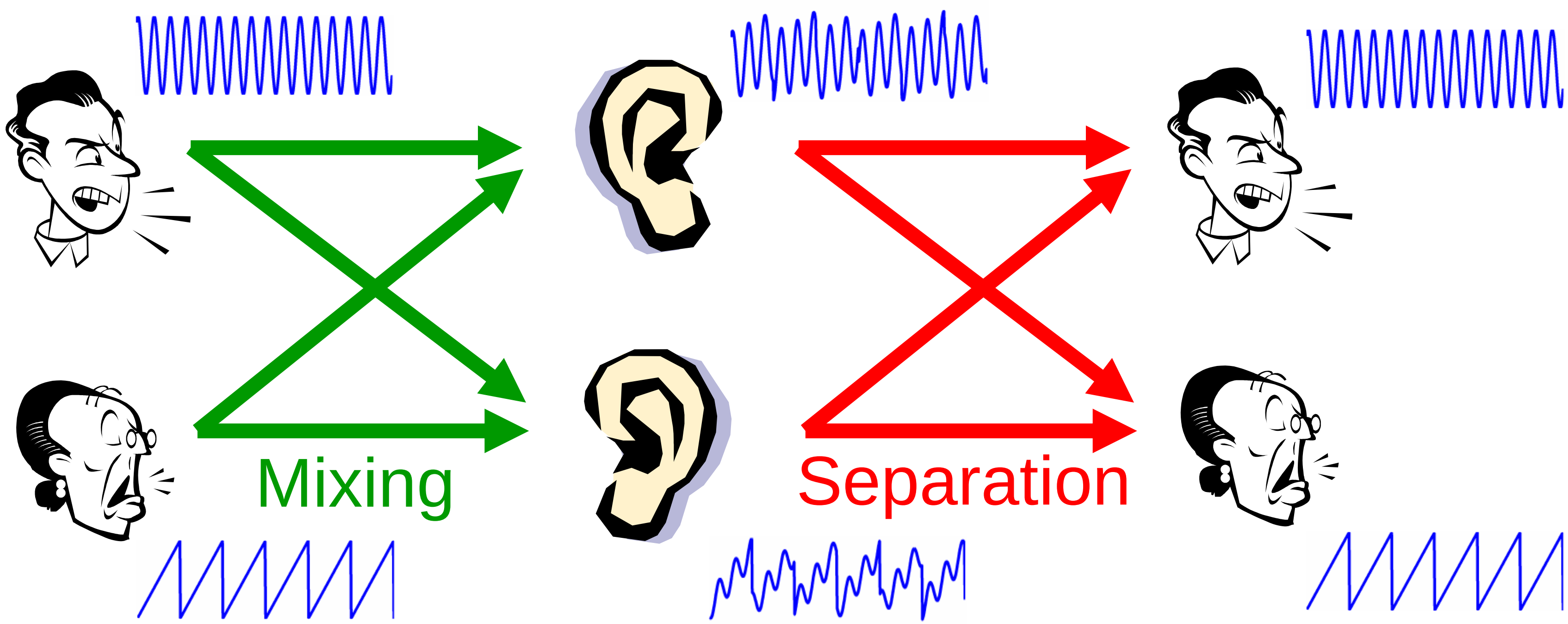
Blind Source Separation

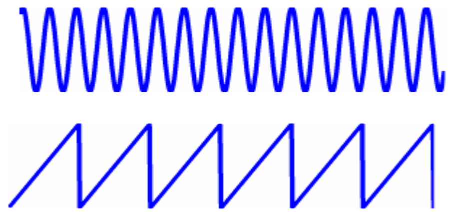
- Cocktail-party problem:



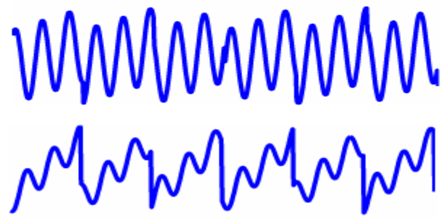
- We want to separate mixed signals into original ones.

