{\omega \mu_0} \int E_{1y}{}^* E_{2y} dx$$
$$\cong \frac{2\omega\varepsilon_0 n}{k_0} \int E_{1y}{}^* E_{2y} dx$$

$$\kappa_{11} = \omega\varepsilon_0 \int (N^2 \text{-} N_1{}^2) E_1^* \cdot E_1 dx = \omega\varepsilon_0 \int_{D-b}^{D+b} (n_2{}^2 \text{-} n_0{}^2) |E_{1y}|^2 dx$$

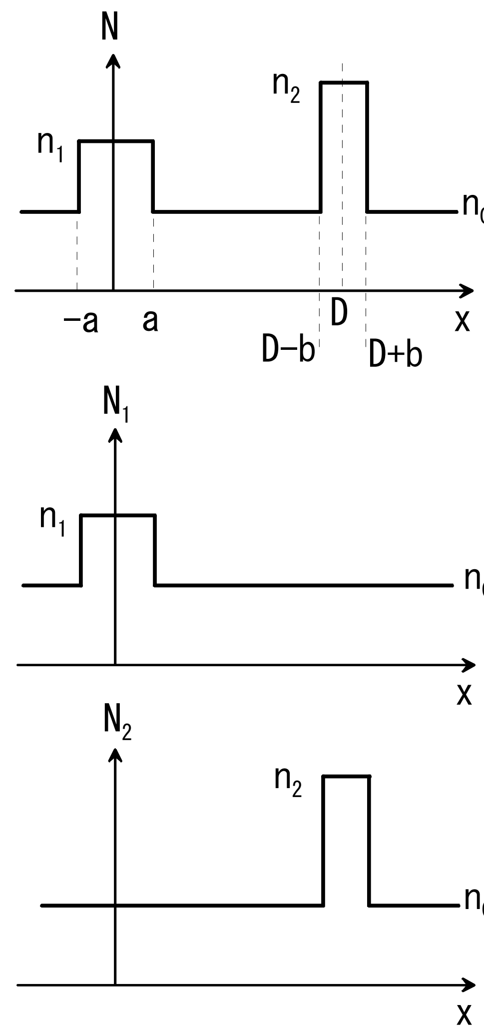
$$\kappa_{12} = \omega\varepsilon_0 \int (N^2 \text{-} N_2{}^2) E_1^* \cdot E_2 dx = \omega\varepsilon_0 \int_{-a}^{a} (n_1{}^2 \text{-} n_0{}^2) E_{1y}{}^* E_{2y} dx$$

field intensity : core region=1, clad region= $\eta (<< 1)$

$$\kappa_{11} \propto \omega\varepsilon_0 (n_2^2 - n_0^2) \eta^2$$

$$\kappa_{12} \propto \omega\varepsilon_0 (n_1^2 - n_0^2) \eta$$

$$C_{12} \propto (\omega\varepsilon_0 n / k_0) \eta \qquad n / k_0 << 1$$



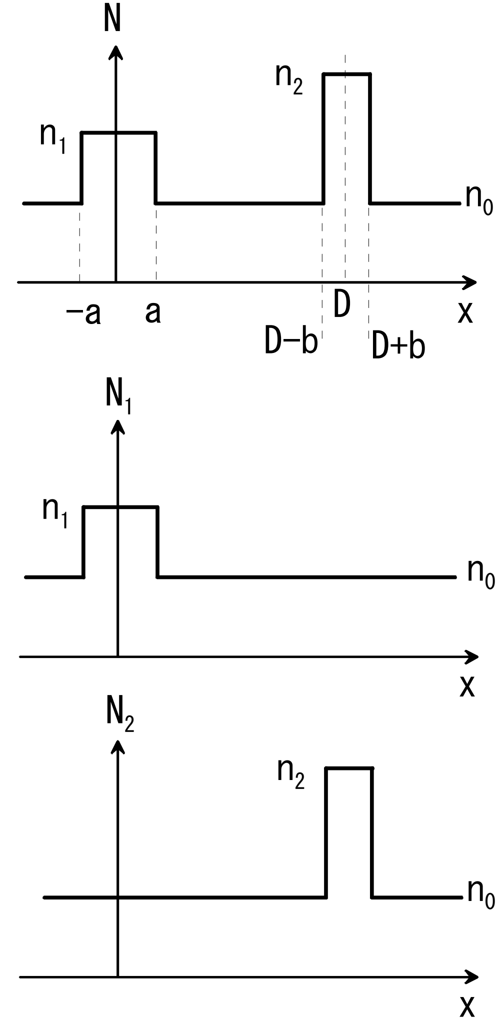
8.3 Co-directional mode coupling

$$C_{12} = 0 \quad \kappa_{21} = \kappa_{12}{}^*$$

coupled mode equation:

