TokyoTech (Tokyo Institute of Technology), HMA (History of Mathematics and Astronomy)

Lecture note 7: (2019)

(Mathematics and astronomy in the Islamic World and Medieval Europe)

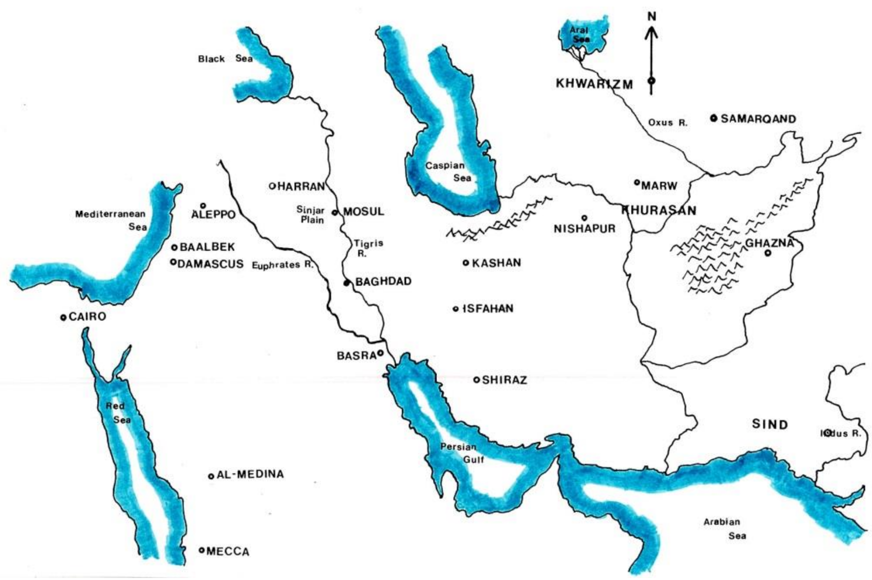
Lecturer: Yukio Ôhashi (大橋由紀夫)

3-5-26, Hiroo, Shibuya-ku, Tokyo, 150-0012, JAPAN.

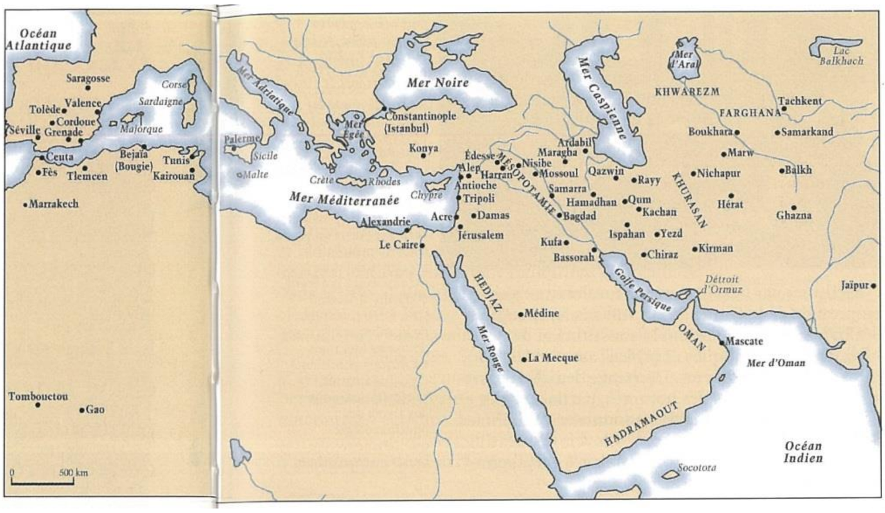
(日本、150-0012、東京都渋谷区広尾 3-5-26)

E-mail: yukio-ohashi@chorus.ocn.ne.jp

Mathematics and astronomy in the Islamic World



(From Berggren (1986))



(From Jacquart (2005) (in French))

(I) Astronomy and daily life of Muslims

Basically, when new crescent moon is observed in the western sky in the evening, a new Islamic lunar-month begins from the sunset. So, the observation of the moon is very important. One year consists of 12 lunar months regardless seasons.

There are five prayers called "ṣalāt" daily, and time keeping is very important for muslims. So, several sundials and water clocks were made.

There is an interesting record of the prayer time in the East India (now Indonesia) in the early 20th century (Izutsu, Toshihiko (井筒俊彦), 1942: *Higashi-Indo ni-okeru Kaikyō Hōsei* (東印度に於ける回教法制, Islamic Law in East India, in Japanese), Tokyo, Tōa-kenkyūjo (東亜研究所), (also reissued in his *Arabia Tetsugaku* (アラビア哲学, Arabic Philosophy, in Japanese), Tokyo, Keio University Press (慶應義塾大学出版会), 2011), pp.29-30), as follows.

(1) ẓuhr (after the time when the sun is going to move from its highest point and before the shadow becomes the same length (or double length) as the height of itself): Starting from 12 noon in East India (while starting shortly after 12 AM in Mecca).

(2) ʿaṣr (in the afternoon after ẓuhr and before the sun is going to set): Done around half past 4 PM in East India (while done about 3 hours after ẓuhr in Mecca).

(3) maghrib (after complete sunset within evening twilight): Done around 6 PM in East India (while done right after sunset in Mecca).

(4) ʿishā (at night after evening twilight and before morning twilight (before sleeping)): Done around half past 8 PM in East India (while done about two hours after maghrib in Mecca).

(5) ṣubḥ (after ishā and before sunrise): Done between 4 AM and 5 AM in East India (while done about one and a half hour before sunrise in Mecca).

The direction of the Kaʿba in Mecca is called "Qibla". The prayers should be done towards this direction. So, mathematical geography developed in the Islamic World.

It may also mentioned here that arithmetic and algebra were used for inheritance in the Islamic World.

(II) Mathematics in the Islamic World

(A) Numerals and arithmetic

The Indian system of numerals (decimal place value system with a symbol of zero) was introduced into the Islamic World around the 8th century or so. The following is the modern Arabic numerals.

Modern Arabic numerals:

٠	١	٢	٣	٤	٥	٦	٧	٨	٩
0	1	2	3	4	5	6	7	8	9

Al-Uqlīdisī wrote The Book of Chapters on Hindu Arithmetic (952-953 CE), and used decimal fractions (probably for the first time). For example, in the case of 163.85, a short vertical mark was placed above "3". (See below.)

§ 3. The Discovery of Decimal Fractions

Today we use not sexagesimal but decimal fractions to represent the fraction remaining after a division, and it now appears these were a contribution of the Islamic world. Evidence for this claim is contained in *The Book of Chapters on Hindu Arithmetic*, written in Damascus in the years A.D. 952–953 by Abu'l-Ḥasan al-Uqlīdisī. The name "al-Uqlīdisī" indicates that the author earned his living copying manuscripts of Euclid ("Uqlīdis" in Arabic), but beyond this we know nothing of the life of a man who seems to have been the first to use decimal fractions, complete with the decimal point, and therefore the first to write numbers as we do. Since al-Uqlīdisī specifically states in the preface to his book that he has taken great pains to include the best methods of all previous writers on the subject, it is hard to be sure that decimal fractions were his own discovery, but their complete absence in Indian sources makes it fairly certain that they were a discovery of Islamic scientists.

Al-Uqlīdisī is also proud of the fact that he has collected ways of performing on paper, with ink, algorithms usually performed by arithmeticians on the dust board, and in his *Book of Chapters* he gives the following reasons for abandoning the dust board in favor of pen and paper.

> Many a man hates to show the dust board in his hands when he needs to use this art of calculation (Hindu arithmetic) for fear of misunderstanding from those present who see it in his hands. It is unbecoming him since it is seen in the hands of the good-for-nothings earning their living by astrology in the streets.

It seems that the street astrologers could be recognized by their use of the dust board, and al-Uqlīdisī urges the use of pen and paper to escape being taken for a mendicant fortune teller.

Al-Uqlīdisī's text contains four parts, of which the first two deal with the elementary and advanced parts of Hindu arithmetic, but it is in the second part where decimal fractions first appear. This is in the section on doubling and halving numbers, where he introduces them as one of the three ways of halving an odd number. The first way is the one described by Kūshyār who, to halve 5625, considered it as degrees or *dirhams* and wrote the result as in Fig. 2.3, where the lower 30 could be interpreted as *fulūs* or minutes. The second way is one al-Uqlīdisī calls numerical, and describes as follows:

> ... halving one in any place is five (in the place) before it, and this necessitates that when we halve an odd number we make half of the unit five before it and we put over the units' place a mark by which we distinguish the place. So

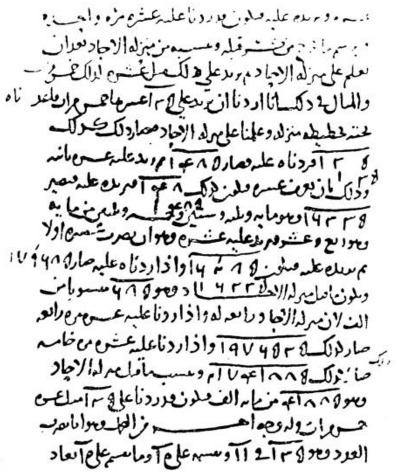


Fig. 2.6

> the value of the unit's place is tens to that before it. Now the five may be halved just as whole numbers are halved, and the value of the units' place in the second halving becomes hundreds and this may continue indefinitely.

When al-Uqlīdisī writes of places in a numeral being "before" other places he is referring to the direction of Arabic writing, which is from right to left. Thus in 175 the 5 would be before the 7. As an example of what he has explained al-Uqlīdisī gives the results of halving 19 five times as $\overset{|}{0}59375$, where, he says, "the place of the units is hundred-thousands to what is in front of it". Figure 2.6 shows the Arabic text of al-Uqlīdisī's work and the use of the decimal point (in the form of the short vertical mark pointing out the unit's place). If the reader uses the forms of the numerals given earlier, he will have no trouble identifying the various numerals in that figure.

(From Berggren (1986), pp.36 – 37)

(B) Algebra

The famous mathematician al-Khwārizmī (ca.780 – ca.850 CE) wrote *The Condensed Book on the Calculation of al-Jabr and al-Muqābala*. This is a widely circulated work of algebra. The word "al-jabr" means the operation of restoring a quantity subtracted from one side of the equation to the other side to make it positive, and is the origin of the modern word "algebra". The word "al-muqābala" means replacing two terms of the same type, but on different sides of an equation, by their difference on the side of the larger. (See below.) The author's name "al-Khwārizmī" was transliterated as "Algorismus" in Latin, and this is the origin of the modern word "algorithm".

§3. Al-Khwārizmī's Algebra

The Name "Algebra"

Out of this dual heritage of solutions to problems asking for the discovery of numerical and geometrical unknowns Islamic civilization created and named a science—algebra. The word itself comes from the Arabic word "al-jabr", which appears in the title of many Arabic works as part of the phrase "al-jabr wa al-muqābala". One meaning of "al-jabr" is "setting back in its place" or "restoring", and the ninth century algebraist al-Khwārizmī, although he is not always consistent, uses the term to denote the operation of restoring a quantity subtracted from one side of the equation to the other side to make it positive. Thus replacing $5x + 1 = 2 - 3x$ by $8x + 1 = 2$ would be an instance of "al-jabr". The word "wa" just means "and", and it joins "al-jabr" with the word "al-muqābala", which means in this context replacing two terms of the same type, but on different sides of an equation, by their difference on the side of the larger. Thus, replacing $8x + 1 = 2$ by $8x = 1$ would be an instance of "al-muqābala".

Clearly, with the two operations any algebraic equation can be reduced to one in which a sum of positive terms on one side is equal either to a sum of positive terms involving different powers of x on the other, or to zero. In particular, any quadratic equation with a positive root can be reduced to one of three standard forms:

$$px^2 = qx + r, \quad px^2 + r = qx, \quad \text{or} \quad px^2 + qx = r, \quad \text{with } p, q, r \text{ all positive,}$$

a condition that runs through the whole medieval period in Islamic mathematics. We shall meet it again in the work of ʿUmar al-Khayyāmī, and it is the rule in Western mathematics as well through the early sixteenth century. Thus the science of "al-jabr wa al-muqābala" was, at its beginning, the science of transforming equations involving one or more unknowns into one of the above standard forms and then solving this form.

Basic Ideas in Al-Khwārizmī's Algebra

One of the earliest writers on algebra was Muḥammad b. Mūsā al-Khwārizmī, whose treatise on Hindu reckoning we referred to in Chapter 2. His work on algebra, *The Condensed Book on the Calculation of al-Jabr wa al-Muqābala*, enjoyed wide circulation not only in the Islamic world but in the Latin West as well.