Formulation

- Source signals:
  - Speaker 1: $s_1^{(1)}, s_2^{(1)}, \ldots, s_n^{(1)}$
  - Speaker 2: $s_1^{(2)}, s_2^{(2)}, \ldots, s_n^{(2)}$



- Mixed signals:
  - Left ear: $x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}$
  - Right ear: $x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}$



$$x_i^{(1)} = m_{11} s_i^{(1)} + m_{12} s_i^{(2)}$$
$$x_i^{(2)} = m_{21} s_i^{(1)} + m_{22} s_i^{(2)}$$

Formulation (cont.)

- In matrix form:

$$\boldsymbol{x}_i = \boldsymbol{M}\boldsymbol{s}_i$$

$$\boldsymbol{x}_i = \begin{pmatrix} x_i^{(1)} \\ x_i^{(2)} \end{pmatrix}$$

$$\boldsymbol{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \qquad \boldsymbol{s}_i = \begin{pmatrix} s_i^{(1)} \\ s_i^{(2)} \end{pmatrix}$$

- More generally
  - $\boldsymbol{x}_i, \boldsymbol{s}_i$: d-dimensional vectors
  - $\boldsymbol{M}$: d-dimensional matrix.

Problem

$$\boldsymbol{x}_i = \boldsymbol{M}\boldsymbol{s}_i$$

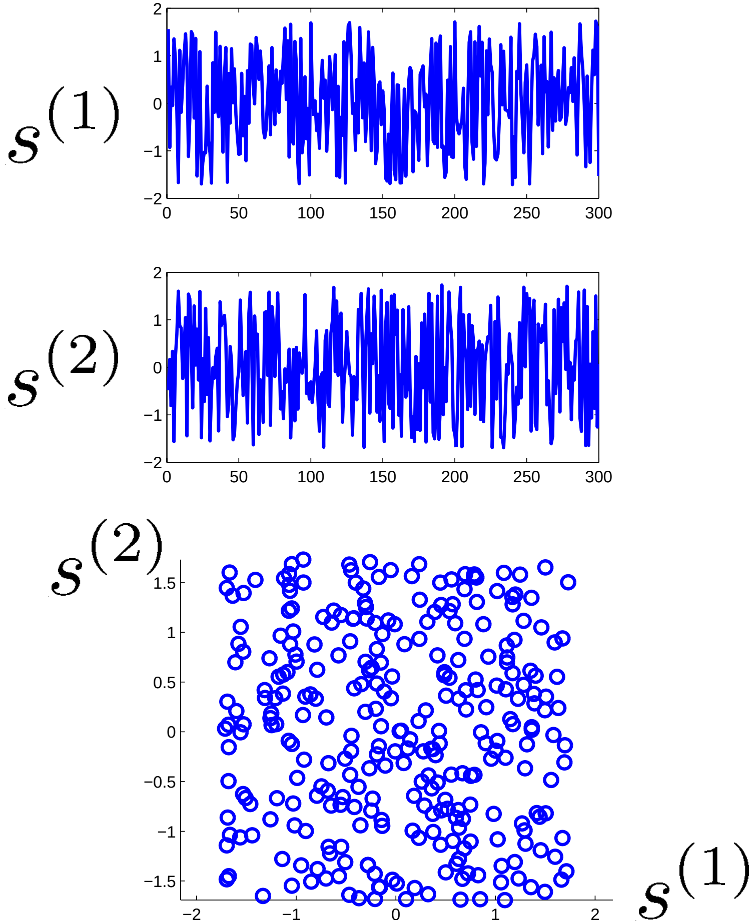
- We want to estimate $\{\boldsymbol{s}_i\}_{i=1}^n$ from $\{\boldsymbol{x}_i\}_{i=1}^n$.
- Approach: Estimate $\boldsymbol{M}$, and use its inverse for obtaining $\{\widehat{\boldsymbol{s}}_i\}_{i=1}^n$.

$$\widehat{\boldsymbol{s}}_i = \widehat{\boldsymbol{M}}^{-1}\boldsymbol{x}_i$$

- In BSS, the followings may not be important:
  - Permutation of separated signals
  - Scaling of separated signals
- Therefore, we estimate $\widehat{\boldsymbol{M}}^{-1}$ up to permutation and scaling of rows.

