Urban Environmental Engineering 4

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Lecture in the last time

- Diffusion
  - Molecular diffusion
  - Turbulent diffusion
  - Dispersion

Extension to turbulent diffusion

$$\frac{\partial C}{\partial t}+\frac{\partial uC}{\partial x}+\frac{\partial vC}{\partial y}+\frac{\partial wC}{\partial z}=D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)C$$

Substituting $C=\overline{C}+C', u=\overline{u}+u'$ and so on,

What will happen to the original equation ?

Q4: Show that the above equation can be extended to the following equation in the case of turbulent diffusion.

$$\frac{\partial C}{\partial t}+\frac{\partial uC}{\partial x}+\frac{\partial vC}{\partial y}+\frac{\partial wC}{\partial z}=(D+K)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)C$$

Separation into fluctuations and time smoothed value

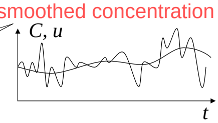
$$\frac{\partial C}{\partial t}+\frac{\partial uC}{\partial x}+\frac{\partial vC}{\partial y}+\frac{\partial wC}{\partial z}=D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)C$$

In order to make the discussion simple, we shall consider one dimensional system

Time smoothed concentration is not the completely smoothed concentration but the concentration neglecting high frequency composition.

$$\frac{\partial C}{\partial t}+\frac{\partial uC}{\partial x}=D\frac{\partial^2}{\partial x^2}C$$

Time-smoothed concentration



Substituting $C=\overline{C}+C', u=\overline{u}+u'$

Fluctuation

Time smoothing of the original equation

The flow in environment is in most cases turbulent flow. Flow velocity and concentrations are fluctuating due to turbulence. If we want to analyze environmental matters, we want to know the time-smoothed concentration.

$$\frac{\partial (C'+\overline{C})}{\partial t}+\frac{\partial (u'+\overline{u})(C'+\overline{C})}{\partial x}=D\frac{\partial^2}{\partial x^2}(C'+\overline{C})$$

Taking time average

$$\overline{\frac{\partial (C'+\overline{C})}{\partial t}}+\overline{\frac{\partial (u'+\overline{u})(C'+\overline{C})}{\partial x}}=D\overline{\frac{\partial^2}{\partial x^2}(C'+\overline{C})}$$

Calculations

$$\overline{\frac{\partial (C'+\overline{C})}{\partial t}}+\overline{\frac{\partial (u'+\overline{u})(C'+\overline{C})}{\partial x}}=D\overline{\frac{\partial^2}{\partial x^2}(C'+\overline{C})}$$

1. We can exchange the sequence of partial derivative and time smoothing.
2. Time-smoothing of fluctuation is zero.
3. Double time-smoothing equals to single time-smoothing

$$\overline{\frac{\partial (C'+\overline{C})}{\partial t}}=\overline{\frac{\partial C'}{\partial t}}+\overline{\frac{\partial \overline{C}}{\partial t}}=\frac{\partial \overline{C'}}{\partial t}+\frac{\partial \overline{\overline{C}}}{\partial t}=\frac{\partial \overline{C}}{\partial t}$$

Calculations

$$\frac{\partial\overline{(C'+\overline{C})}}{\partial t}+\frac{\partial\overline{(u'+\overline{u})(C'+\overline{C})}}{\partial x}=D\frac{\partial^2}{\partial x^2}\overline{(C'+\overline{C})}$$

1. We can exchange the sequence of partial derivative and time smoothing.
2. Time-smoothing of fluctuation is zero.
3. Double time-smoothing equals to single time-smoothing

$$\frac{\partial\overline{(C'+\overline{C})}}{\partial t}=\frac{\partial\overline{C'}}{\partial t}+\frac{\partial\overline{\overline{C}}}{\partial t}=\frac{\partial\overline{C'}}{\partial t}+\frac{\partial\overline{\overline{C}}}{\partial t}=\frac{\partial\overline{C}}{\partial t}$$

Calculations (cont.)