According to al-Khwārizmī there are three kinds of quantities: *simple numbers* like 2, 13 and 101, then *root*, which is the unknown, x, that is to be found in a particular problem, and *wealth*, the square of the root, called in Arabic *māl*. (A possible advantage of thinking of the square term as representing wealth is that al-Khwārizmī can then interpret the number term as *dirhams*, a local unit of currency. Another word used for "root" by many writers is "thing". In these terms al-Khwārizmī could list the six basic types of equations as:

(1) Roots equal numbers ($nx = m$).
(2) Māl equal roots ($x^2 = nx$).
(3) Māl equal numbers ($x^2 = m$).
(4) Numbers and māl equal roots ($m + x^2 = nx$).
(5) Numbers equal roots and māl ($m = nx + x^2$).
(6) Māl equals numbers and roots ($x^2 = m + nx$).

All equations involving only the three basic quantities and having a positive solution could be reduced to one of these three types, the only ones with which al-Khwārizmī concerns himself.

Al-Khwārizmī's Discussion of $x^2 + 21 = 10x$

In following al-Khwārizmī's discussion of type (4) above we shall use modern notation to render his verbal account. He discusses this type in terms of the specific example $x^2 + 21 = 10x$, which he describes as "māl and 21 equals 10 roots", as follows (translation adapted from F. Rosen):

> Halve the number of roots. It is 5. Multiply this by itself and the product is 25. Subtract from this the 21 added to the square (term) and the remainder is 4. Extract its square root, 2, and subtract this from half the number of roots, 5. There remains 3. This is the root you wanted, whose square is 9. Alternately, you may add the square root to half the number of roots and the sum is 7. This is (then) the root you wanted and the square is 49.

Notice that al-Khwārizmī's first procedure is simply a verbal description of our rule

$$\frac{10}{2} - \sqrt{\left(\frac{10}{2}\right)^2 - 21},$$

and his second procedure describes the calculation of $5 + \sqrt{5^2 - 21}$, but since all quantities are named in terms of their role in the problem whenever they appear (For example, "5" is called "the number of roots"), his description of the solution is quite as general, if not so compact, as our

$$\frac{n}{2} \pm \sqrt{\left(\frac{n}{2}\right)^2 - m}.$$

In fact, al-Khwārizmī's generality is reflected in the remarks that continue

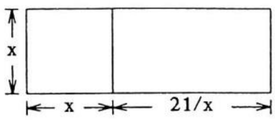


Fig. 4.2

those quoted above:

> When you meet an instance which refers you to this case, try its solution by addition, and if that does not work subtraction will. In this case, both addition and subtraction can be used, which will not serve in any other of the three cases where the number of roots is to be halved.
>
> Know also that when, in a problem leading to this case, you have multiplied half the number of roots by itself, if the product is less than the number of dirhams added to māl, then the case is impossible. On the other hand, if the product is equal to the *dirhams* themselves, then the root is half the number of roots.

In the first of the above paragraphs al-Khwārizmī recognizes that the case we are dealing with is the only one where there can be two positive roots. In the second paragraph he remarks that there is no solution when what we call the discriminant is less than zero and he says that when $(n/2)^2 = m$ the only solution is $n/2$. Finally, he remarks that in the case $px^2 + m = nx$ it is necessary to divide everything by p to obtain $x^2 + (m/p) = (n/p)x$, which can be solved by the previous method. This shows, by the way, that his coefficients are not restricted to whole numbers.

What distinguishes al-Khwārizmī and his successors from earlier writers on problems of the above sort is that, following the procedures for obtaining the numerical solutions, he gives proofs of the validity of these same procedures, proofs that interpret $x^2 + 21$, for example, as a rectangle consisting of a square (x^2) joined to a rectangle of sides x and $21/x$ (Fig. 4.2).

(From Berggren (1986), pp.102 - 104)

After al-Khwārizmī, algebra further developed in the Islamic World.

(C) Geometry

The Greek works of geometry, the *Elements* of Euclid, Archimedes' works, the *Conics* of Apollonios, etc., were translated into Arabic, and were studied and developed in the Islamic world.

The Greek original text from vol.5 through vol.7 of Apollonios' Conics was lost, and only its Arabic translation is extant.

The conic section is very important in astronomy. For example, the loci of the tip of gnomon shadow are hyperbolas (or a straight line on equinoctial days), and appear in horizontal sundials. (See below.)

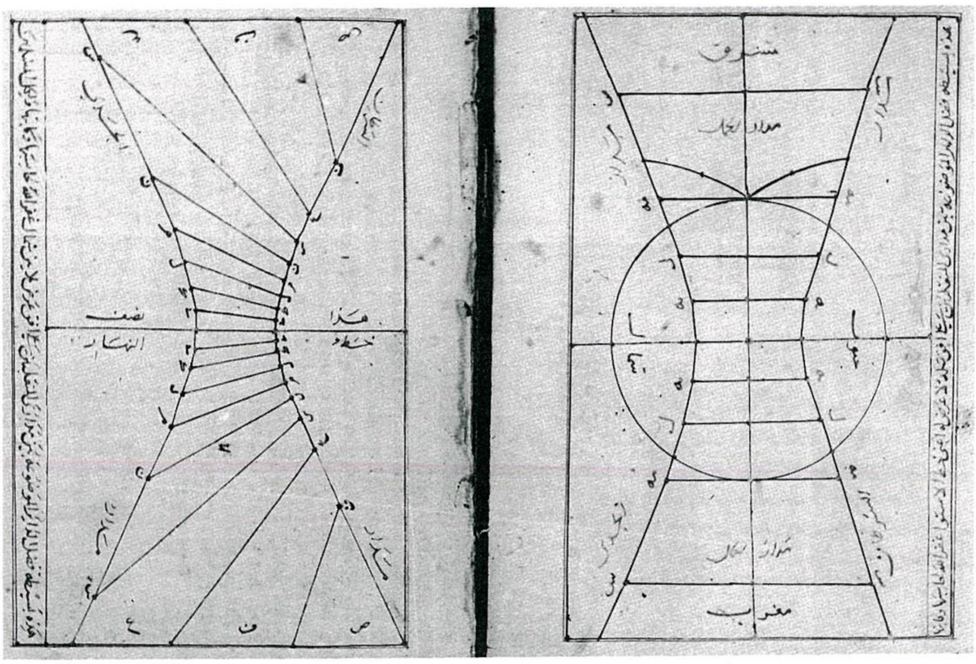


Plate 3.1. Diagrams of a sundial for the latitude of Cairo and a universal sundial in a fifteenth century Egyptian treatise on sundial theory by the *muwaqqit* al-Karādīsī. (A *muwaqqit* is a person who determines the times of prayer in Islam.) The two hyperbolas represent the paths of the shadow of the pointer (*miqyās*) at the two solstices. (Taken from MS Cairo Dār al-kutub riyāḍa 892. Courtesy of the Egyptian National Library.)

(From Berggren (1986), p/86)

(D) Plane trigonometry

In ancient Greece, the relationship between chord and arc of a circle was used for astronomical calculations. In India, the relationship between a half chord and its corresponding arc was used for astronomical calculations. It is the origin of trigonometry.

The half chord is called "jyā" or "jīva" in Sanskrit. It was transliterated as "jayb" in Arabic, and then translated as "sinus" in Latin. This is the origin of the word "sine".

The trigonometry was well developed in the Islamic World. (See the following example.)

§4. Naṣīr al-Dīn's Proof of the Sine Law

Naṣīr al-Dīn introduces the Sine Law for plane triangles to provide a basic tool for solving them, and in this section we shall see how he proves the law and how he applies it to find unknown parts of triangles from known ones.

The Sine Law. If ABC is any triangle then c/b = Sin C/Sin B.

Figure 5.11 illustrates the case when one of the angles B or C is obtuse, and Fig. 5.12 the case when neither B nor C is obtuse, so that one of them is acute. In either case prolong CA to D and BA to T so each is 60 units long and, with centers B, C, draw the circular arcs TH and DE. If we now drop perpendiculars TK and DF to the base BC, extended if necessary, then TK = Sin B and DF = Sin C. (In the case of Fig. 5.11 both of these statements are obvious, but in the case of Fig. 5.10 the reader must remember that

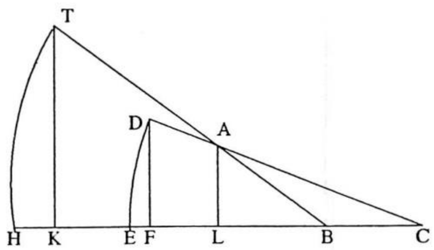


Fig. 5.11

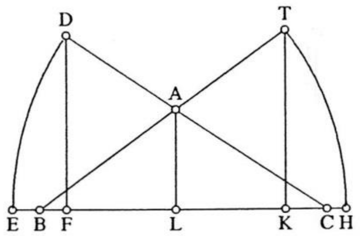


Fig. 5.12

Sin($\sphericalangle$B) = Sin(180° − $\sphericalangle$B).) Now draw AL perpendicular to BC. Since triangles ABL, TBK are similar AB/AL = TB/TK, and since triangles ACL and DCF are similar, AL/AC = DF/DC but DC = 60 = TB, so, if we multiply the left and right sides, respectively, of these two proportions, we obtain the proportion AB/AC = DF/TK. Therefore c/b = Sin C/Sin B, and this proves the Sine Theorem.

Since Naṣīr al-Dīn's sine function is simply 60 times the modern one, the above theorem holds for the modern function as well. We may re-write the theorem as $c/\sin C = b/\sin B = a/\sin A$, a form it is often given in today, and it may be most easily remembered as the statement that in a given triangle the ratio of any side to the sine of the opposite angle is constant.

(From Berggren (1986), pp.138 – 139)

(E) Spherical trigonometry

In ancient India, only plane trigonometry was used for spherical astronomy. In the Islamic World, spherical astronomy was started to be used.

The following is an application of the spherical trigonometry to the determination of Qibla as an example.

Finding the Direction of Mecca

Until now we have always looked to the heavens for inspiration or context for spherical trigonometry. Ironically, it was a religious concern that diverted the eyes of trigonometers downward to the Earth. The practice of Islam requires the faithful to perform five tasks, known as the "Five Pillars." Astronomers cannot help much with three of them (profession of faith, alms, and the *hajj*—the pilgrimage to Mecca). The other two—fasting during daylight hours during the month of Ramadan, and the five daily prayers—require technical assistance if they are to be obeyed strictly. Consider the monthly fast. The Arabic calendar is lunar, so each month begins when the lunar crescent reappears from behind the Sun after New Moon. Miss the crescent on a particular day, and you may end up violating the fasting requirement unawares. Muslim scientists worked hard attempting to predict the first appearance of the lunar crescent, with varying degrees of success.

But scientists were really able to justify their incomes with the times of prayer, which are regulated by the position of the Sun in the sky. When the moment occurs, worshippers are enjoined to face the Kaʿba,

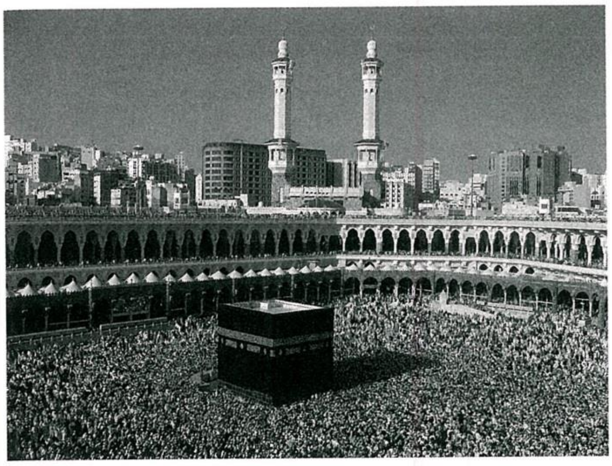


Figure 4.5. The Kaʿba, the most sacred site in Islam and destination of the *hajj* (pilgrimage). © iStockphoto.com / Aidar Ayazbayev.

the most sacred site of Islam. The Kaʿba, a cubical building (figure 4.5) that houses the Black Stone, is the destination of the pilgrimage that Muslims are asked to embark upon once in their lives. The direction of the Kaʿba—the *qibla*—serves several purposes besides the daily prayers, including determining the direction in which Muslims should face when they are buried. Modern technology is challenging the meaning of the qibla; a conference met in 2006 to decide the direction of prayer while in space. In practice, however, the injunction to face Mecca has not been taken as seriously as the scientists might have liked. Legal scholars often carried more weight than scientists, which may account for the wide variety of mosques' actual orientations.