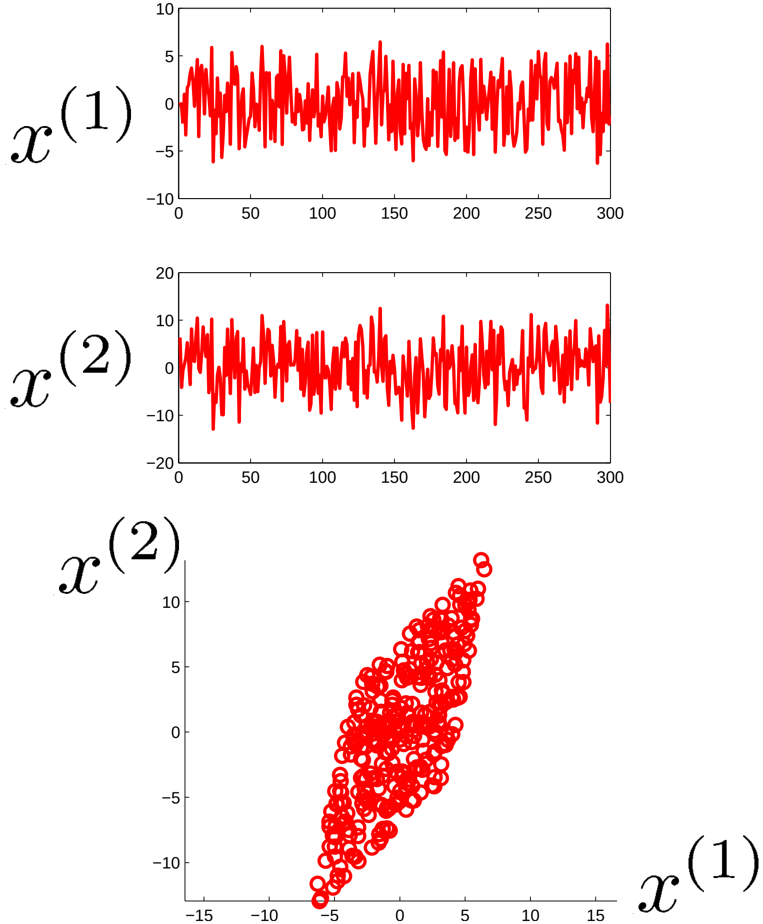
Example

Source signals (uniform)



Mixed signals

$$\boldsymbol{M} = \begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix}$$



Assumptions

- $\{\boldsymbol{s}_i\}_{i=1}^n$ are i.i.d. random variables with mean zero and covariance identity:

$$\frac{1}{n}\sum_{i=1}^n \boldsymbol{s}_i = \boldsymbol{0} \qquad \frac{1}{n}\sum_{i=1}^n \boldsymbol{s}_i\boldsymbol{s}_i^\top = \boldsymbol{I}_d$$

- $\{s^{(j)}\}_{j=1}^d$ are mutually independent:

$$P(s^{(1)}, s^{(2)}, \ldots, s^{(d)}) = P(s^{(1)})P(s^{(2)})\cdots P(s^{(d)})$$

- $\{s^{(j)}\}_{j=1}^d$ are non-Gaussian.
- $\boldsymbol{M}$ is invertible.
- BSS under source independence is called independent component analysis.