$$\frac{\partial\overline{(u'+\overline{u})(C'+\overline{C})}}{\partial x}=\frac{\partial\overline{u'C'}}{\partial x}+\frac{\partial\overline{u'\overline{C}}}{\partial x}+\frac{\partial\overline{\overline{u}C'}}{\partial x}+\frac{\partial\overline{\overline{u}\,\overline{C}}}{\partial x}$$

$$=\frac{\partial\overline{u'C'}}{\partial x}+\frac{\partial\overline{u'}\,\overline{C}}{\partial x}+\frac{\partial\overline{u}\,\overline{C'}}{\partial x}+\frac{\partial\overline{\overline{u}\,\overline{C}}}{\partial x}=\boxed{\frac{\partial\overline{u'C'}}{\partial x}}+\frac{\partial\overline{u}\,\overline{C}}{\partial x}$$

In the case of Navier Stokes equation, we considered the correlation between u' and v' and we introduced a concept of Reynolds Stress. If no correlation is found between u' and C', the term $\overline{u'C'}$ will be zero. However, as we considered in the Reynolds stress, we introduce the following relationship as an analogy with Fick's law.

$$\overline{u'C'}=-K\frac{\partial\overline{C}}{\partial x}$$

Diffusion equation in Turbulent Flow

$$\frac{\partial C}{\partial t}+\frac{\partial uC}{\partial x}=D\frac{\partial^2}{\partial x^2}C$$

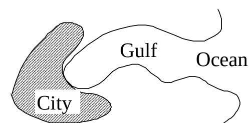
$$\downarrow$$

$$\frac{\partial\overline{C}}{\partial t}+\frac{\partial\overline{u}\,\overline{C}}{\partial x}=D\frac{\partial^2}{\partial x^2}\overline{C}+\frac{\partial}{\partial x}K\frac{\partial}{\partial x}\overline{C}$$

If K is independent from x and if we extend into 3-D equation with the assumption that the turbulence is independent from directions,

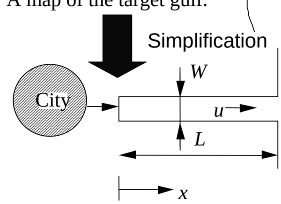
$$\frac{\partial\overline{C}}{\partial t}+\frac{\partial\overline{u}\,\overline{C}}{\partial x}+\frac{\partial\overline{v}\,\overline{C}}{\partial y}+\frac{\partial\overline{w}\,\overline{C}}{\partial z}=(D+K)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\overline{C}$$

Concentration Profile in a gulf



A map of the target gulf.

Simplification



Assuming that

1) Depth, length and width of the gulf is h, L and W.
2) The City discharges wastewater to the gulf with the flow rate of Q and concentration of certain pollutant of Ci
3) Concentration the pollutant outside the gulf assumed to be 0.
4) No reaction takes place in the gulf.

Concentration Profile in a gulf (cont.)

Question

The mass flux in the gulf in steady state can be written as

$$J=uC(x)-D\frac{dC(x)}{dx}$$

The average velocity in the gulf is $u=\dfrac{Q}{Wh}$

1) Derive the equation $Q(C(x)-C_i)=hWD\dfrac{dC(x)}{dx}$
2) Give the solution which satisfies at the mouth of the gulf to the ocean.
3) Draw a figure which shows concentration profile and give an explanation of the effect of Q/hWD on the profile.

Answer

$$(C(x)-C_0)=\frac{D}{u}\bullet\frac{dC(x)}{dx}$$ can be written as

$$(C(x)-C_0)=\frac{D}{u}\bullet\frac{d(C(x)-C_0)}{dx}$$

This is a very simple first order differential equation and can be solved as

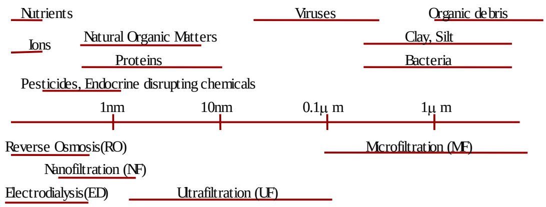
$$(C(x)-C_0)=A\exp\left(\frac{u}{D}x\right)$$ Boundary condition: $C(L)=0$