On the face of it the qibla does not seem difficult to calculate. Since the positions of both Mecca and the worshipper are given, we know the local latitude φ_L, $\varphi_M = 21.67°$, and the difference in longitude. So we would seem to have a right triangle on the Earth's surface with values for the lengths of the two sides adjacent to the right angle (figure 4.6). Unfortunately, the bottom side representing the difference in longitude

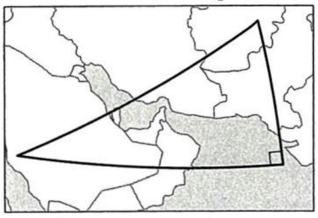


Figure 4.6. The qibla problem.

is not a great circle arc, but rather an arc of a circle of latitude. Thus the shape in figure 4.6 is not even a triangle.

The earliest solutions to the qibla problem were approximate, even as crude as assuming that figure 4.6 is actually a planar right-angled triangle. Around AD 900 precise solutions based on spherical trigonometry (originally, Menelaus's Theorem) started to appear. As one might expect, al-Bīrūnī's classic work of mathematical geography, *Determination of the Coordinates of Cities* (from which we took his measure of the circumference of the Earth), goes into the matter in some depth. He gives no less than four precise solutions. Two of them apply constructions that go beneath the surface of the sphere, and so might be influenced by Indian methods. The other two probably use the latest spherical trigonometric methods of al-Bīrūnī's time, such as the Rule of Four Quantities and the Law of Sines. We're not quite sure of this assertion because al-Bīrūnī simply states the relations needed to solve the problem, not telling us precisely what theorems he used to get there.

All four of al-Bīrūnī's methods determine the qibla for the city of Ghazna, now Ghazni in eastern Afghanistan. In his time Ghazna was one of the most important cities in the world: the capital of the Ghaznavid Empire, a Persian dynasty that lasted two centuries and at its peak incorporated most of modern-day Iran, Afghanistan, Pakistan, and several surrounding countries. To give the reader a taste of ancient and medieval diagrams, we have reproduced al-Bīrūnī's diagram (with a couple of trivial modifications) in figure 4.7. Although it looks two-dimensional, appearances are deceiving. Imagine that you are looking directly down on Ghazna from above the celestial sphere. All the curves on the figure (even the two straight lines) are great circle arcs on the celestial sphere seen from above, so G is the zenith directly above Ghazna. The line connecting north and south through G, actually a great circle called the *meridian* of Ghazna, passes through the north pole P; the outer circle is Ghazna's horizon. M is the point on the celestial sphere that an observer at Mecca would perceive as the zenith. $\widehat{WM}$ connects the west point on the horizon to M, and extends to A on the meridian. $\widehat{PMB}$ is the meridian of Mecca.

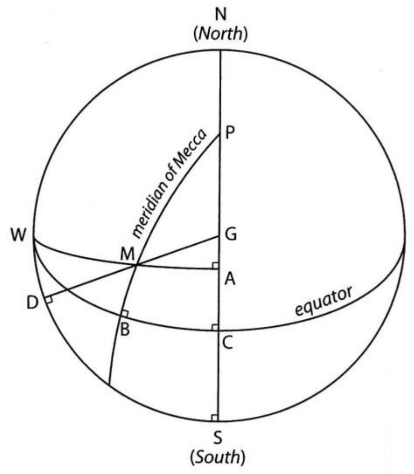


Figure 4.7. Graphic of al-Bīrūnī's determination of the qibla.

→Al-Bīrūnī's geographical coordinates for Ghazna and Mecca were $\varphi_L = 33.58°$, $\varphi_M = 21.67°$, and a longitude difference of $\Delta\lambda = 27.37°$. Now φ_L is the altitude $\widehat{NP}$ of the North Pole, the northernmost segment of Ghazna's meridian; but both $\widehat{NG}$ and $\widehat{PC}$ are 90°, so $\widehat{GC} = \varphi_L = 33.58°$. So the arc from the worshipper's zenith perpendicularly down to the equator is the local latitude. This fact must also apply to the zenith of Mecca, so $\widehat{MB} = \varphi_M = 21.67°$. Finally, the difference in longitude is equal to the angle at the North Pole between the two zeniths, so $\measuredangle MPG = \widehat{BC} = 27.37°$. Now that we have transferred all the data onto arcs in the diagram, we are ready to begin the actual mathematics.

We shall use nothing but the Rule of Four Quantities. Starting with configuration *CAPMB* we have →(See below.)

$$\frac{\sin\widehat{PM}}{\sin\widehat{MA}} = \frac{\sin\widehat{PB}}{\sin\widehat{BC}}, \quad \text{or} \quad \frac{\sin(90° - \varphi_M)}{\sin\widehat{MA}} = \frac{1}{\sin\Delta\lambda},$$

so $\sin\widehat{MA} = \cos\varphi_M \sin\Delta\lambda$, which gives the "modified longitude" $\widehat{MA} = 25.29°$. Our second configuration is *WMACB*, from which we get

$$\frac{\sin\widehat{WM}}{\sin\widehat{MB}} = \frac{\sin\widehat{WA}}{\sin\widehat{AC}} \quad \text{or} \quad \frac{\sin(90° - \widehat{MA})}{\sin\varphi_M} = \frac{1}{\sin\widehat{AC}},$$

so $\sin\widehat{AC} = \sin\varphi_M / \cos\widehat{MA}$, and we have the "modified latitude" $\widehat{AC} = 24.11°$. Then $\widehat{GA} = \widehat{GC} - \widehat{AC} = \varphi_L - 24.11° = 9.47°$.

With the modified longitude and latitude in hand, we turn our attention to the outer horizon circle for Ghazna, which is where the qibla resides. It will take two steps. Firstly, from *WMASD*,

$$\frac{\sin\widehat{WM}}{\sin\widehat{MD}} = \frac{\sin\widehat{WA}}{\sin\widehat{AS}} \quad \text{or} \quad \frac{\sin(90° - \widehat{MA})}{\sin\widehat{MD}} = \frac{1}{\sin(90° - \widehat{GA})},$$

so $\sin\widehat{MD} = \cos\widehat{MA}\cos\widehat{GA}$, which gives $\widehat{MD} = 63.10°$. Our final step applies the Rule of Four Quantities to figure *GMDSA*:

$$\frac{\sin\widehat{GM}}{\sin\widehat{MA}} = \frac{\sin\widehat{GD}}{\sin\widehat{DS}} \quad \text{or} \quad \frac{\sin(90° - \widehat{MD})}{\sin\widehat{MA}} = \frac{1}{\sin\widehat{DS}},$$

so $\sin\widehat{DS} = \sin\widehat{MA} / \cos\widehat{MD}$. This gives us the qibla, because $\widehat{DS} = 70.79°$ is the number of degrees west of south that we must turn to face Mecca.→

There is nothing special about Mecca in the above calculations. We could use the same reasoning to find the direction to any destination. So scientists now had a means to determine the direction from any place on the Earth's surface to any other. Granted, the calculations are not simple, but once they are automated they work quite smoothly. Nevertheless a small industry arose to generate tables of the qibla for any location within the Arabic-speaking world, so that the faithful would be spared the pain of lengthy trigonometric calculation. The best of these tables was a set composed by Shams al-Dīn al-Khalīlī, an astronomical timekeeper employed by the Umayyad mosque in Damascus. Its sixteen pages contain almost 3000 entries of the qibla for every degree of latitude and difference in longitude for all Earthly locations that mattered. The effort involved must have been Herculean.

Rule of Four Quantities:

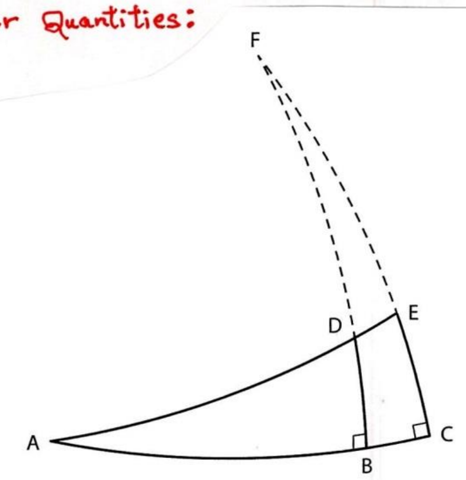


Figure 4.1. The Rule of Four Quantities and Abū Naṣr's second theorem.

~~Our second claimant is a familiar face:~~ Abū Naṣr Manṣūr ibn ʿAlī ibn ʿIrāq, al-Bīrūnī's teacher and discoverer of the polar triangle. His original work on the subject, the *Book of the Azimuth*, is preserved only by a quotation in al-Bīrūnī's *Keys to Astronomy*. In it he proposes two new theorems, both based on the same diagram (figure 4.1):

Rule of Four Quantities: $\dfrac{\sin\widehat{BD}}{\sin\widehat{CE}} = \dfrac{\sin\widehat{AD}}{\sin\widehat{AE}}$.

Abū Naṣr's Second Theorem: $\dfrac{\sin\widehat{DF}}{\sin\widehat{EF}} = \dfrac{\sin\widehat{AD}}{\sin\widehat{AB}}$.

At first it appears that these theorems are nothing more than corollaries to Menelaus, and in a mathematical sense they are.

Proof of the Rule of Four Quantities: Apply Menelaus's conjunction theorem to figure 4.1; we get $\frac{1}{\sin\widehat{CE}} = \frac{1}{\sin\widehat{BD}} \cdot \frac{\sin\widehat{AD}}{\sin\widehat{AE}}$.

(From Van Brummelen (2013), pp.66 – 71, and 59 – 60.)

(III) Historical development of mathematics and astronomy in the Islamic World

(A) The best days of the Abbasid Caliphate (around the mid-8th century – around the mid-10th century)

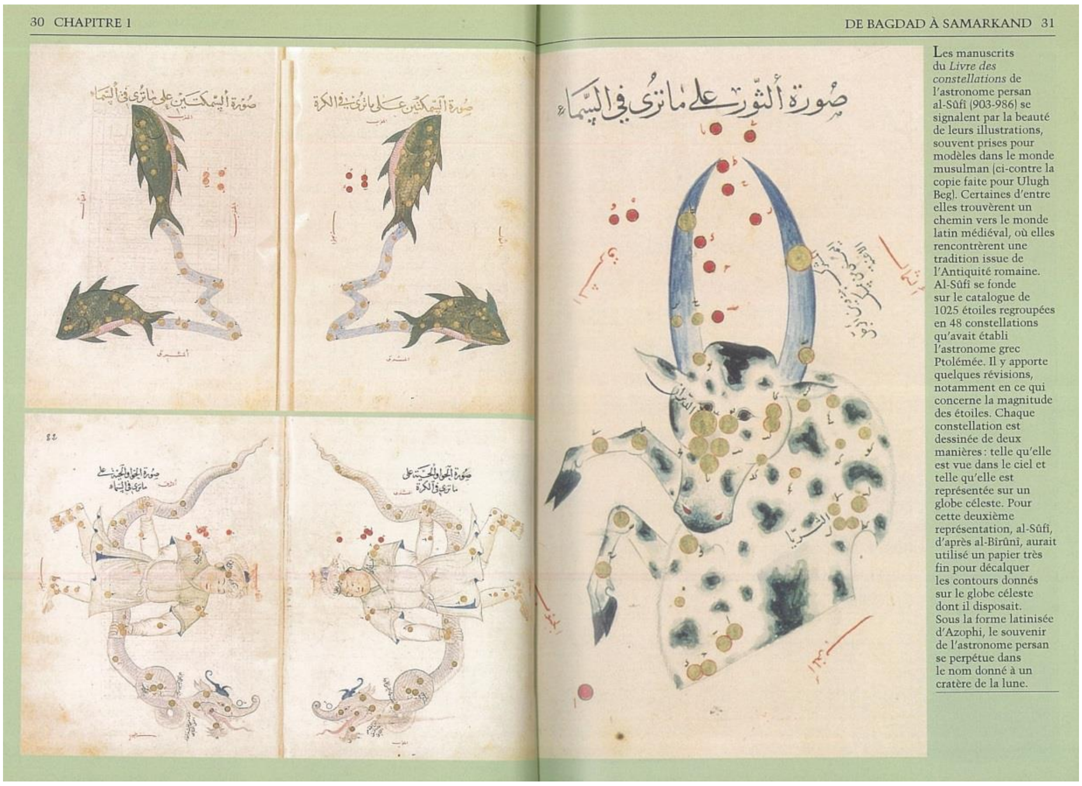
From the mid-8th century, several Greek mathematical and astronomical works were translated into Arabic. Thābit ibn Qurra (d.901 CE) translated several important works.

In this period, al-Khwārizmī (ca.780 – ca.850 CE) wrote work on Indian system of arithmetic, the famous work on algebra, etc.

