2 ULTIMATE LIMIT STATE

2.1 Flexural Capacity of RC Beam

2.1.1 Behavior of RC Beam under Flexure

For a RC beam subjected to flexural moment, its behavior can be classified into three stages.

Stage I (before flexural cracking of concrete)

- The distribution of stress and that of strain are linear and the strains of concrete and steel at the same position are the same.

- The relation of moment and deformation (deflection, rotation or strain of steel) is linear.

- The stress in steel is extremely small and hence, its effect can be neglected.

Stage II (after flexural cracking of concrete and before yielding of steel)

- When the tensile stress of concrete becomes larger than the tensile strength, flexure crack starts at the extreme tension fiber and propagates to the compression side. Flexural cracks in other section than that of the maximum moment also occur with the increase in load, and the crack width is increasing.

- The resistance in tensile part of the section is mostly carried by reinforcing steel and hence the strain in steel increases rapidly.

- The strain distribution along the height of the cross section is almost linear.

Stage III (after yielding of steel)

- **After yielding of steel, the strain in steel increases very rapidly and the stress in steel keeps a constant value equal to the yield stress. The moment at the point of yielding of steel is called as the yield moment, M_y.**
- **Due to the balance of forces, the increase in resisting load is very small and only influenced by the arm length between the total compressive and tensile forces.**
- **The cross section fails when the strain in concrete reaches the ultimate compressive strain. This state is called the ultimate state and the moment at this stage is called the flexural ultimate moment, M_u (Fig. 2.1).**