The solution is: $C(x)=C_0-C_0\exp\left(\frac{u}{D}(x-L)\right)$

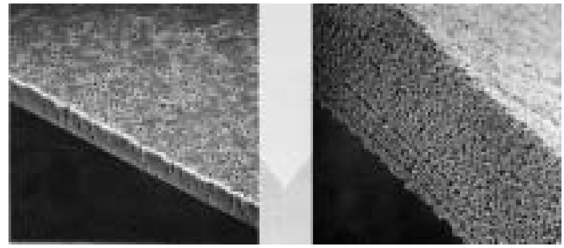
Q: Draw a figure to show graphically the solution when

$\dfrac{u}{D}\to\infty$ $\qquad$ $\dfrac{u}{D}\to 0$

Solute size and Separation size



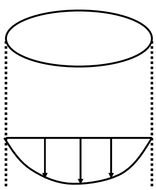
Porous Symmetric Membranes



Nuclepore Membrane   Cellulose Membrane

From Catalog of Nuclepore

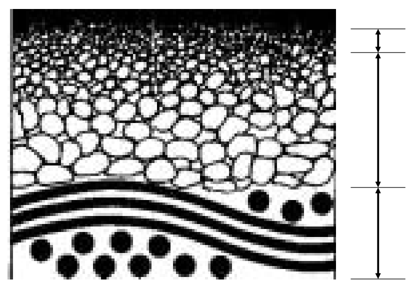
Model on flow in membrane pores



$$J_v = \frac{A_k \Delta P d_p^2}{32 \mu \Delta z}$$

J_v : Volume Flux
A_k : Pore Open Ratio
ΔP : Applied Pressure
d_p : Pore Diameter
μ : Viscosity
Δz : Membrane Thickness

Asymmetric Membranes



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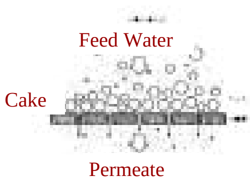
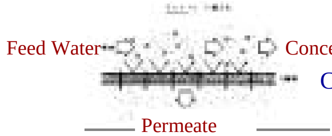
Crossflow and Deadend



Deadend Filtration

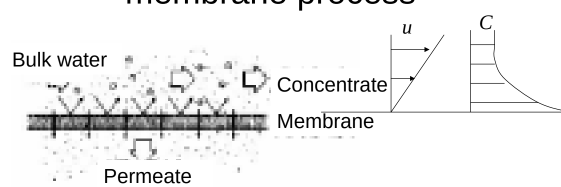
Suitable for low SS
Periodical Flushing



Crossflow Filtration

Continuous Operation

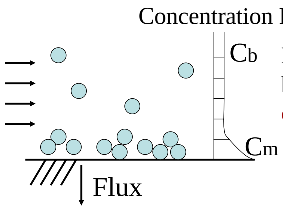
Concentration polarization in membrane process



Question 2 : Hydrodynamic boundary layer is formed on a plate placed in a fluid stream. In the same way, concentration boundary layer is also formed on the surface if we consider a membrane process. Which is thicker, hydrodynamic boundary layer or concentration boundary layer?

Hint: Compare kinematic viscosity and molecular diffusivity in water system. In air system kinematic viscosity is roughly the same as molecular diffusivity.

Concept of Film Theory



Difference in concentration between Cb and Cm is called concentration polarization

$$J_v = k \ln\left(\frac{C_m}{C_b}\right)$$

k: Mass transfer coefficient

Mass Transfer Coefficient

$$k = \frac{\text{D (Diffusion coefficient)}}{\delta\text{(Thickness of polarization layer)}}$$