Al-Battānī (ca.858 – 929 CE) made accurate observations, and developed Ptolemaic geocentric astronomy.

(B) Development in Egypt and Spain besides West Asia (around the mid-10th century – around the 12th century)

Aṣ-Ṣūfī (903 – 986 CE) wrote a work on Greek constellations with several figures. (See below.)



(From Jacquart (2005), pp.30 – 31)

Ibn Yūnus (d.1009 CE) made astronomical observations at Cairo, Egypt, and made the astronomical work *al-zīj al-Kabīr al-Ḥākimī*.

Ibn al-Haytham (ca.965 – ca.1039 CE) studied optics.

Al-Bīrūnī (973 – ca.1050 CE) wrote the astronomical work *Masʿudic Canon*, and some other astronomical works. And also, his *India* is a very important account of Indian culture.

Al-Zarqālī (1029 – 1087 CE) made astronomical observations in Toledo, Spain, and made the astronomical work *Toledan Tables*.

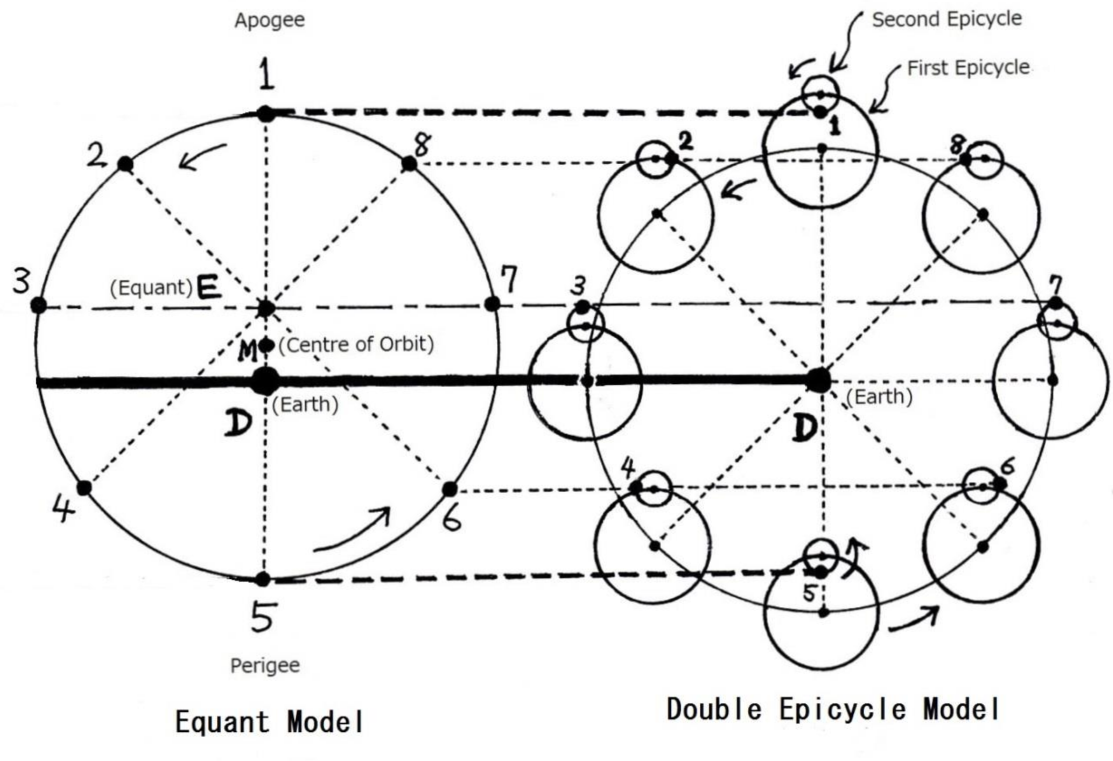
ʿOmar Khayyām joined the work to compile a Persian solar calendar called "Jalālī calendar" (adopted from 1079).

(C) Development in Central Asia besides West Asia (around the mid-13th century – around the mid-15th century)

Persian astronomer Naṣīr ad-Dīn aṭ-Ṭūsī (1201 – 1274 CE) made astronomical observations at Marāgha Observatory in Iran, and made the astronomical work *Ilkhanic Tables*.

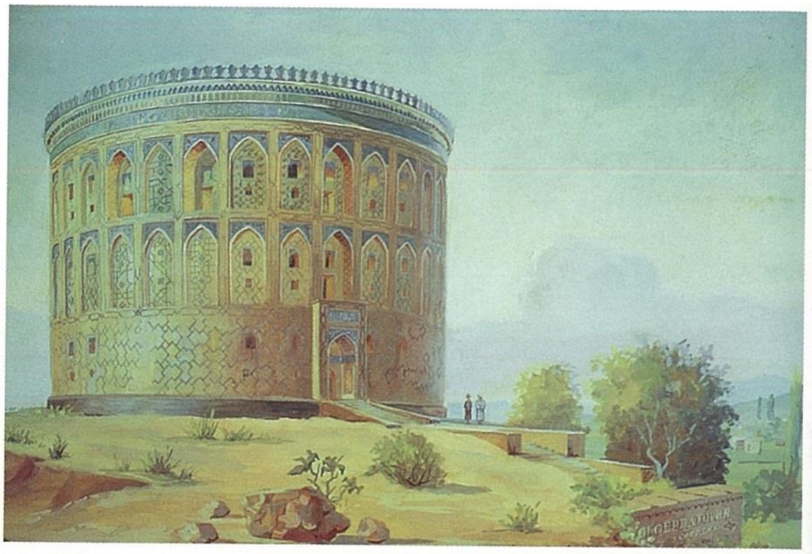
Aṭ-Ṭūsī's disciple ash-Shīrāzī (1236 – 1311 CE) also developed astronomy.

Ibn ash-Shāṭir (1306 – 1375 CE) made the double epicycle model.



Ulugh Beg (1394 – 1449 CE) made astronomical observations at his Samarqand Observatory.

The following is a conjectured picture of the Samarqand Observatory (1420 CE).

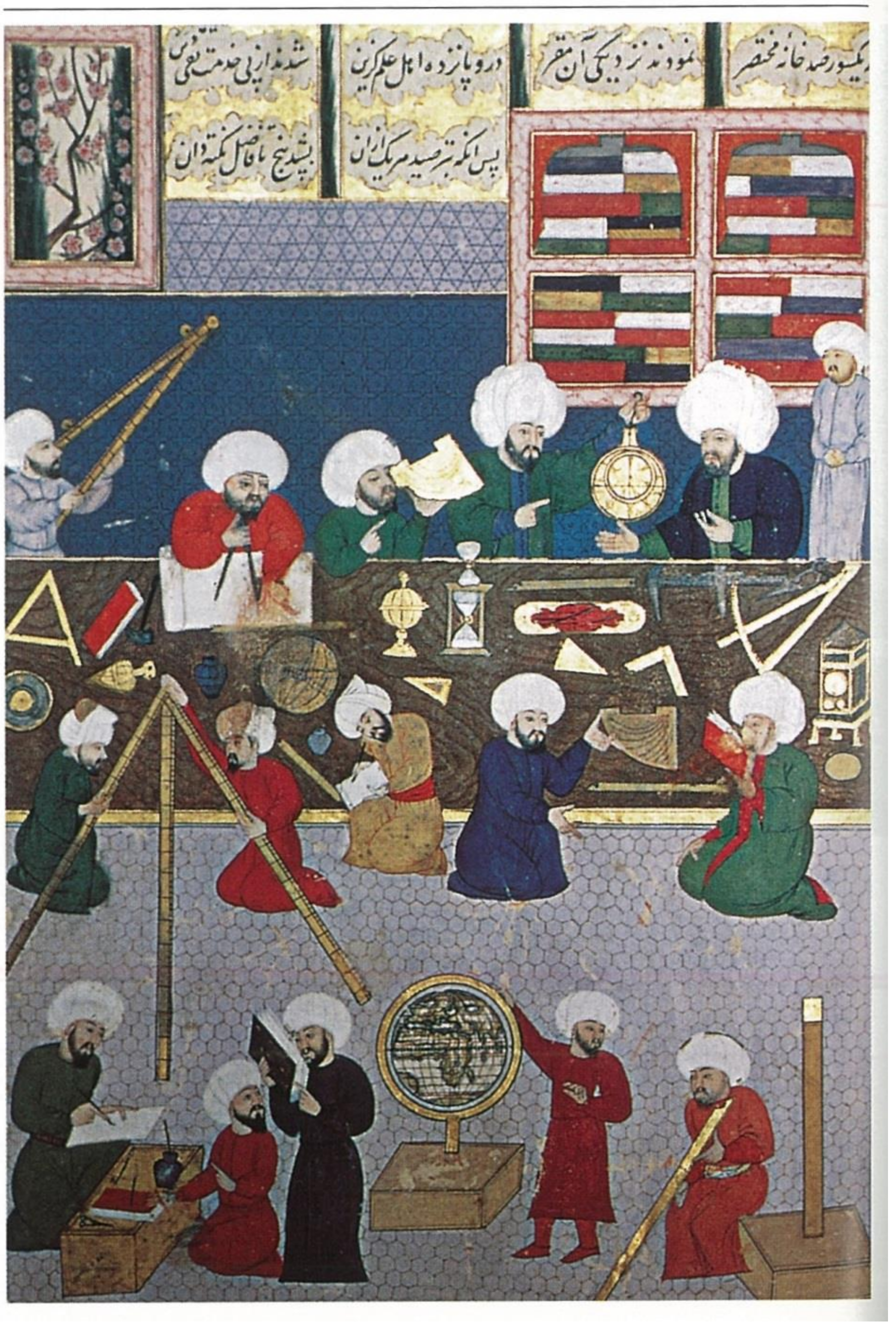


(From Jacquart (2005), p.28)

(D) Development in Turkey, Iran, India etc. (around the 16th century – the 18/19th century)

In this period, modern science developed in Europe etc., but Islamic traditional science also developed in Turkey, Iran, India etc.

The following is a picture of the Istanbul Observatory in Turkey (the 16th century).



(From Jacquart (2005), p.48)

Mathematics and astronomy in Medieval Europe

(I) Introduction of Arabic and Greek science into Medieval Europe (12th – 13th centuries)

In the 12th – 13th centuries, Islamic science was introduced to West Europe, and several Arabic texts were translated into Latin.

And also, Greek texts were introduced from Byzantine Empire, and Greek texts were translated into Latin.

(II) Establishment of universities (from the 12th century)

From the 12th century, universities were created in Europe.

In their faculty of arts, seven liberal arts were studied.

Seven liberal arts:

"trivium": grammar, rhetoric and logic,

"quadrivium": arithmetic, geometry, astronomy and acoustic.

And also, Aristotelian philosophy (natural philosophy, ethics and metaphysics) were studied.

After studying liberal arts, students could go to the faculty of theology, law or medicine.

(III) Medieval European astronomy

Sacrobosco

Johannes de Sacrobosco (the first half of the 13th century) wrote the *Sphaera*, which is a compendium of Ptolemaic astronomy. (The detail of mathematical astronomy is not described there.) This work was widely read.

Alfonso X

Alfonso X of Castile, Spain (reign 1252 – 1284) composed the *Alfonsine Tables*. It was widely used.

Nicolaus Cusanus

Nicolaus Cusanus (1401 – 1464 CE) wrote in his *De docta ignorantia* (1440) that the universe if infinite, and that there is no centre and the earth may move.

Peuerbach and Johannes Müller

Georg von Peuerbach (1423 – 1461 CE) and his disciple Johannes Müller (Latin name Regiomontanus) (1436 – 1476 CE) wrote the *Epitoma in Almagestum Ptolemaei* (1496), which is a work on mathematical astronomy

(IV) Mathematics for commerce

Fibonacci

Leonardo da Pisa (= Fibonacci) (ca.1170 – ca.1250 CE) traveled several places with his father who was a merchant, and studied mathematics. He wrote the *Liber abaci* (1202), which is a systematic compilation of mathematics at his time.

Schools of practical mathematics

From the end of 13th century to the beginning of the 16th century, there were schools of practical mathematics for commerce etc.

Luca Pacioli

Luca Pacioli (ca.1445 – 1517 CE) wrote the compilation of mathematics *Summa de Arithmetica, Geometria, Proportion et Proportionalita* (1494). It is well known for its description of bookkeeping.

(V) Mechanics (14th century)

Some people were trying to overcome Aristotelian mechanics.

