2 Time Series model

2.1 Expression

We consider a system represented by a state space model as follows.

$$\left\{ \begin{array}{rcllll} \dot{x}(t) & = & Ax(t) & + & Bu(t) & , \quad x(0) = x_0 \\ y(t) & = & Cx(t) & & & \end{array} \right. \tag{1}$$

We can obtain the solution of (1) as (2), and the transfer function of the system is represented by (3), where s and p means Laplace operator and differential operator ($= \frac{d}{dt}$) respectively.

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{2}$$

$$Y(s) = G(s)U(s) \quad or \quad y(t) = G(p)u(t) \tag{3}$$

Here, we assume system noise $v(t)$ and measurement noise $w(t)$.

$$\left\{ \begin{array}{rcllll} \dot{x}(t) & = & Ax(t) & + & Bu(t) & + \quad Dv(t) \\ y(t) & = & Cx(t) & + & w(t) & \end{array} \right. \tag{4}$$

We assume $v(t)$ and $w(t)$ are independent white noises that have following properties.

$$E[v(t)] = E[w(t)] = 0$$

$$E[v(t)w^T(t)] = 0$$

$$E[v(t)v^T(t+\tau)] = Q\delta(\tau)$$

$$E[w(t)w^T(t+\tau)] = R\delta(\tau)$$

$E[\cdot]$ represents an expectation, and $\delta(t)$ means Dirac's delta function. Q and R denote simmetory positive definite matrix and positive semidefinite matrix, respectively.

2.2 Forward and Backward Shift Operators

Here, we consider a discretized system whose sampling time is T.

$$\left\{ \begin{array}{ccccccc} x(k+1) & = & F(T) & x(k) & + & H(T) & u(k) \\ y(k) & = & C(T) & x(k) & + & D(T) & u(k) \end{array} \right. \tag{5}$$

, where

$$F(T) = e^{AT}, H(T) = \left(\int_0^T e^{A\lambda} d\lambda \right) B.$$

If we assume $T << 1$,

$$F(T) \simeq I + AT, H(T) \simeq BT.$$

Here, we introduce forwardand backward operators. Forward and backward operators q and q^{-1} are defined as follows.

$$\begin{aligned} qy(k) &= y(k+1) && (6) \\ q^{-1}y(k) &= \begin{cases} y(k-1) & (k \geq 1) \\ 0 & (k = 0) \end{cases} && (7) \end{aligned}$$

$$\begin{aligned} q^{i}y(k) &= y(k+i) && (8) \\ q^{-1}y(k) &= \begin{cases} y(k-i) & (k \geq i) \\ 0 & (0 \leq k < i) \end{cases} && (9) \end{aligned}$$

Then we consider stochastic noise-added model represented by following euqation.

$$\begin{aligned} y(k) &= \frac{B(q^{-1})}{A(q^{-1})}u(k) + n(k) && (10) \\ A(q^{-1}) &= 1 + a_1 q^{-1} + \cdots + a_n q^{-n} \\ B(q^{-1}) &= b_0 + b_1 q^{-1} + \cdots + b_m q^{-m} \end{aligned}$$

$$n(k) = H(q^{-1})m(k) \tag{11}$$

, where $m(k)$ represents white noise. Then $H(q^{-1})$ is called as shaping filter.

$$\begin{aligned} H(q^{-1}) &= \frac{D(q^{-1})}{C(q^{-1})} && (12) \\ C(q^{-1}) &= 1 + c_1 q^{-1} + \cdots + c_p q^{-p} \\ D(q^{-1}) &= 1 + d_1 q^{-1} + \cdots + d_r q^{-r} \end{aligned}$$

Therefore, output $y(k)$ is represented by following equation. This system is called "time series model".

$$y(k) = \frac{B(q^{-1})}{A(q^{-1})}u(k) + \frac{D(q^{-1})}{C(q^{-1})}m(k) \tag{13}$$

2.3 ARMA model

When $D(q^{-1}) = 1$, $n(k)$ becomes

$$n(k) = -c_1 n(k-1) - c_2 n(k-2) \cdots - c_p n(k-p) + m(k). \tag{14}$$

It is called AR model (Auto-Regressive model).