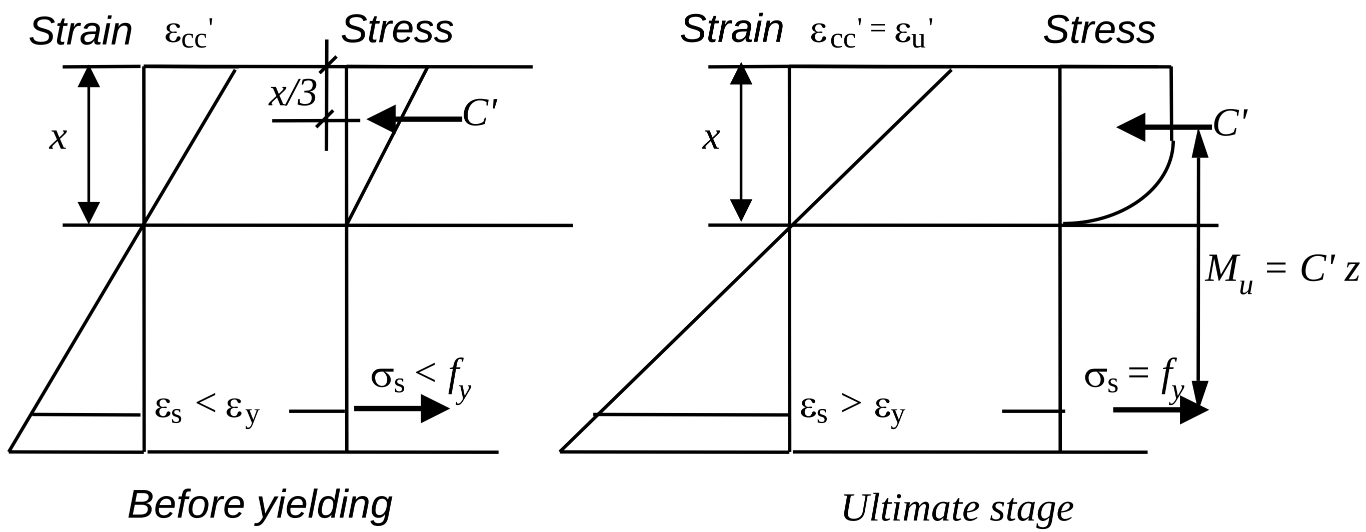


Fig. 2.1 Stress and strain behaviors of RC beams in various stages

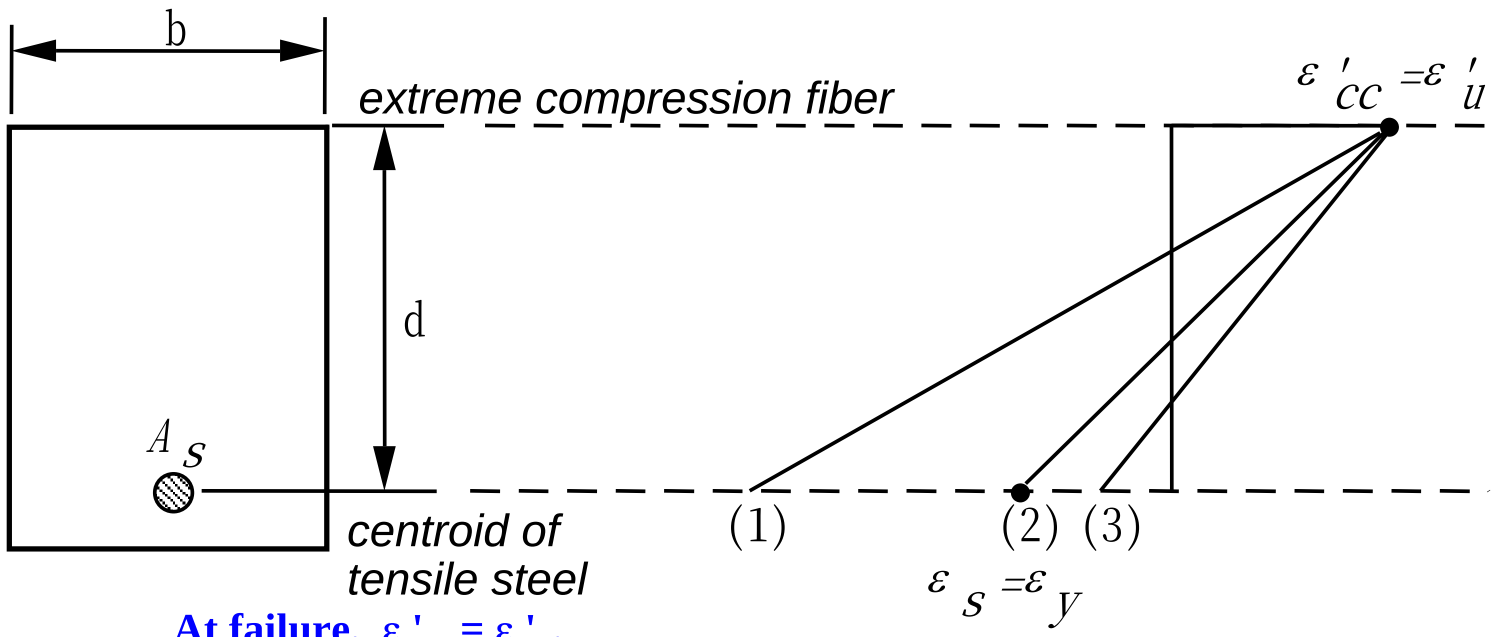
The rough estimate for the strain in steel is as follows:

strain at stage III = 10 × (strain at stage II)

strain at stage II = 10 × (strain at stage I)

In the case of large amount of tensile reinforcing steel, before reaching the stage III, the strain in concrete might reach the ultimate strain, i.e., the steel does not yield even at the ultimate state. This type of failure is not preferable in the design of reinforced concrete due to two reasons: (1) The steel is not effectively used and only slight deformation is expected up to failure. (2) The ductility of the cross section defined by the area under the load-deformation curve is small, and therefore a brittle failure occurs.

Types of both failures are called: "flexural tension failure" for the case of yielding and "flexural compression failure" for the case of no yielding, respectively. The parameter controlling the failure mode is reinforcement ratio "p" defined as $p = A_s /(b\ d)$, where A_s : the area of tensile steel, b : the width of cross section, and d : the effective depth measured from the extreme compression fiber to the centroid of the tension steel (Fig. 2.2).