Data Sphering

- Sphering (or pre-whitening):

$$\widetilde{\boldsymbol{x}}_i = \boldsymbol{C}^{-\frac{1}{2}}\boldsymbol{x}_i \qquad \boldsymbol{C} = \frac{1}{n}\sum_{j=1}^{n}\boldsymbol{x}_j\boldsymbol{x}_j^\top$$

- Then

$$\widetilde{\boldsymbol{x}}_i = \widetilde{\boldsymbol{M}}\boldsymbol{s}_i \qquad \widetilde{\boldsymbol{M}} = \boldsymbol{C}^{-\frac{1}{2}}\boldsymbol{M}$$

- Now we want to estimate $\widetilde{\boldsymbol{M}}$ from $\{\widetilde{\boldsymbol{x}}_i\}_{i=1}^n$, and obtain $\{\widehat{\boldsymbol{s}}_i\}_{i=1}^n$ by

$$\widehat{\boldsymbol{s}}_i = \boldsymbol{W}\widetilde{\boldsymbol{x}} \qquad \boldsymbol{W} = \widetilde{\boldsymbol{M}}^{-1}$$

Example

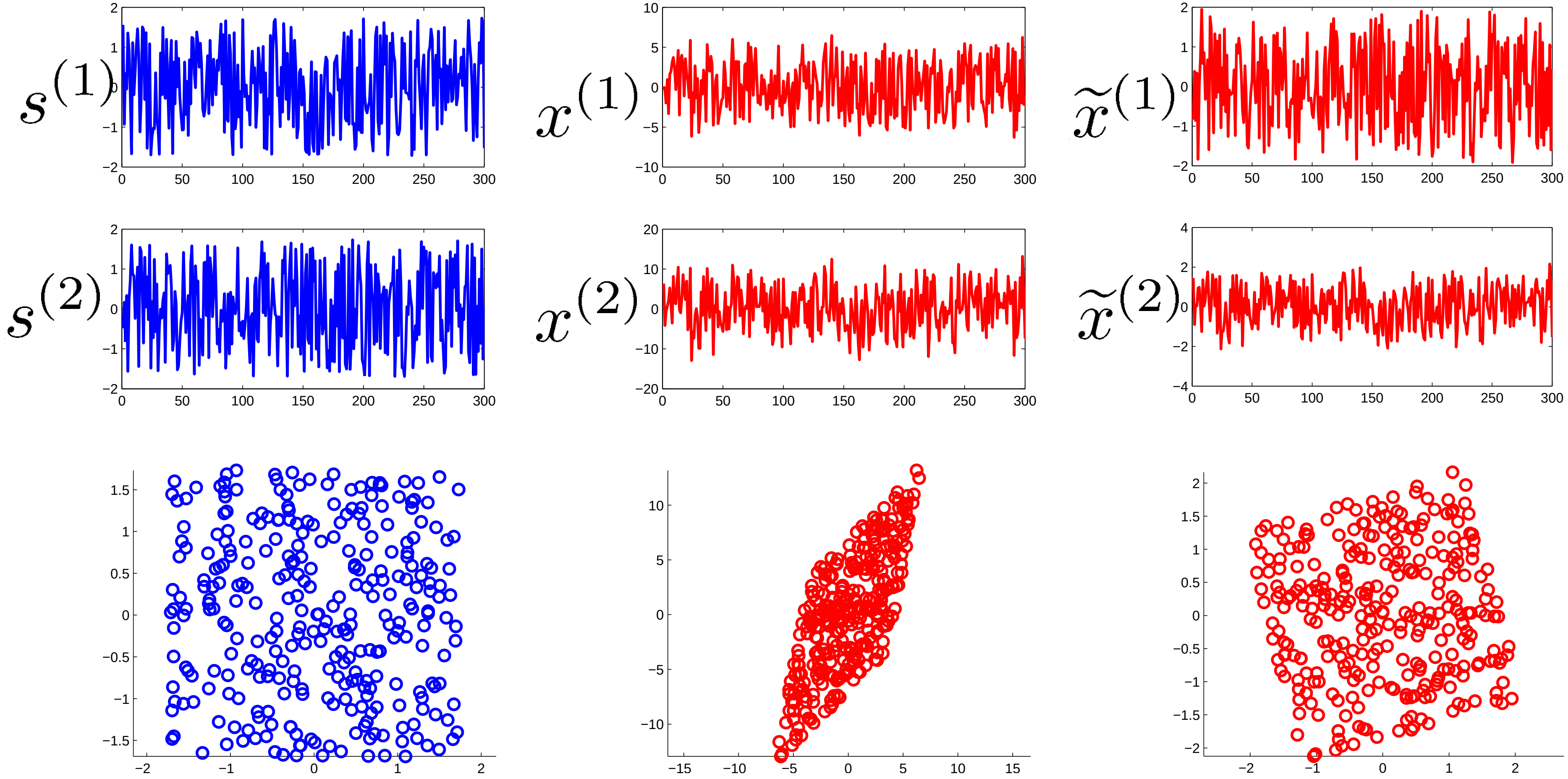
Source signals (uniform)