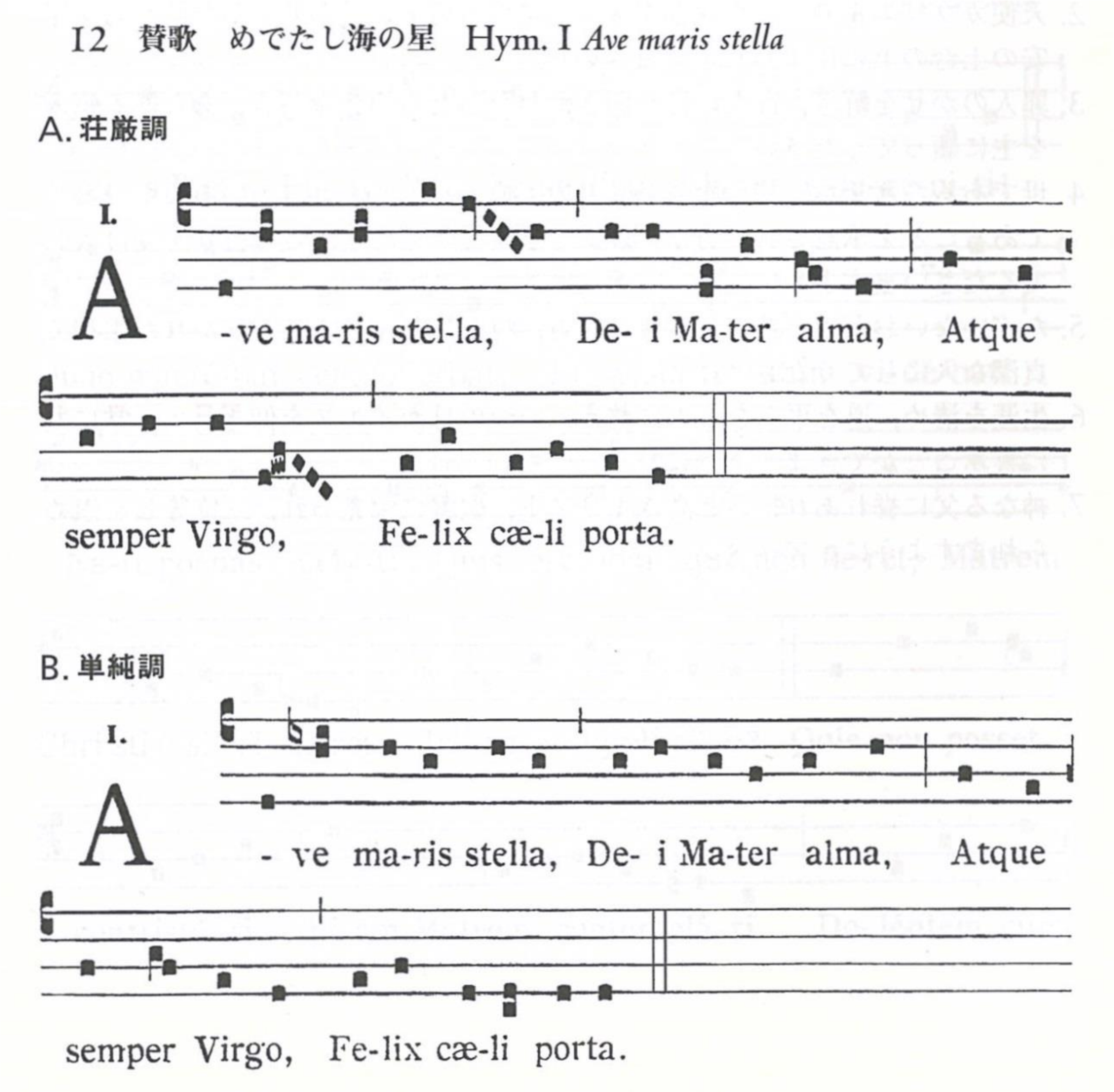
One is Thomas Bradwardine (ca.1290 – 1349 CE) of Merton College of Oxford University.

Others are Jean Buridan (ca.1300 – ca.1358 CE) and Nicole Oresme (ca.1320 – 1382 CE) of the University of Paris.

They used mathematics for the study of mechanics.

(VI) Quantification of music

The mathematical theory of musical scales was already developed since ancient Greece. In ancient Greece, music was monophony, and its rhythm need not be mathematical. The following is an example of the medieval monophonic music *Gregorian chant*.



(From Toeda, Masako: 十枝正子（編著）『グレゴリア聖歌選集』(Cantus Gregoriani Selecti)、サン パウロ, 2004, p.203.)

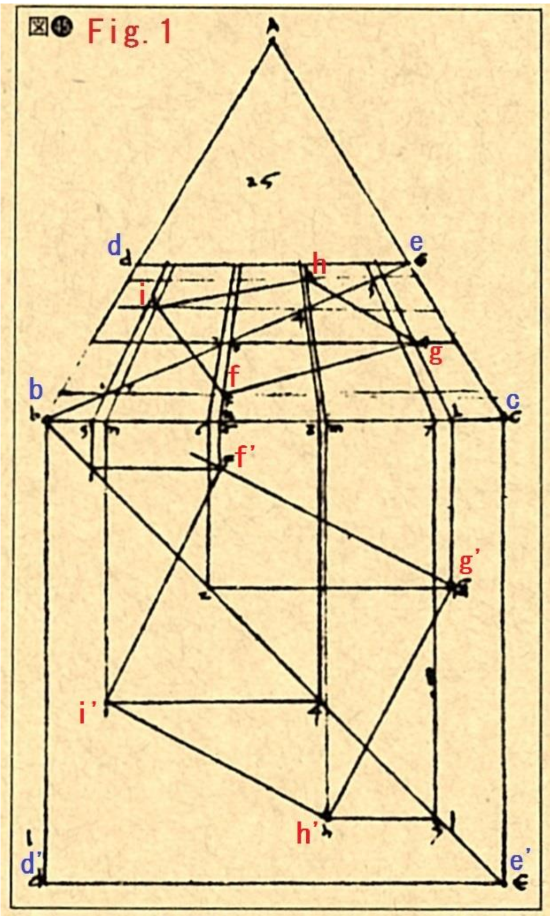
From the 9th or 10th century, polyphony was made, and the theory of rhythm was developed since the second half of the 13th century. In order to express rhythm, “black mensural notation” (the 14th century – the mid-15th century) and “white mensural notation” (the mid-15th century – the late 16th century) were used in the early period. The following is an example of the “white mensural notation” of Ave maris stella of Josquin des Prés (ca.1440 – 1521).

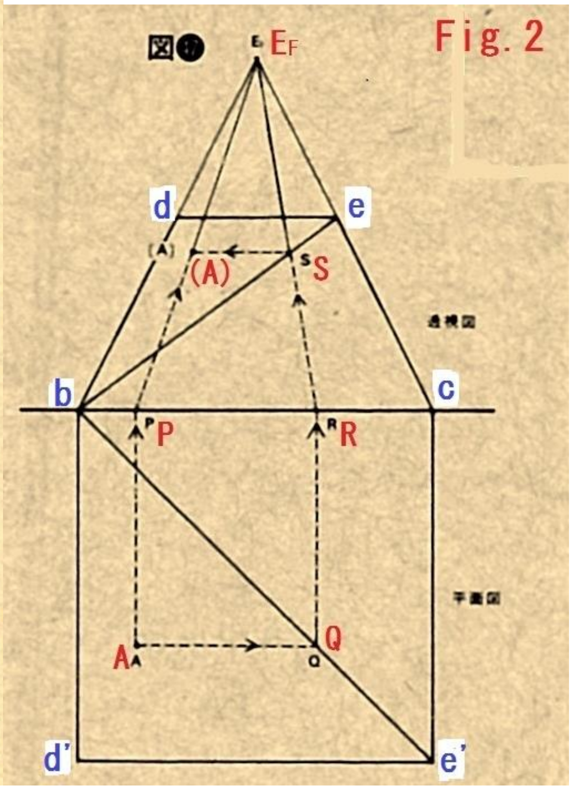
(From Minagawa, Tatsuo: 皆川達夫『楽譜の歴史』、音楽之友社, 1985, pp.44 – 45.)

(VII) Geometry for drawings

At the time of Renaissance (around the 14th – 16th centuries), fine art was highly developed. In order to draw realistic pictures, “perspective” was developed. It is a kind of applied geometry. Early theory of perspective was developed by Brunelleschi (1377 – 1446 CE) and Alberti (1404 – 1472 CE).

In a manuscript written by Piero della Francesca (1418? – 1492 CE), the method of perspective is clearly explained. (See Fig.1 below.)





(From: Yokoyama, Tadashi: 横山正（解説、翻訳、製図）『アール・ヴィヴァン叢書、空間の発見①、ヴィアトールの透視図法 1505』、リブロポート, (1981), p.81)

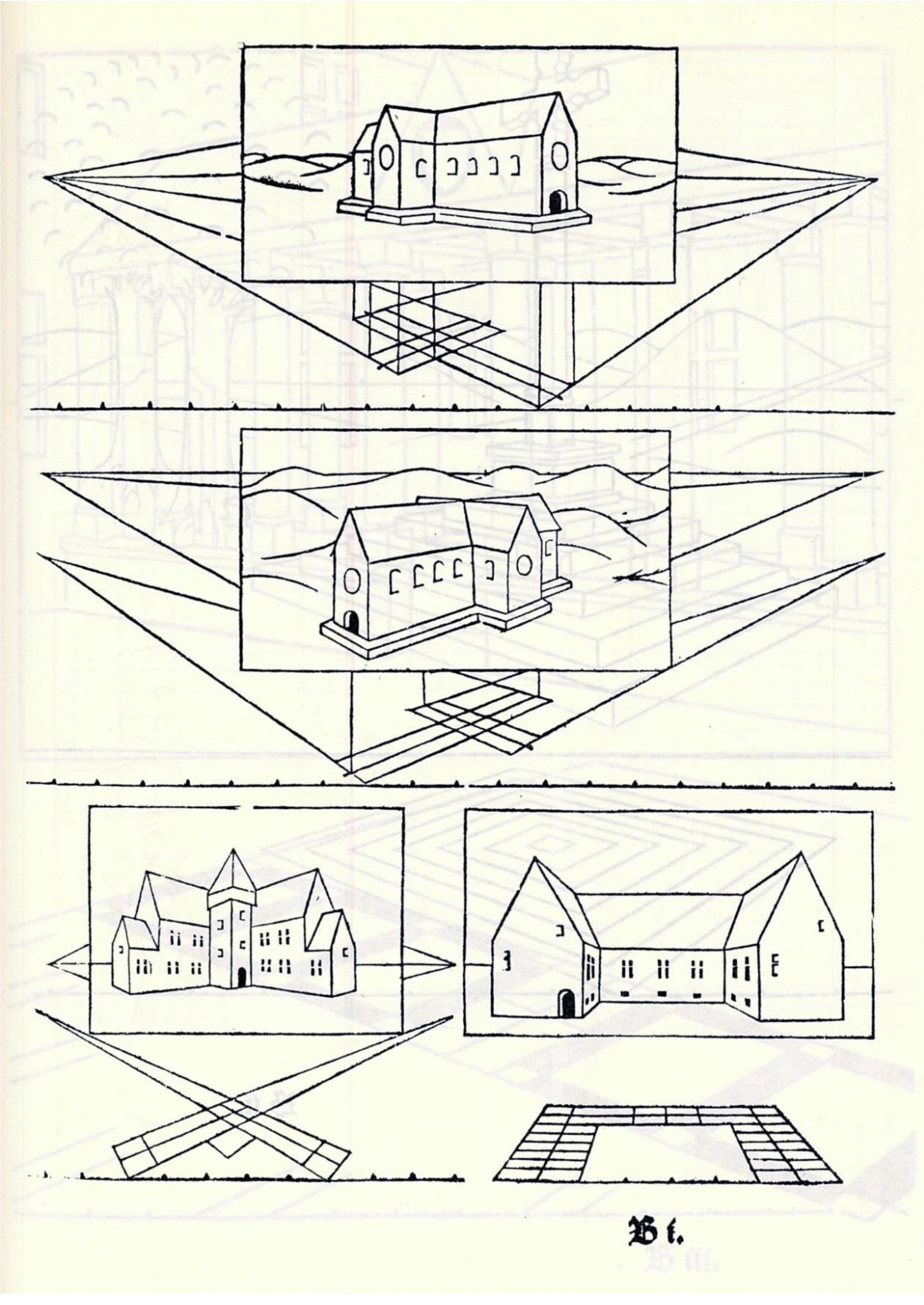
In the above Fig.1, a square ***fghi*** on a ground ***bcde*** is drawn. Firstly, a vertical square ***bcd'e'*** is drawn, and a reversed square ***f'g'h'i'*** is drawn in it. Then, as in Fig.2, from the point A (which corresponds to the point ***i***), two sets of lines are drawn:

(1): A ⇒ P ⇒ (A).