At failure, $\varepsilon'_{cc} = \varepsilon'_u$,

(1) flexural tension failure $\varepsilon_s > \varepsilon_y$

(2) balanced failure $\varepsilon_s = \varepsilon_y$

(3) flexural compression failure $\varepsilon_s < \varepsilon_y$

Fig. 2.2 Definitions of failure modes for RC beams in terms of strain in tensile reinforcement

The reinforcement ratio at the border of both flexural tension and compression failures is called as the balanced reinforcement ratio p_b.

For flexural tension failure, flexural capacity M_u is strongly influenced by the reinforcement ratio (p) and yield strength (f_y) of tension steel, rather than those of compression steel (p', f_y') and concrete strength (f'_c).

In flexural compression failure, on the contrary M_u is affected by p', f_y', f'_c rather than p, f_y.

In the design of reinforced concrete beams subjected to flexural moment, the reinforcement ratio of tension steel p should not be too large to prevent flexural compression failure as previously mentioned. Also, p should not be too small because reinforced concrete beams might fail after the first flexural crack occurs.

2.1.2 General Method to Calculate Stresses in RC Beams

For ordinary reinforced concrete beams, the flexural tension failure occurs, and the relation among M_{cr}, M_y, and M_u are as follows:

$$M_{cr} \leq M_y \leq M_u \qquad (2.1)$$

where, M_{cr} : the flexural cracking moment, M_y : the yielding moment,
M_u : the ultimate moment (= the flexural capacity).

In the *stage I*, concrete can be assumed to be elastic and the effect of reinforcing steel can be neglected. Therefore,

$$M_{cr} = \frac{f_b \; I_g}{y_t} \qquad (2.2)$$

where, I_g : moment of inertia of gross cross section of concrete about centroid axis (neutral axis)

f_b : flexural strength of concrete

y_t : distance from the neutral axis to the extreme tension fiber

After concrete cracks (stage II), the stresses in concrete subjected to compression can be calculated by assuming the function of strain according to uniaxial stress-strain relation as follows:

$$\sigma'_c = f_1\,(\varepsilon'_c) \qquad (2.3)$$

$$\sigma_s = f_2\,(\varepsilon_s) \qquad (2.4)$$

$$\sigma'_{sc} = f_3\,(\varepsilon'_{sc}) \qquad (2.5)$$

where, σ'_c, ε'_c : compressive stress and strain of concrete

σ_s, ε_s : tensile stress and strain of tensile steel

σ'_{sc}, ε'_{sc} : compressive stress and strain of compression steel

The strain distribution is assumed to be linear as follows:

$$\varepsilon'_{cy} = \varepsilon'_{cc}\,\frac{y}{x} \qquad (2.6)$$

$$\varepsilon_s = \varepsilon'_{cc}\,\frac{d-x}{x} \qquad (2.7)$$

d − y was wrong!

$$\varepsilon'_{sc} = \varepsilon'_{cc}\,\frac{x-d'}{x} \qquad (2.8)$$

where, ε'_{cc} : the value of compressive strain of concrete at the extreme compression fiber (positive)

ε'_{sc} : the strain in compression steel (positive in compression)

ε'_{cy} : the value of compressive strain of concrete at the distance *y* from the neutral axis (positive in compression)