When $C(q^{1}) = 1$, $n(k)$ becomes

$$n(k) = m(k) + d_1 m(k-1) + \cdots + d_r m(k-r). \tag{15}$$

It is called MA model (Moving Average model).

Generally, $n(k)$ is

$$n(k) = -c_1 n(k-1) - c_2 n(k-2) \cdots - c_p n(k-p) + m(k) + d_1 m(k-1) + \cdots + d_r m(k-r). \tag{16}$$

It is called ARMA model (Auto-Regressive Moving Average Model).

2.4 Linear Diophantine equation

We consider following system, where $A(p)$ and $B(p)$ are coprime polynomials.

$$\begin{aligned} y(t) &= \frac{B(p)}{A(p)} u(t) \quad (17) \\ A(p) &= p^n + a_{n-1}p^{n-1} + \cdots + a_1 p + a_0 \\ B(p) &= b_m p^m + \cdots + b_1 p + b_0 \end{aligned}$$

When $Q(p)$ (order n) and $D(p)$ (order $n-m$) are monic stability polynomial, we can find a uniq pair of $R(p)$ and $H(p)$ that satisfy (18).

$$R(p)A(p) + H(p)B(p) = Q(p)(b_m A(p) - D(p)B(p)) \tag{18}$$

, where

$$\begin{aligned} R(p) &= r_{n-1}p^{n-1} + \cdots + r_1 p + r_0 \\ H(p) &= h_{n-1}p^{n-1} + \cdots + h_1 p + h_0. \end{aligned}$$

Equation (18) is called Diophantine equation.

By multiplying $B^{-1}(p)y(t)$ to (18) and introducing (17), we obtain (19).

$$D(p)y(t) = b_m u(t) - \frac{R(p)}{Q(p)} u(t) - \frac{H(p)}{Q(p)} y(t) \tag{19}$$

This equation gives non-minimal realization of the system (17). Figure 1 illustrates the block diagram of a non-minimal realization.

Arrange equation (18) as

$$(b_m Q(p) - R(p))\, A(p) = (H(p) + Q(p)D(p))\, B(p), \tag{20}$$

then we set $E(p)$ and $F(p)$ as

$$\begin{aligned} E(p)B(p) &= b_m Q(p) - R(p) \quad (21) \\ F(p) &= -H(p), \quad (22) \end{aligned}$$

equation (20 becomes

$$E(p)A(p)B(p) = (Q(p)D(p) - F(p))\, B(p). \tag{23}$$

Therefore, we obtain Egardt's identity (24).

$$Q(p)D(p) = A(p)E(p) + F(p) \tag{24}$$

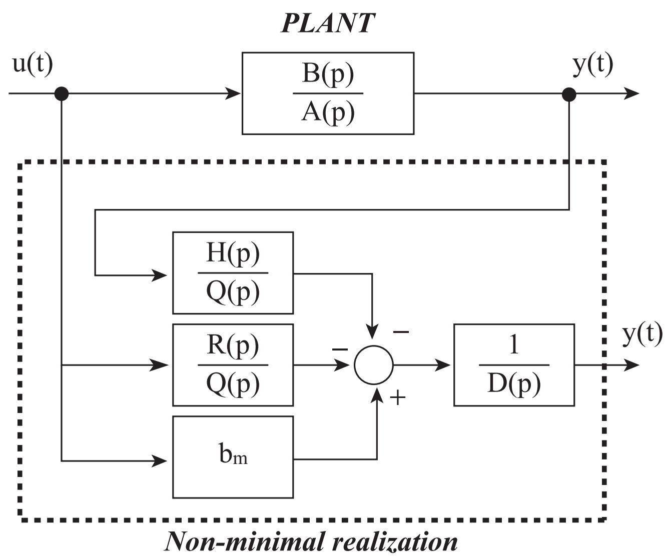

Figure 1: Block diagram of a non-minimal realization