(2): A ⇒ Q ⇒ R ⇒ S ⇒ (A).

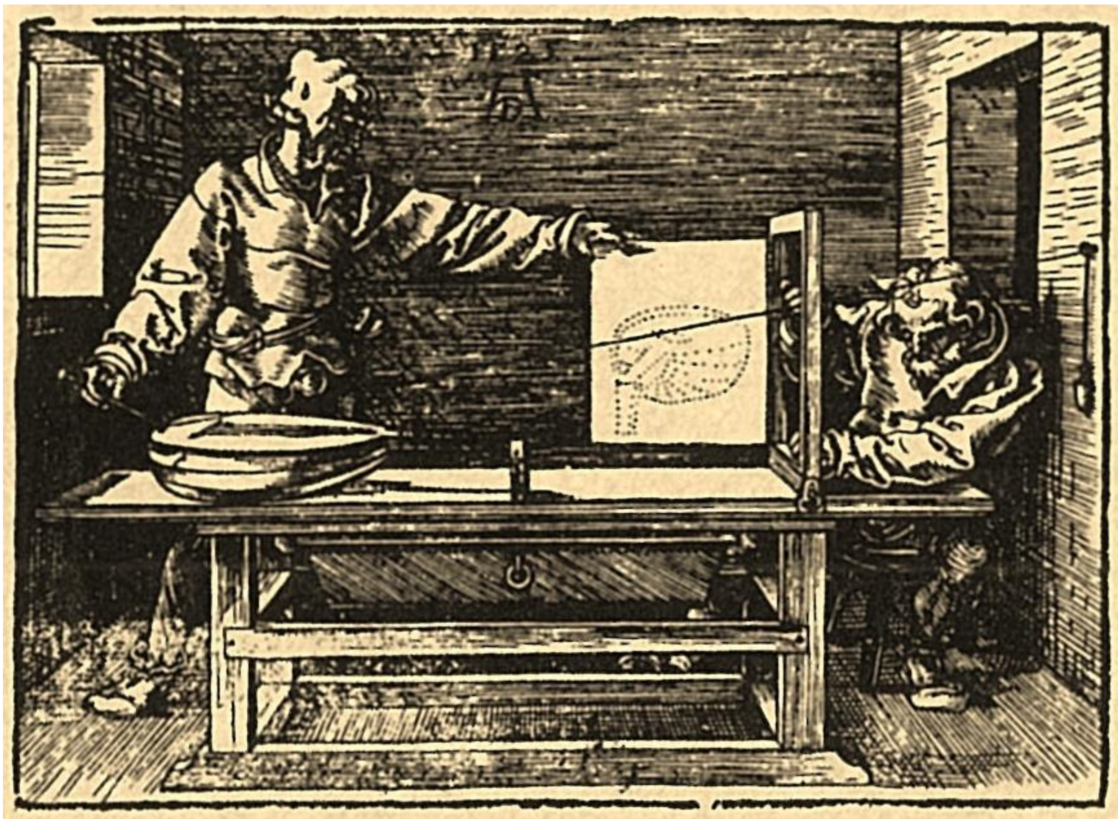
The cross point (A) corresponds to the point ***i*** in Fig.1.

The following is from *De artificiali perspectiva* (1505 CE) of Viator



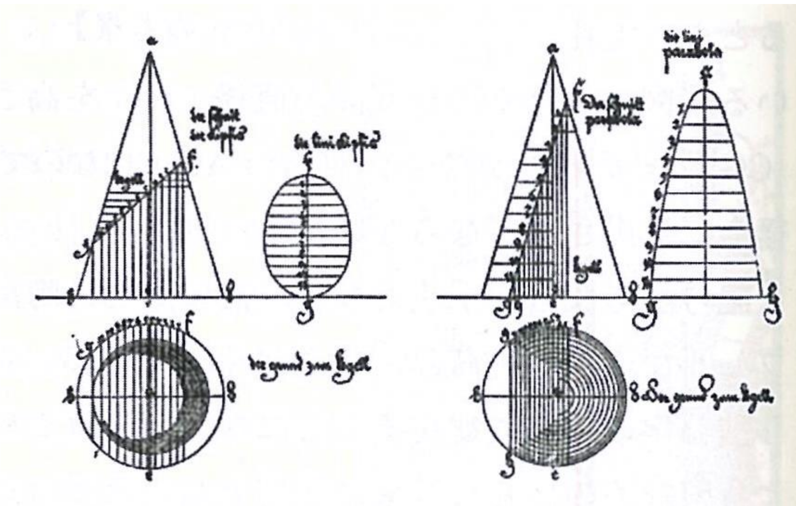
（From: Yokoyama, Tadashi: 横山正（解説、翻訳、製図）『アール・ヴィヴァン叢書、空間の発見①、ヴィアトールの透視図法 1505』、リブロポート, (1981))

The following is from a text book of drawing (1525 CE) by Dühler 81471 – 1528 CE).



(From: Yokoyama, Tadashi: 横山正（解説、翻訳、製図）『アール・ヴィヴァン叢書、空間の発見①、ヴィアトールの透視図法 1505』、リブロポート, (1981), p.68)

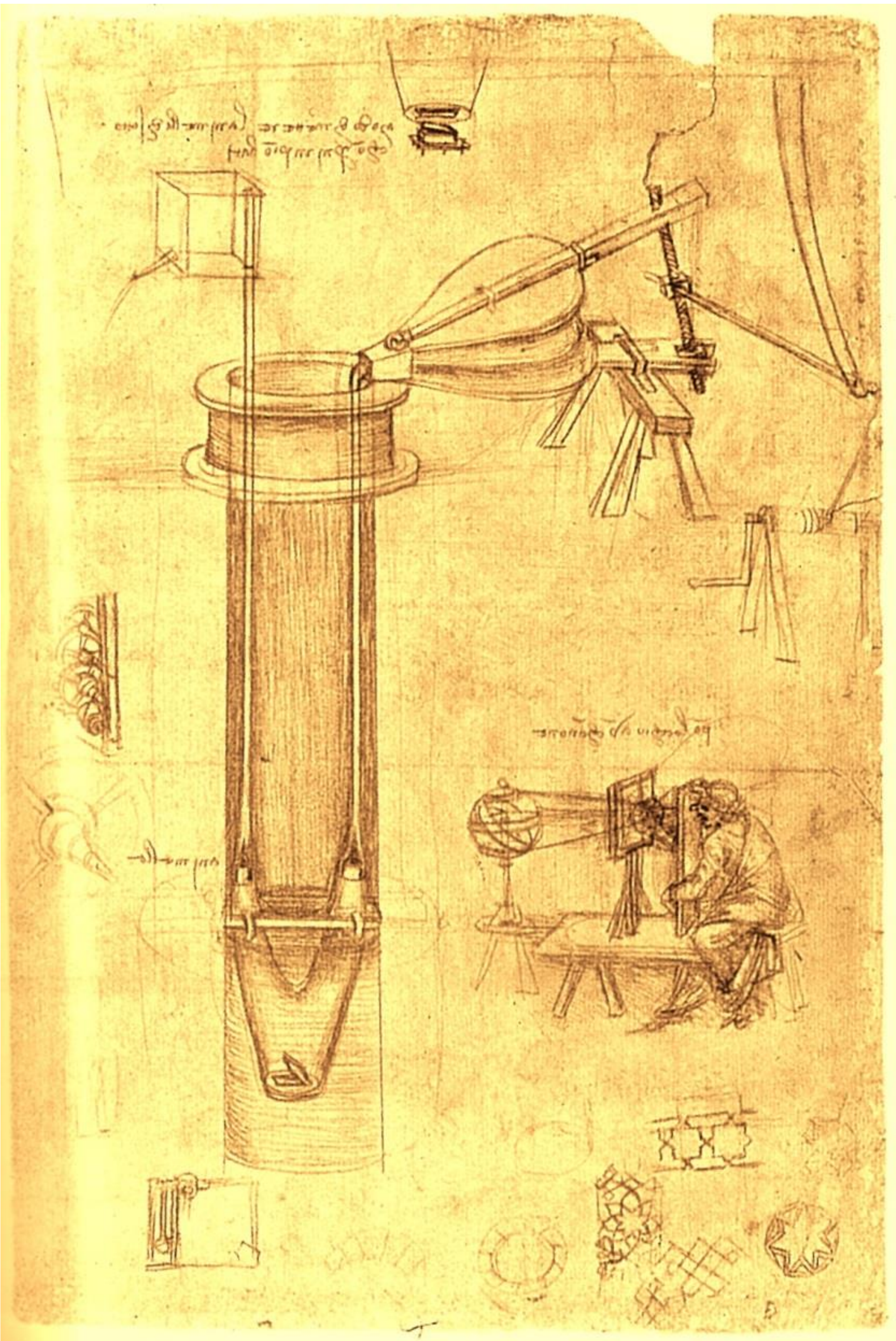
Dühler's method to draw ellipse and parabola:



【図⑦　デューラーによる楕円とパラボラの作図法】

(From: 三浦伸夫『数学の歴史』、放送大学教育振興会, 2013, p.152)

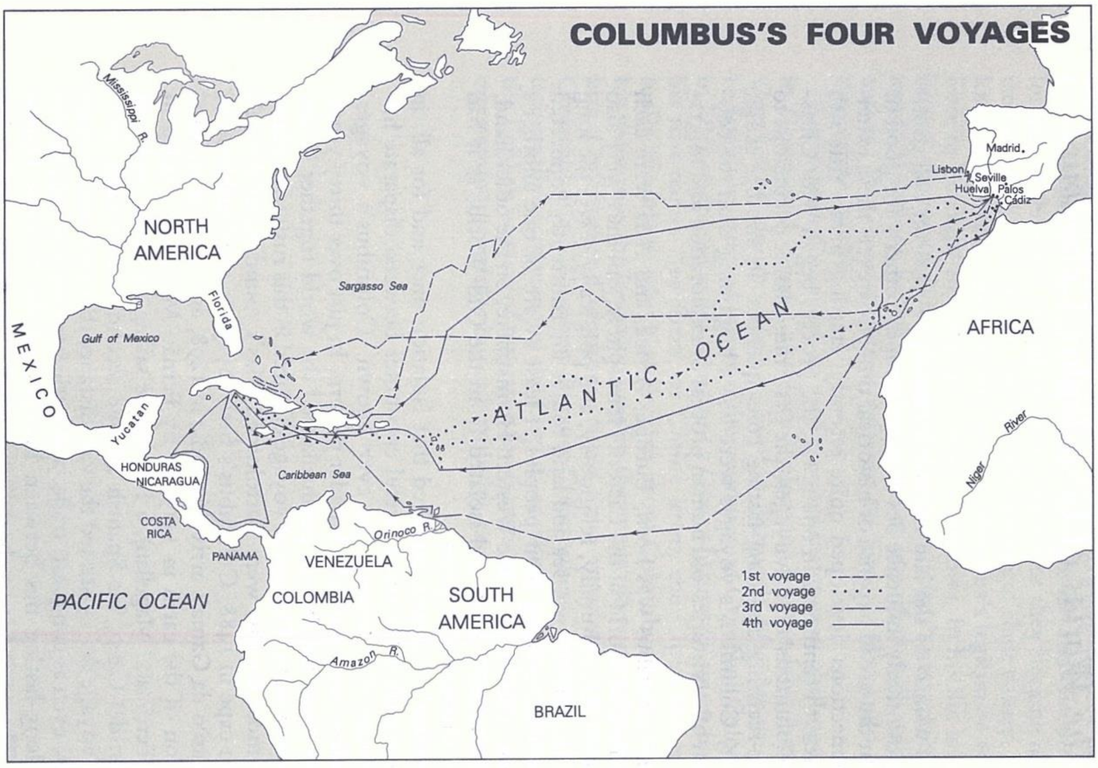
From a manuscript of Leonard da Vinci (1452 – 1519 CE)



(From *Leonard da Vinci, The Complete Works*, UK, David & Charles, 2006, p.355)

(VIII) Navigation

Columbus (1451 – 1506) arrived at America in 1492 CE. At that time latitude could be determined by astronomical observations, but longitude could not be determined accurately. So, it was usual to sail westwards or eastwards keeping the same latitude.



Columbus's approximate routes are marked in both directions for the first, second, and fourth voyages. Only his outbound route is marked for the third voyage, because he returned to Spain under arrest, rather than in command of the voyage. (Map prepared by the Cartography Laboratory, Department of Geography, University of Minnesota.)

(From Phillips and Phillips: *The Worlds of Christopher Columbus*, Cambridge, Cambridge University Press, 1992, p.2)

At that time, magnetic compass, quadrant, and mariner's astrolabe were used for navigation.