***d* : the effective depth**

***x* : distance from the extreme compression fiber to the neutral axis (Fig. 2.3).**

***d'* : distance from the extreme compression fiber to the centroid of compression steel**

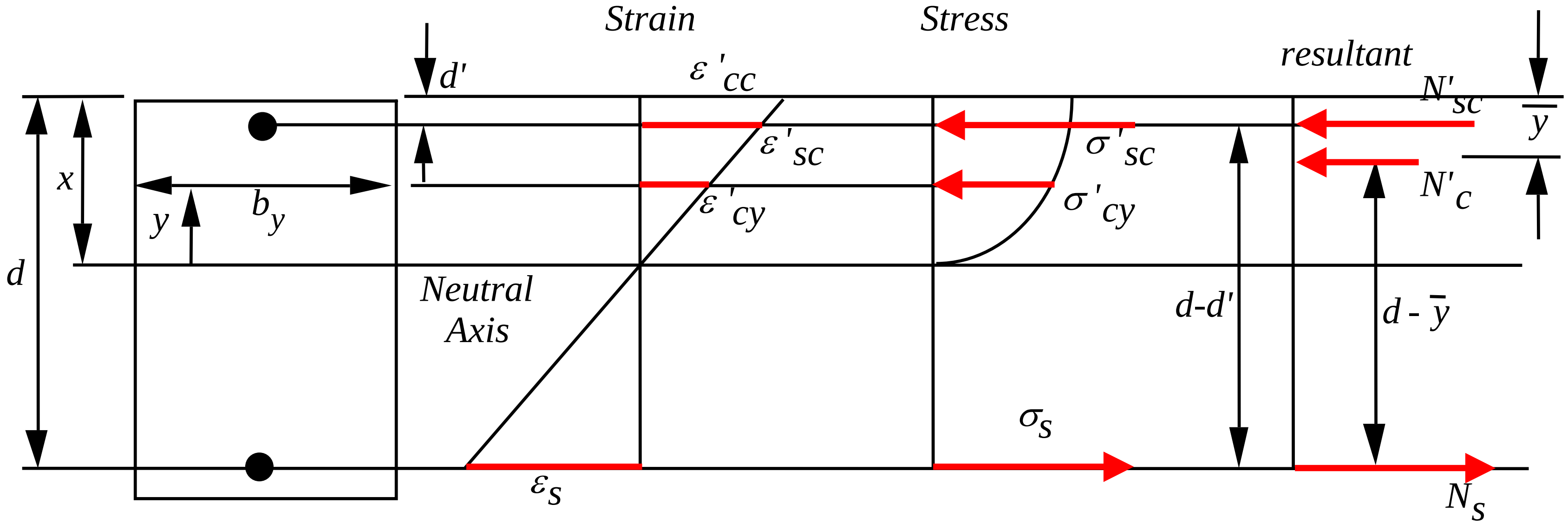


Fig. 2.3 Definitions of notations

The relations between internal forces and stresses are as follows:

$$N'_c = \int_0^x b_y \, \sigma'_{cy} \, dy = \frac{x}{\varepsilon'_{cc}} \int_0^{\varepsilon'_{cc}} b_y \; f_1\left(\varepsilon'_{cy}\right) d\varepsilon'_{cy} \qquad \textbf{(2.9)}$$

$$N_s = A_s \, \sigma_s = A_s \; f_2\left(\varepsilon_s\right) = A_s \; f_4\left(\varepsilon'_{cc}, x, d\right) \qquad \textbf{(2.10)}$$

$$N'_{sc} = A_s' \sigma'_{sc} = A_s' \, f_3\left(\varepsilon'_{sc}\right) = A_s' \, f_5\left(\varepsilon'_{cc}, x, d'\right) \qquad \textbf{(2.11)}$$

***f_4* and *f_5* are the functions obtained from Eqs.(2.4), (2.7) and (2.5), (2.8), respectively. In these relationships,**

N'_c	**: resultant internal compressive force of concrete**
N_s	**: resultant internal tensile force of tension steel**
N'_{sc}	**: resultant internal compressive force of compression steel**
b_y	**: width of section at distance "y" from the neutral axis**
A_s	**: area of tension steel**
A'_s	**: area of compression steel**

From the equilibrium of forces in the case of pure bending (*M* : external moment applied),

$$N'_c + N'_{sc} - N_s = 0 \qquad \textbf{(2.12)}$$

$$M = N'_c\left(d - \overline{y}\right) + N'_{sc}\left(d - d'\right) \qquad \textbf{(2.13)}$$

where, $\overline{y}$: the distance from the extreme compression fiber to N'_c

$$\overline{y} = x - \frac{\int_0^x b_y \, \sigma'_{cy} \, y \, dy}{N'_c} = x - \left(\frac{x}{\varepsilon'_{cc}}\right)^2 \frac{1}{N'_c} \int_0^x b_y \, f_1(\varepsilon'_{cy}) \varepsilon'_{cy} \, d\varepsilon'_{cy} \qquad \textbf{(2.14)}$$

By substituting Eqs.(2.9)-(2.11) into Eqs.(2.12) and (2.13), the unknown values of ε'_{cc}, x, M are determined from two equations with one given value for either of them.