$$\frac{dA}{dz} = -j\kappa_{12} B(z) e^{-j(\beta_2 - \beta_1)z}$$

$$\frac{dB}{dz} = -j\kappa_{21} A(z) e^{j(\beta_2 - \beta_1)z}$$



$$\begin{cases} A(z) = (a_1 e^{jqz} + a_2 e^{-jqz}) e^{-j\delta z} \\ B(z) = (b_1 e^{jqz} + b_2 e^{-jqz}) e^{j\delta z} \end{cases}$$

$$\begin{cases} A(0) = a_1 + a_2 \\ B(0) = b_1 + b_2 \end{cases}$$

$$\begin{bmatrix} A(z) \\ B(z) \end{bmatrix} = \begin{bmatrix} (\cos qz + j\frac{\delta}{q}\sin qz) e^{-j\delta z} & -j\frac{\kappa_{12}}{q}\sin qz e^{-j\delta z} \\ -j\frac{\kappa_{12}{}^*}{q}\sin qz e^{j\delta z} & (\cos qz - j\frac{\delta}{q}\sin qz) e^{j\delta z} \end{bmatrix} \begin{bmatrix} A(0) \\ B(0) \end{bmatrix}$$

$$q = \sqrt{|\kappa_{12}|^2 + \delta^2}$$

Co-directional coupling

input from guide-1: $A(0) = A_0 \quad B(0) = 0$

$$\begin{cases} A(z) = A_0(\cos qz + j\dfrac{\delta}{q}\sin qz)e^{-j\delta z} \\ B(z) = A_0(-j\dfrac{\kappa_{12}^*}{q})\sin qz e^{j\delta z} \end{cases}$$

$$\frac{P_a}{P_0} = \frac{|A(z)|^2}{|A_0|^2} = \cos^2 qz + \frac{\delta^2}{q^2}\sin^2 qz = 1 - \frac{|\kappa_{12}|^2}{q^2}\sin^2 qz$$

$$\frac{P_b}{P_0} = \frac{|\kappa_{12}|^2}{q^2}\sin^2 qz = \frac{1}{1+\dfrac{\delta^2}{|\kappa_{12}|^2}}\sin^2 qz$$

coupling length: $L_c = \dfrac{\frac{\pi}{2}}{q} = \dfrac{\pi}{2\sqrt{|\kappa_{12}|^2 + \delta^2}}$ ➡ $\dfrac{P_b}{P_0} = \dfrac{1}{1+\dfrac{\delta^2}{|\kappa_{12}|^2}}$

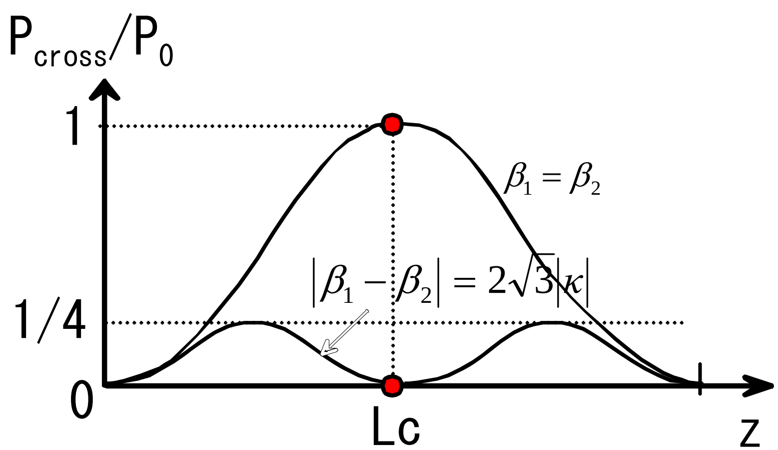
phase matched ($\delta = 0$) ➡ $L_c = \dfrac{\pi}{2|\kappa_{12}|}$, $\dfrac{P_b}{P_0} = 1$

Co-directional coupling

coupled power: $$\frac{P_{cross}}{P_{in}} = \frac{1}{1+\frac{(\beta_2-\beta_1)^2}{4|\kappa_{12}|^2}} \sin^2 \sqrt{|\kappa_{12}|^2 + \frac{(\beta_2-\beta_1)^2}{4}}\, z$$

symmetric coupler $\beta_1 = \beta_2$: $$\frac{P_{cross}}{P_{in}} = \sin^2 |\kappa_{12}| z$$

asymmetric coupler: $|\beta_1 - \beta_2| = 2\sqrt{3}|\kappa|$ ➡ $$\frac{P_{cross}}{P_{in}} = \frac{1}{4}\sin^2 2|\kappa| z$$



cross state: $\beta_1 = \beta_2$

bar state: $|\beta_1 - \beta_2| = 2\sqrt{3}|\kappa|$