The altitude of heavenly bodies, when they cross the meridian, was observed to obtain latitude. For example, the altitude of the celestial north pole of the altitude of the midday sun at equinoctial days corresponds to the latitude of the observer. If the observation of the sun is made in days other than equinoctial days, the result is corrected by the sun's declination.

The quadrant and the mariner's astrolabe:

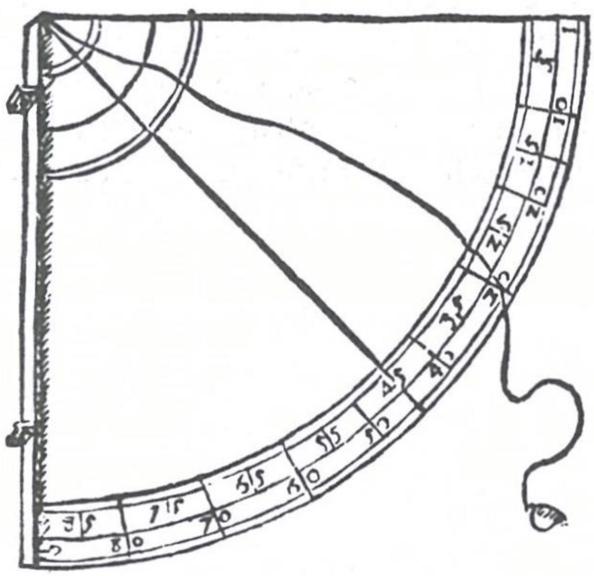


Abb. 24. Quadrant, Diego García de Palacio, Instrucción Naútica.

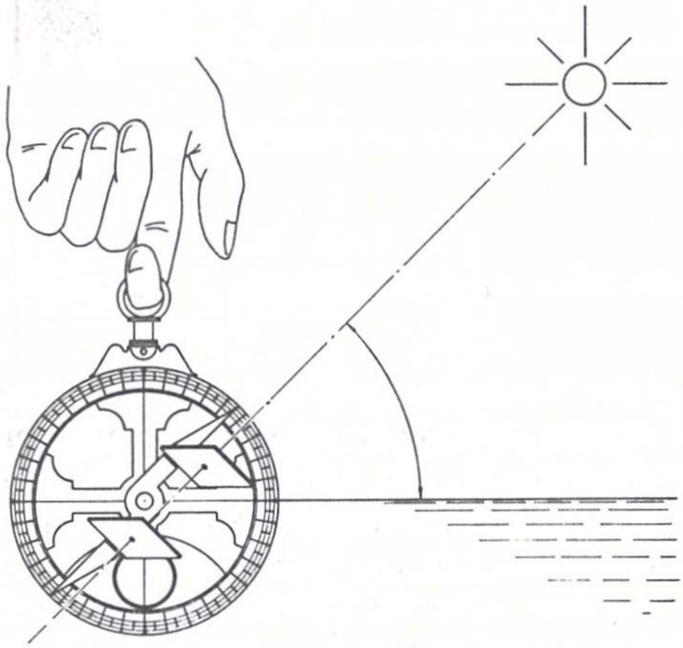
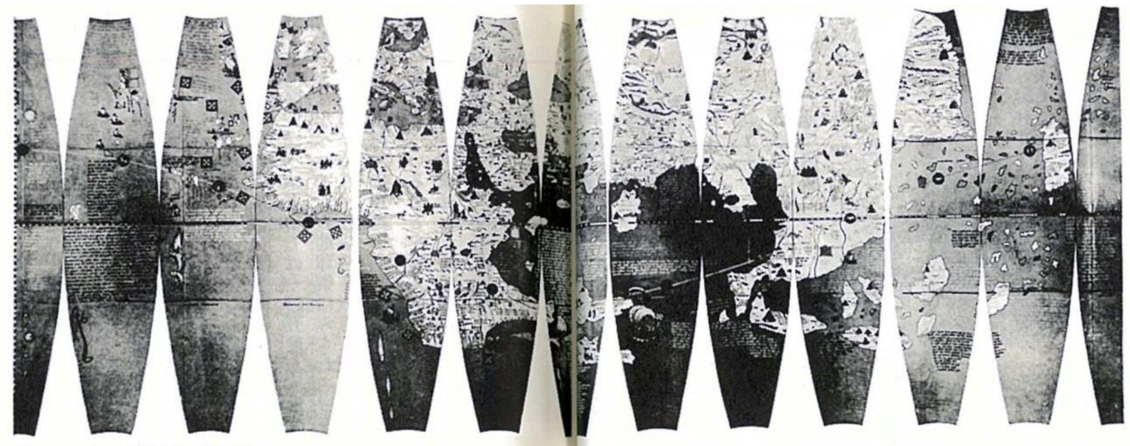


Abb. 25. See-Astrolab C. Plath, 1862–1962 Seite 56. Die Abbildung eines alten spanischen Geräts befindet sich auf dem Umschlag.

(From: Freiesleben, Hans-Christian: *Geschichte der Navigation*, Wiesbaden, Franz (There is a Japanese translation of this book: フライエスレーベン（坂本賢三訳）『航海術の歴史』、岩波書店, 1983)

(IX) Cartography

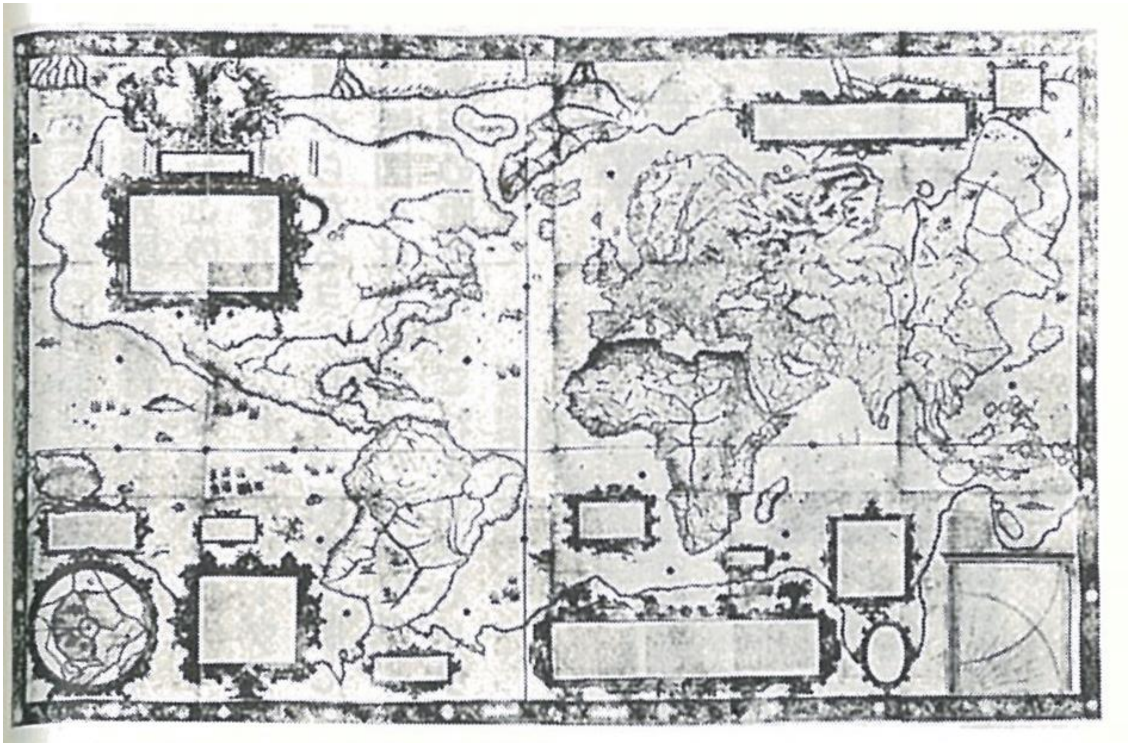
Martin Behaim (1459 – 1507) made the first terrestrial globe in 1492.



25　ベハイムの地球儀の世界図

Behaim's globe (From: 織田武雄『地図の歴史』、講談社学術文庫, 2018, pp.76 – 77)

Mercator (1512 – 1594) devised Mercator's projection, and made maps.



55　1569年のメルカトルの世界図

Mercator's world map (1569) (From: 織田武雄『地図の歴史』、講談社学術文庫, 2018, p.144)

Appendix ----- The principle of astrolabe

The astrolabe is a sophisticated instrument for the observation of the altitude of heavenly body, and the graphical astronomical calculations for, for example, the determination of time etc.　The astrolabe is originated in the ancient Hellenistic World, and well developed in the Islamic World etc.

La plaque de base, ou « mère », d'un astrolabe donne sur son rebord extérieur, terminé par l'anneau de suspension, les degrés du cercle. Les disques, ou « tympans » (quatre ici), s'empilent sur la « mère », porteurs des coordonnées d'une latitude déterminée. Est ensuite superposée, très ajourée, la carte céleste, ou « araignée ». Percées d'un trou central, ces différentes parties sont maintenues par un pivot, qui porte sur le dos de l'astrolabe une règle de visée, l'« alidade ».

59 Astrolabe construit par ibn-Khalaf al- Iraqî, Irak, Xᵉ siècle. BnF, Paris.

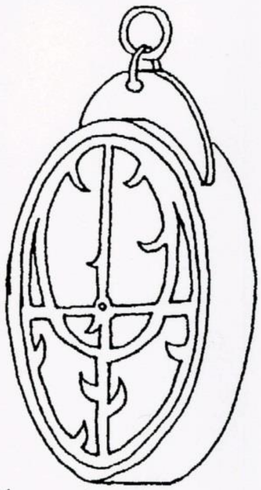
イブン・ハラフ・アル・イラーキーが製作したアストロラーベ――アストロラーベの基礎となる板，つまり母体盤の外側の縁にはぐるりと度が刻まれており，上部にはつり輪がついている。母体盤の上には特定の緯度のデータが記された盤（ここでは４枚）が，さらにその上にはくりぬかれた星図が重ねられている。これらはみな中央に穴が開いていて，軸によって固定されている。

(From Jacquart (2005), p.59 etc.)

The following is from Ôhashi (1997).

a)　**The astrolabe and the stereographic projection**

The astrolabe is a disklike instrument to observe the altitude of a heavenly body, and calculate time, *lagna*, etc. graphically.[5] (see Fig.1.)

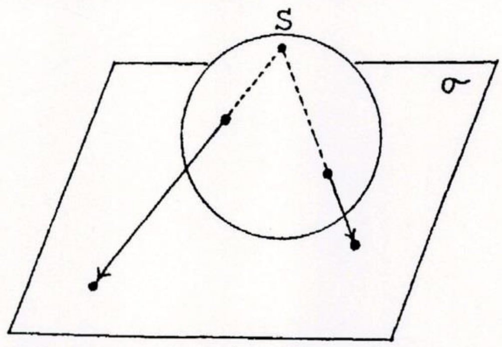


The Astrolabe　　Fig. 1

On the front side of the astrolabe is a disc (or discs changeable for different latitudes), on which the Tropic of Cancer, the equator, the Tropic of Capricorn, the horizon, the parallels of altitude, the vertical circles, the hour circles etc., for the observer's latitude, are drawn by the stereographic projection. Above this disc is put a spider which has the ecliptic circle and some pointers of fixed stars. This spider can be rotated around the centre of the disc which is the projection of the celestial pole.

The back side of the astrolabe is usually divided into four quardrants, each of which is used for graphical calculation etc. On the back side is attached an alidade in order to observe the altitude.

The stereographic projection[6] is a projection of a sphere from one of its points S onto the plane σ which is parallel to the tangent plane of the point S (see. Fig. 2) .The stereographic projection has two important properties, viz :



The Stereographic Projection

Fig. 2

(1) The preservation of circles :

The circles lying on a sphere are projected onto the plane as circles or, if the circles on the sphere pass through the projection centre, as straight lines.

(2) Conformality :

The stereographic projection maps the angles between the curves lying on a sphere as equal-to-them angles between the curves projected onto the plane.

For the convenience of later discussions, let us briefly review the proof of these two properties.[7]

Proof of the first property :

Let *AB* be a diameter of a circle (see Fig. 3) and *CD* be a perpendicular dropped from an arbitrary point C on the circle onto the diameter AB. Then, from Euclid (vi.8), we have the following equation :

$$AD \cdot DB = CD^2, \qquad \text{--------} \ (1)$$

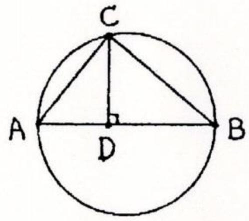


Fig. 3

and conversely, if this equation holds for any pont C on a curve for a segment AB, the curve is a circle.