2.1.3 Calculation of Flexural Capacity

(1) Assumption for uniaxial stress - strain relationship

The following relationships are stipulated in JSCE's specification.

$$\begin{aligned} \sigma'_c &= k_3 \, f'_c \left[2\left(\frac{\varepsilon'_c}{\varepsilon'_o}\right) - \left(\frac{\varepsilon'_c}{\varepsilon'_o}\right)^2 \right] && (0 \le \varepsilon'_c \le \varepsilon'_o) \\ &= k_3 \, f'_c && (\varepsilon'_o \le \varepsilon'_c \le \varepsilon'_u) \end{aligned} \qquad \textbf{(2.15)}$$

$$\begin{aligned} \sigma_s &= E_s \, \varepsilon_s && (\varepsilon_s \le \varepsilon_y) \\ &= f_y && (\varepsilon_y < \varepsilon_s) \end{aligned} \qquad \textbf{(2.16)}$$

where, $\varepsilon_y = f_y / E_s$

$$\sigma'_{sc} = E_s\,\varepsilon'_{sc} \qquad (\varepsilon'_{sc} \le \varepsilon'_y)$$
$$= f'_y \qquad (\varepsilon'_y < \varepsilon'_{sc}) \qquad \textbf{(2.17)}$$

where, $\varepsilon'_y = f'_y/E_s$

Where, **f'_c : compressive strength of concrete**

$k_3 = 0.85$, $\varepsilon'_o = 0.002$, $\varepsilon'_u = 0.0035$

f_y, f'_y : yield strength of tension and compression steel

$E_s = 2.0 \times 10^5$ N/mm^2 (Fig. 2.4)

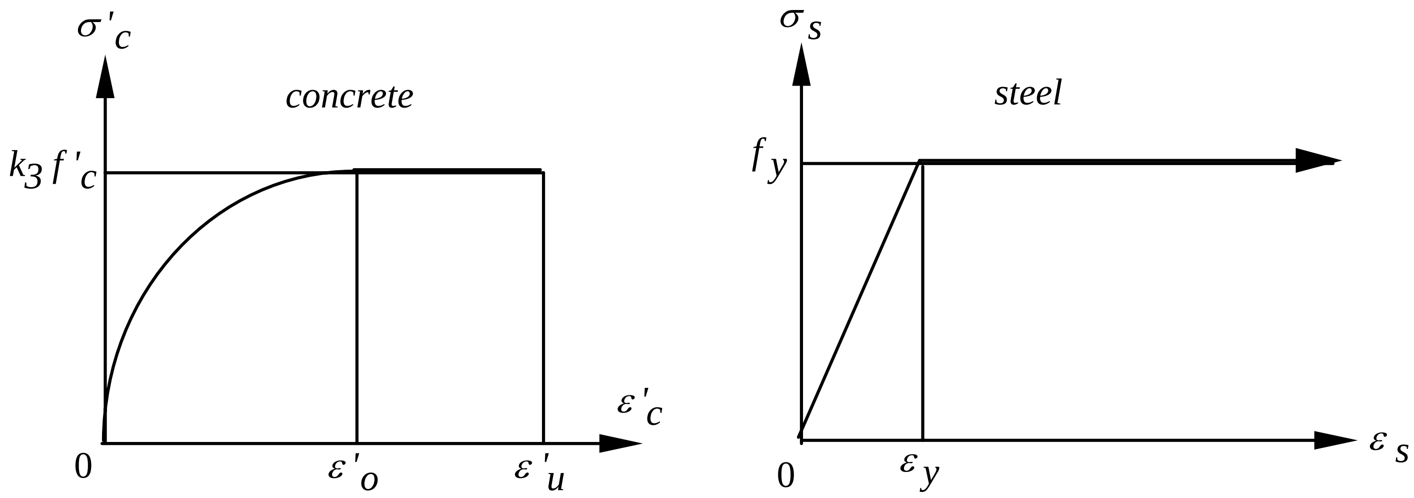


Fig. 2.4 Stress-strain relationships for concrete and steel to be applied to the design