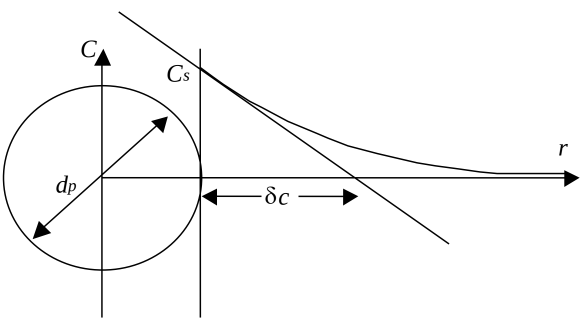
Laminar

$$Sh = 1.62\, Re^{0.33}\, Sc^{0.33} \left(\frac{d_h}{L}\right)^{0.33}$$

Turbulent

$$Sh = 0.023\, Re^{0.8}\, Sc^{0.33}$$

Melting sphere (No flow)



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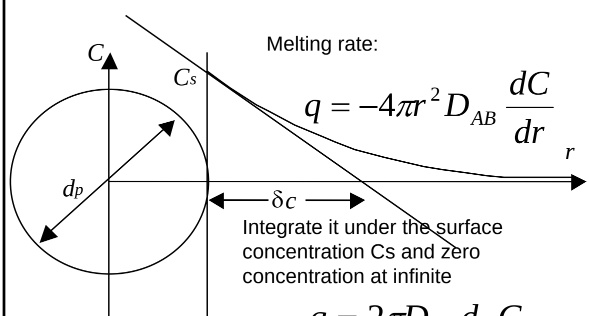


Melting rate:

$$q = -4\pi r^2 D_{AB} \frac{dC}{dr}$$

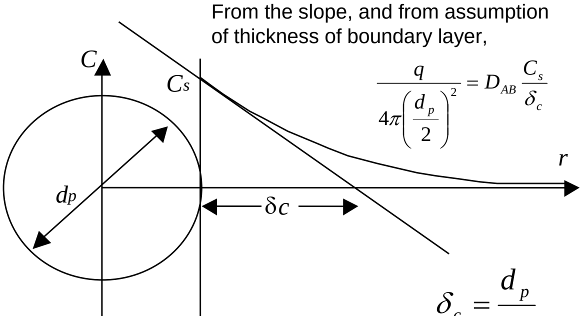
Integrate it under the surface concentration Cs and zero concentration at infinite

$$q = 2\pi D_{AB} d_p C_s$$

Melting sphere (No flow)

From the slope, and from assumption of thickness of boundary layer,

$$\frac{q}{4\pi\left(\frac{d_p}{2}\right)^2} = D_{AB}\frac{C_s}{\delta_c}$$

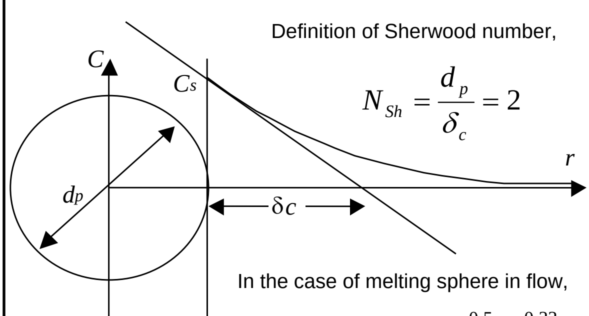


$$\delta_c = \frac{d_p}{2}$$

Melting sphere

Definition of Sherwood number,

$$N_{Sh} = \frac{d_p}{\delta_c} = 2$$



In the case of melting sphere in flow,

$$N_{Sh} = 2 + 0.6 N_{\mathrm{Re}}^{0.5} N_{Sc}^{0.33}$$

Concentration

Concentration is most commonly measured or expressed as mass density. Here we consider a binary system which consists of two substances, for example, A (Salt) and B (Water).

$$\rho = C_A + C_B \quad [kg / m^3]$$

Density of Fluid

If you want to use other concentrations than mass density, modification is necessary to diffusion equation.

Concentrations other than mass density.

Mass fraction, Molar concentration,
Volume fraction, Mole fraction

Q1. Which concentration do we use when we say….

1) Salt concentration in sea is 3% 2) BOD is 10 mg/L 3) BOD is 10 ppm

4) Oxygen in air is 21%

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