Mixed signals

$$\boldsymbol{M} = \begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix}$$

Sphered signals

$$\widetilde{\boldsymbol{x}}_i = \boldsymbol{C}^{-\frac{1}{2}} \boldsymbol{x}_i$$



Orthogonal Matrix

- Since
$$\widetilde{\boldsymbol{C}} = \frac{1}{n}\sum_{i=1}^{n}\widetilde{\boldsymbol{x}}_i\widetilde{\boldsymbol{x}}_i^\top = \boldsymbol{I}_d$$
$$\widetilde{\boldsymbol{C}} = \widetilde{\boldsymbol{M}}\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{s}_i\boldsymbol{s}_i^\top\right)\widetilde{\boldsymbol{M}}^\top = \widetilde{\boldsymbol{M}}\widetilde{\boldsymbol{M}}^\top$$

  $\widetilde{M}$ is an orthogonal matrix.

- Therefore,

$$\widehat{\boldsymbol{s}}_i = \boldsymbol{W}\widetilde{\boldsymbol{x}}$$

$$\boldsymbol{W} = \widetilde{\boldsymbol{M}}^{-1} = \widetilde{\boldsymbol{M}}^\top \equiv (\boldsymbol{w}^{(1)}|\boldsymbol{w}^{(2)}|\cdots|\boldsymbol{w}^{(d)})^\top$$