(2) Ultimate moment for flexural tension failure

In the case of no compression steel, i.e., A'_s = 0, for the flexural tension failure, the state of strains can be expressed as follows:

$$\varepsilon_s > \varepsilon_y\,, \quad \varepsilon'_{cc} = \varepsilon'_u \qquad \textbf{(2.18)}$$

Also, when the rectangular cross section is considered, b_y becomes constant and b_y = b. Substituting Eqs.(2.15) - (2.18) into Eqs.(2.9) - (2.11), the following results can be obtained.

$$N'_c = \frac{b\,x}{\varepsilon'_u} k_3\, f'_c \left[\int_0^{\varepsilon'_o} \left\{ 2\left(\frac{\varepsilon'_{cy}}{\varepsilon'_o}\right) - \left(\frac{\varepsilon'_{cy}}{\varepsilon'_o}\right)^2 \right\} d\varepsilon'_{cy} + \int_{\varepsilon'_o}^{\varepsilon'_u} d\varepsilon'_{cy} \right]$$

$$= b\,x \left(1 - \frac{\varepsilon'_o}{3\varepsilon'_u}\right) k_3\, f'_c = b\,x\,k_1\,k_3\, f'_c \qquad \textbf{(2.19)}$$

where, $k_1 = \left(1 - \dfrac{\varepsilon'_o}{3\varepsilon'_u}\right)$

$$N_s = A_s\, f_y \qquad \textbf{(2.20)}$$

$$N'_{sc} = 0 \qquad \textbf{(2.21)}$$

Substituting Eqs.(2.19) - (2.21) into Eq.(2.12) (the equilibrium condition), *x* can be calculated.

$$x = \frac{A_s \ f_y}{b \ k_1 \ k_3 \ f'_c} = \frac{p \ f_y}{k_1 \ k_3 \ f'_c} d \qquad \textbf{(2.22)}$$

where, *p* : reinforcement ratio of tension steel (= $A_s/(b\ d)$)
Then, from Eqs. (2.14) and (2.19),

$$\overline{y} = \left[1 - \frac{1 - (\varepsilon'_o / \varepsilon'_u)^2 / 6}{2 \ k_1} \right] x = k_2 \ x \qquad \textbf{(2.23)}$$

where, $k_2 = \left[1 - \frac{1 - (\varepsilon'_o / \varepsilon'_u)^2 / 6}{2 \ k_1} \right]$

From Eq. (2.13),

$$M_u = N'_c \ (d - \overline{y})$$

$$M_u = A_s \ f_y \ d \left(1 - \frac{k_2}{k_1 \ k_3} \ p \ \frac{f_y}{f'_c} \right) \qquad \textbf{(2.24)}$$

After substituting values of ε'_o = 0.002, ε'_u = 0.0035, k_1 and k_2 can be calculated as follows:

$$k_1 = 0.810, \quad k_2 = 0.416$$

Substituting k_3 = 0.85 and k_1, k_2 into Eq. (2.24) leads to,

$$\frac{M_u}{b\,d^2\,f'_c} = \frac{p\,f_y}{f'_c}\left(1 - 0.60\,\frac{p\,f_y}{f'_c}\right) \qquad \textbf{(2.25)}$$

Eq.(2.25) shows the nondimensional relationship between $M_u/bd^2f'_c$ and pf_y/f'_c.