$\{\boldsymbol{w}^{(j)}\}_{j=1}^{d}$:Orthonormal basis

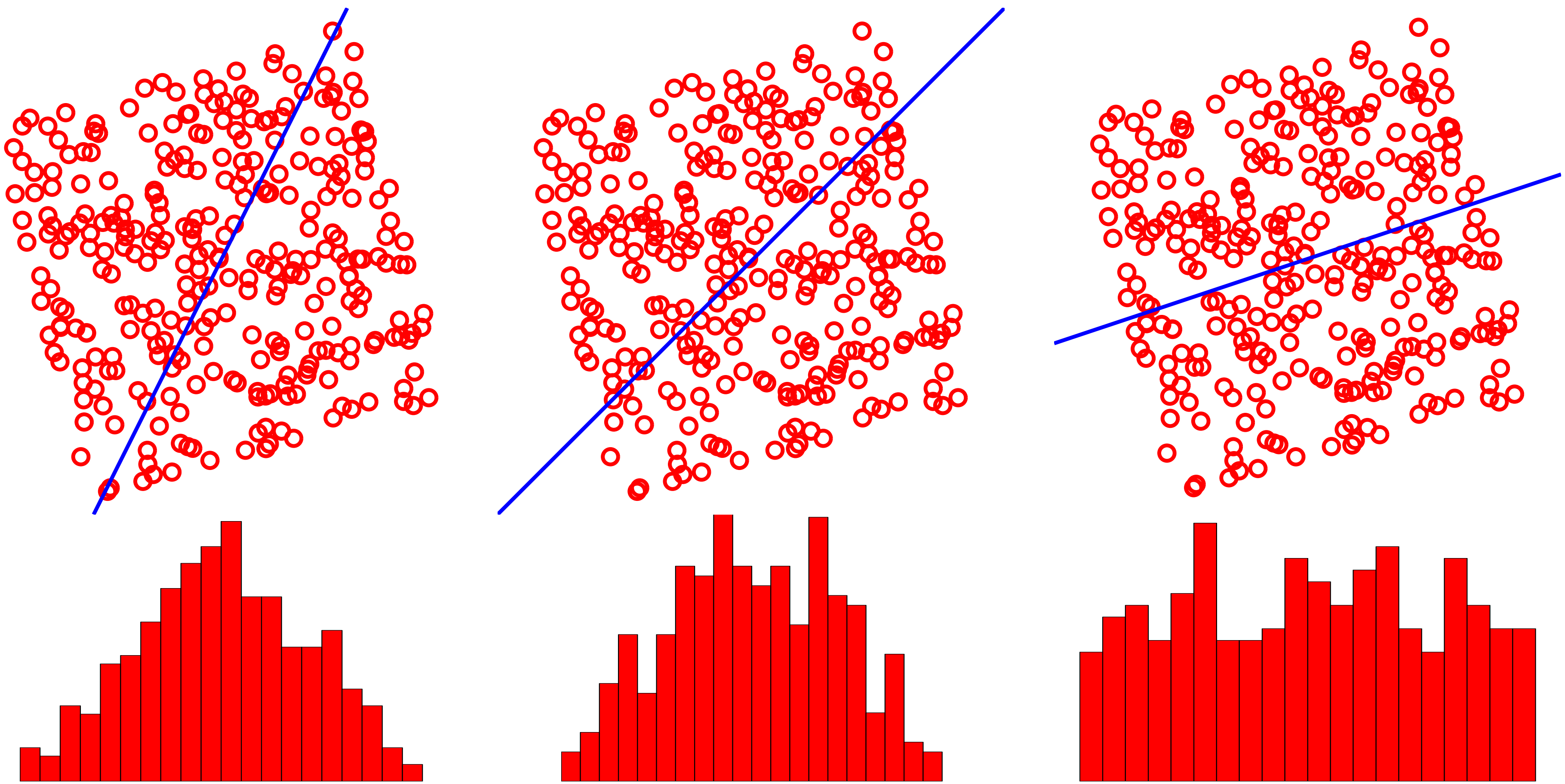
$$\widehat{s}_i^{(j)} = \langle\boldsymbol{w}^{(j)}, \widetilde{\boldsymbol{x}}_i\rangle$$

Non-Gaussian Is Independent

- Now we want to find an ONB $\{\boldsymbol{w}^{(j)}\}_{j=1}^{d}$ such that $\{\widehat{s}^{(j)}\}_{j=1}^{d}$ are independent.
- Central limit theorem: Sum of independent variables tends to be Gaussian.
- Conversely, non-Gaussian variables are independent.
- We find non-Gaussian directions in $\{\widetilde{\boldsymbol{x}}_i\}_{i=1}^{n}$.

Example (cont.)