Flexural compression failure moment can be obtained as follows:

$$N_s = A_s\,\sigma_s = p\,b\,d\,E_s\,\varepsilon'_u\,\frac{d - x}{x}$$

From Eq.(2.19),

$$N'_c = b\,x\,k_1\,k_3\,f'_c$$

$$x = \frac{p\,d\,E_s\,\varepsilon'_u}{2k_1k_3\,f'_c}\left(-1 + \sqrt{1 + \frac{4k_1\,k_3\,f'_c}{p\,E_s\,\varepsilon'_u}}\right)$$

$$M_u = N'_c\left(d - \bar{y}\right)$$

$$Mu = b\,x\,k_1\,k_3\,f'_c\left(d - k_2\,x\right)$$

$$= 0.69\frac{x}{d}\left(1 - 0.416\,\frac{x}{d}\right)b\,d^2\,f'_c = \frac{b\,d^2\,f'_c}{3} \qquad (x/d \approx 0.7)$$

Relating to the shape of stress distribution of concrete, instead of using the parabolic curve as mentioned in the code, *the equivalent stress block*, with the same resultant compressive force and its location can be used because it can produce almost same ultimate moment. The simplest shape for the equivalent stress block is a rectangular one (*Fig. 2.5*).

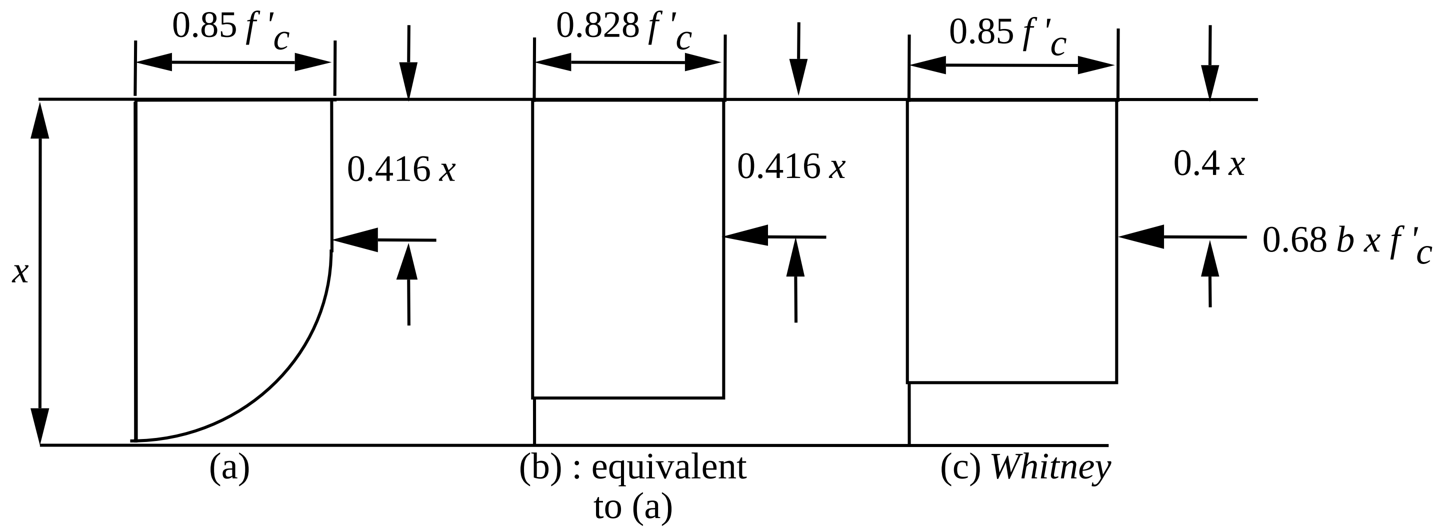


Fig. 2.5 Equivalent stress blocks for flexural compression zone

As shown in *Fig. 2.5*, the stress block (b) is equivalent to the shape (a), and the stress block (c) proposed by *Whitney* is an approximated one. This Whitney's stress block is commonly used in various codes as the equivalent stress block. It is noted that this equivalent stress block concept is theoretically applicable only to a cross-section with a constant width in compressive side, but it is also used to such cross-section shapes as T-section, I-section, and circular section because it can be proved that the difference between the results of actual stress-strain curve and equivalent stress block is very small.

In the case of *double reinforcement*, the above method has to be modified to include the effect of compression steel. However, for ordinary RC beams which are designed to exhibit flexural tension failure, the effect of compression steel is less influential in the calculation of the ultimate moment.

(3) Balanced reinforcement ratio

The *balanced reinforcement ratio* is used to judge whether flexural tension or compression failure occurs. A RC beam with the balanced reinforcement ratio will fail when the ultimate compression strain of concrete and the yield tensile strain of steel occur simultaneously.

$$\varepsilon_y = \varepsilon'_u \frac{d-x}{x} \quad \rightarrow \quad x = \frac{\varepsilon'_u}{\varepsilon'_u + \varepsilon_y} d \qquad \textbf{(2.26)}$$

$$\begin{aligned} p_b &= \frac{\varepsilon'_u}{\varepsilon'_u + \varepsilon_y} k_1 k_3 \frac{f'_c}{f_y} \\ &= \frac{0.69}{1+\varepsilon_y / \varepsilon'_u} \frac{f'_c}{f_y} \\ &= 0.40 \sim 0.48 \frac{f'_c}{f_y} \quad (f_y = 300 \sim 500\,\mathrm{N/mm^2}) \end{aligned} \qquad \textbf{(2.27)}$$

Therefore, if the reinforcement ratio is less than *$0.40f'_c/f_y$*, the flexural tension failure will occur.

(4) Design ultimate flexural capacity (ultimate moment)

In checking the ultimate limit state, partial safety factors are used to determine the design ultimate flexural capacity, M_{ud}.

The design material strength f'_{cd}, f_{yd} and ε_{yd} are obtained from characteristic values and material factors. The member factor γ_b (= 1.15) is used to obtain M_{ud} in Eq. (2.24) as follows:

$$M_{ud} = A_s f_{yd} d \left(1 - 0.60 \, p \frac{f_{yd}}{f'_{cd}} \right) / \gamma_b \qquad \textbf{(2.28)}$$

and from Eq.(2.27),

$$p_{bd} = \frac{0.69}{1 + \varepsilon_{yd} / \varepsilon'_{ud}} \frac{f'_{cd}}{f_{yd}} \qquad \textbf{(2.29)}$$

2.1.4 Stresses in RC Beams in Serviceability Limit State

(1) Doubly reinforced rectangular cross-section

Within *the serviceability limit state*, since stresses in concrete and steel are small, they are considered to be elastic. Eqs. (2.9)-(2.11) can be simplified as follows:

$$N'c = \frac{\varepsilon'_{cc}\ E_c\ b\,x}{2} \tag{2.30}$$

$$N_s = \frac{d - x}{x}\varepsilon'_{cc}\ E_s\ A_s \tag{2.31}$$

$$N'sc = \frac{x - d'}{x}\varepsilon'_{cc}\ E_s\ A'_s \tag{2.32}$$

$$x = \frac{n\left(A_s + A'_s\right)}{b}\left(-1 + \sqrt{1 + \frac{2b\left(d\,A_s + d'\,A'_s\right)}{n\left(A_s + A'_s\right)^2}}\right) \tag{2.33}$$

$$\overline{y} = x/3 \tag{2.34}$$

$$\sigma'_c = \frac{M}{\dfrac{bx\left(d - \overline{y}\right)}{2} + \dfrac{n\,A'_s\left(x - d'\right)\left(d - d'\right)}{x}} \tag{2.35}$$

$$\sigma_s = \frac{d - x}{x}n\,\sigma'_c \tag{2.36}$$

$$\sigma'_{sc} = \frac{x - d'}{x}n\,\sigma'_c \tag{2.37}$$

(2) Singly reinforced rectangular cross-section

Putting A'_s =0, then the following equations are obtained.

$$x = \frac{n A_s}{b}\left(-1+\sqrt{1+\frac{2bd}{n A_s}}\right) \tag{2.38}$$

$$\sigma_s = \frac{M}{A_s\left(d-x/3\right)} \tag{2.39}$$

$$\sigma' c = \frac{2M}{b\,x\left(d-x/3\right)} \tag{2.40}$$