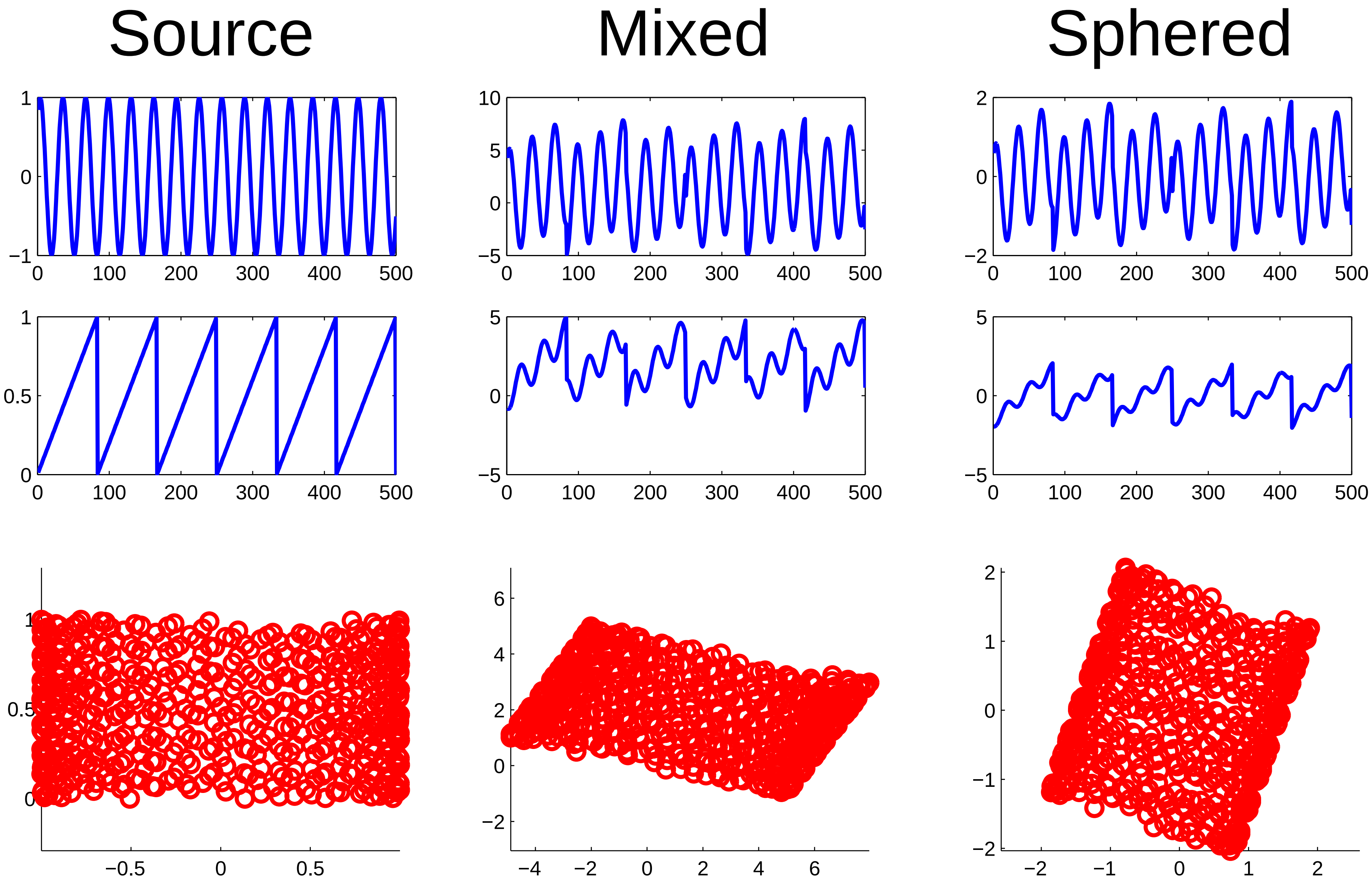
- Non-Gaussian direction is independent.



ICA by Projection Pursuit

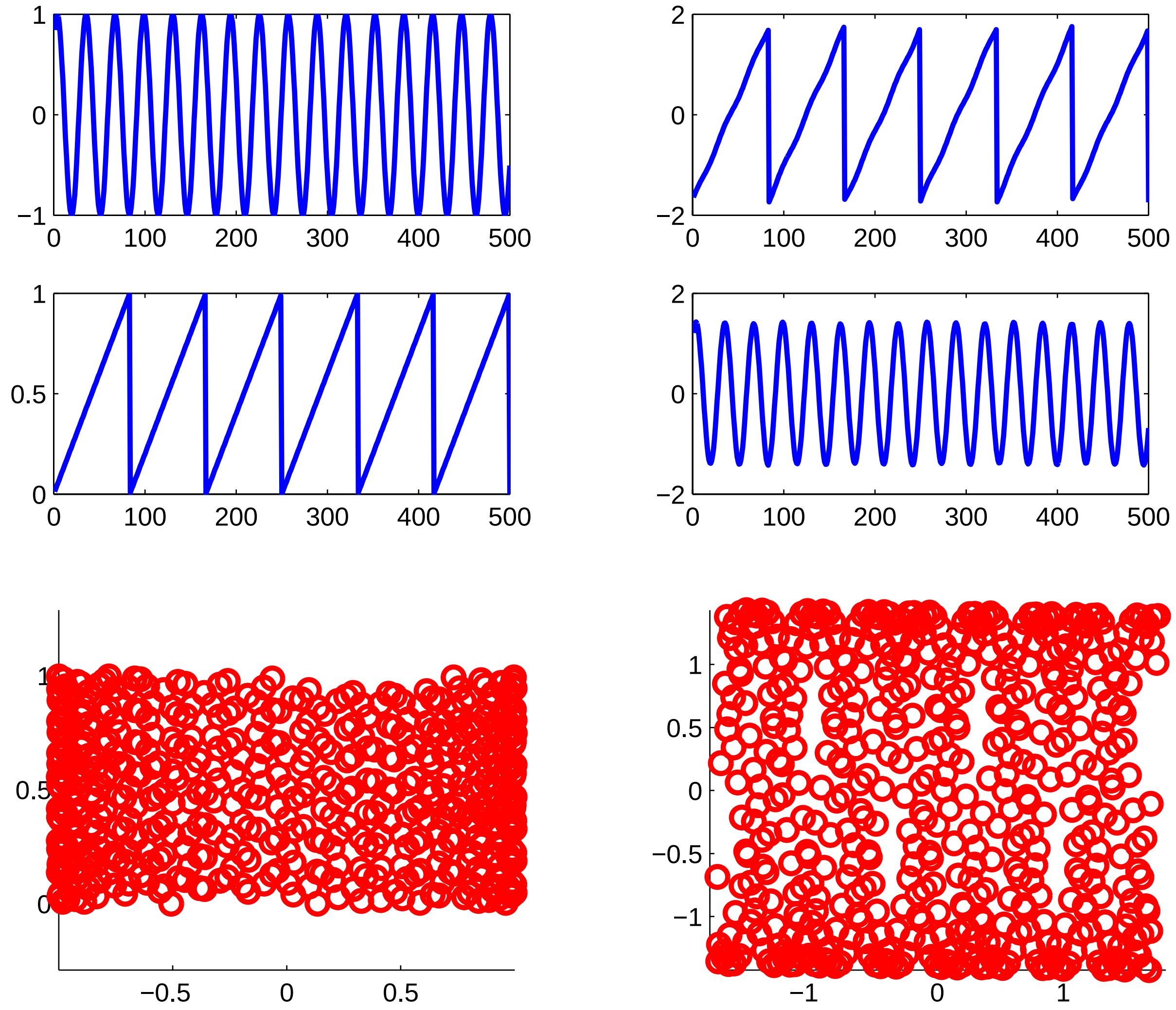
- Finding non-Gaussian directions can be achieved by projection pursuit algorithms!
  - Center and sphere the data.
  - Find non-Gaussian directions by PP.
- As PP algorithm, we may use an approximate Newton-based method, which is called FastICA.

Example 2


Source
Mixed
Sphered
1
0
−1
0
100
200
300
400
500
10
5
0
−5
2
−2
0.5
−0.5
6
4
−4

Example 2 (cont.)

Source | Separated



- Original signals are recovered except for permutation and scaling.