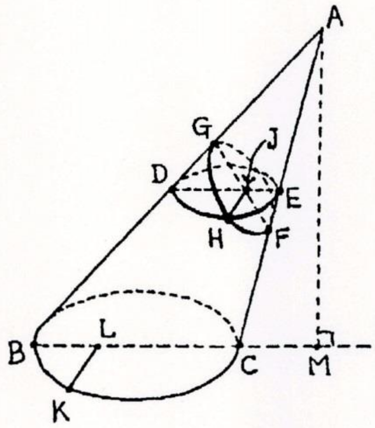


Fig. 4

Now (see Fig. 4), let us consider an oblique circular cone with a vertex A and the base whose diameter is BC, where the straight line BC produced meets the point M which is the foot of the perpendicular dropped from the vertex A onto the plane of the base. Let G be an arbitrary point on the straight line AB. Now, cut the cone by a plane (GHF) which is perpendicular to the plane ABC and passes through the point G in such a way that the angle AGF is equal to the angle ACB, (hence $A\hat{F}G = A\hat{B}C$ also), where F is an intersection of this plane (GHF) and the straight line AC. Then the curve GHF is a circle[8]. Let us prove this fact.

Let H be an arbitrary point on the curve GHF, and J be the foot of the perpendicular dropped from point H onto the straight line GF. Let a segment KL be the perpendicular dropped from an arbitrary point K on the circle *BKC* onto its diameter *BC*. Then :

$HJ \parallel KL,$ --------- (2)

because both are perpendicular to the plane *ABC*.

Now, draw a segment *DE*, which passes through the point *J*, in such a way that :

$DE \parallel BC.$ -------- (3)

Then, from (2) and (3), the plane *DHE* is parallel to the plane *BKC*. Hence the curve *DHE* is a circle. Applying the equation(1), we have :

$DJ \cdot JE = HJ^2.$ -------- (4)

Now, from (2),

$A\hat{D}E = A\hat{B}C$, and

$A\hat{E}D = A\hat{C}B.$

And also,

$A\hat{G}F = A\hat{C}B$, and

$A\hat{F}G = A\hat{B}C.$

Therefore,

$G\hat{D}J = E\hat{F}J$, and

$D\hat{G}J = F\hat{E}J.$

Since $D\hat{J}G = F\hat{J}E$, two triangles *DJG* and *FJE* are similar. Therefore :

$GJ \,/\, DJ = JE \,/\, JF,$

or,

$DJ \cdot JE = GJ \cdot JF.$ ------- (5)

From the equations (4) and (5), we have :

$GJ \cdot JF = HJ^2,$

for an arbitrary point *H* on the curve *GHF*. Therefore, considering the equation (1), the curve *GHF* is a circle.

Lastly, let us consider a stereographic projection of a circle (whose diameter is *MN*) from the point *S* onto the plane σ, where the plane σ tangents to the point *S'* which is

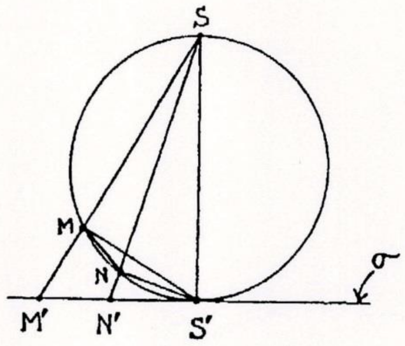


Fig. 5

diametrically opposite to *S*. (see Fig. 5) Let the segment *MN* be a diameter of the circle lying on the plane of the great circle which passes through the point S and the centre of the circle. And also, let *M'* and *N'* be the projections of the points *M* and *N* respectively onto the plane σ. Then, two right-angled triangles *SMS'* and *SS'M'* with a common acute angle *MSS'* are similar, and :

$$SM / SS' = SS' / SM',$$

or,

$$SM \cdot SM' = (SS')^2. \qquad \text{-------} (6)$$

Similarly,

$$SN \cdot SN' = (SS')^2. \qquad \text{--------} (7)$$

From the equations (6) and (7), we have :

$$SM \cdot SM' = SN \cdot SN',$$

or

$$SM / SN = SN' / SM'.$$

So, two triangles *SNM* and *SM'N'* with a common acute angle *MSN* are similar. Hence.

$$\left.\begin{aligned} S\hat{M}N &= S\hat{N}'M', \text{ and} \\ S\hat{N}M &= S\hat{M}'N'. \end{aligned}\right\} \text{------} (8)$$

If we consider an oblique circular cone with the vertex S and the base whose diameter is MN, it is clear from (8) that its section by the plane σ is a circle whose diameter is $M'N'$.

Hence the first property of the stereographic projection has been proved.

Proof of the second property:

The angle between two curves is the angle between their tangents at their intersection. Let a point M be an intersection of two curves on the celestial sphere whose tangents at

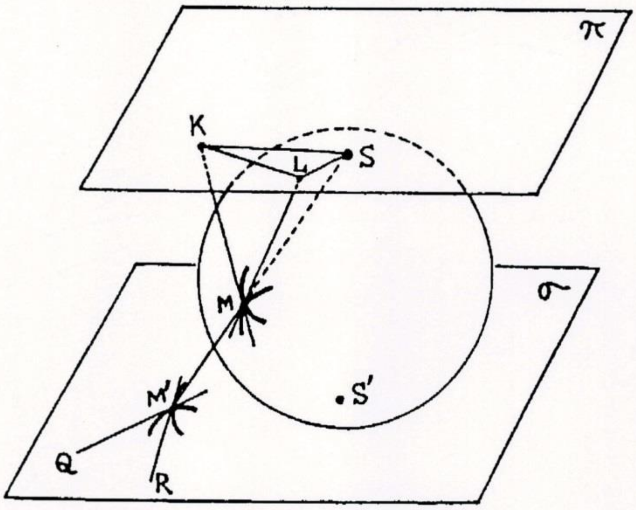


Fig. 6

the point M are MK and ML respectively. (see Fig. 6.) Let the plane σ tangent the point S' which is the opposite of the origin of the projection S. And let M' be the projection of the point M from the point S onto the Plane σ. And also, let $M'Q$ and $M'R$ be the tangents of the two projected curves at the point M'. Let us make a plane π which tangents the point S. And let K and L be the intersections of the plane π with the aforesaid straight lines MK and ML.

Since two tangents to a sphere from the same point are equal, we have:

$KS = KM$, and

$LS = LM$.

Hence, two triangles KSL and KML with the common side KL are mutually equal. Therefore:

$K\hat{S}L = K\hat{M}L$. ------- (9)

Now, we can consider that the tangents M′Q and M′R are the projections of the tangents MK and ML. So, the straight lines M′Q and M′R are the intersections of the plane σ with the planes KSM and LSM respectively, and we have:

$SK \,//\, M'Q$, and

$SL \,//\, M'R$.

Therefore:

$K\hat{S}L = Q\hat{M}'R$. ------- (10)

From the equations (9) and (10), we have:

$K\hat{M}L = Q\hat{M}'R$.

Hence the second property of the stereographic projection has been proved.

b) The construction of the astrolabe

On the front side of the astrolabe is attached a disc or discs which can be changed according to the observer's latitude. On the disc is attached a spider which can be rotated

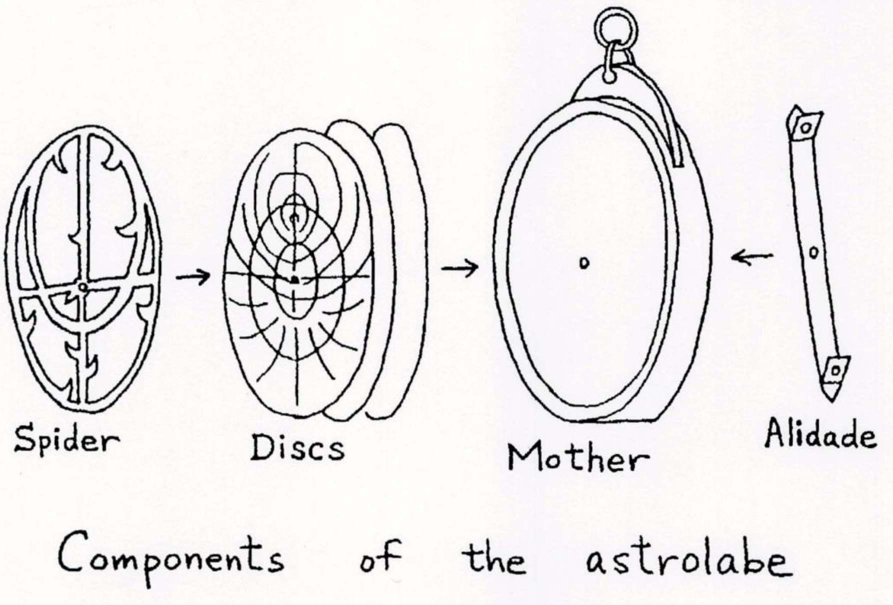


Fig. 7

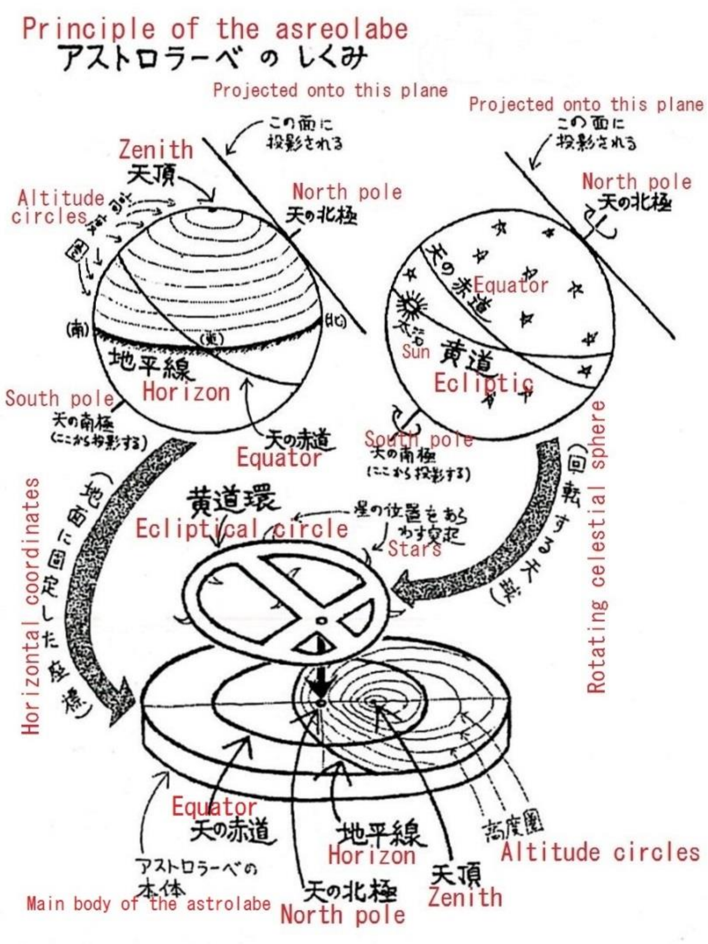
around the centre of the disc. On the back side of the astrolabe is attached an alidade in order to observe the altitude of a heavenly body. (see Fig. 7.)

The celestial sphere is projected onto the disc from the celestial pole by the stereographic projection. The disc represents horizontal coordinates. The spider, which is a net-like object, has an ecliptic circle and some indicators of fixed stars. The spider represents ecliptic coordinates. Therefore, the rotation of the spider is equivalent to the diurnal rotation of the celestial sphere.

Usually the celestial sphere is projected from the south celestial pole onto the disc. In this case, the centre of the disc is the projection of the north celestial pole.

Since the positions of the horizon, parallels of altitude etc. in the equatorial coordinates are different at different latitudes of the observer, one disc with the projection of the horizontal coordinates by the aforesaid method can be used at a particular latitude only. So several discs should be prepared if the astrolabe is to be used at different places.

(From: Ôhashi, Yukio: "Early History of the Astrolabe in India", *Indian Journal of History of Science*, **32**(3), 1997, 199-295; pp.202 – 209)



References

References of the history of science in the Islamic World

Jacquart, Danielle: *L'épopée de la science arabe*, Paris, Gallimard, 2005, (Japanese translation: ダニエル・ジャカール（遠藤ゆかり訳）『アラビア科学の歴史』、「知の再発見」双書 131、創元社, 2006)

Rashed, Roshdi: *Encyclopedia oh the History of Arabic Science*, 3 vols., London and New York, Routledge, 1996.

Gutas, Dimitri: *Greek Thought, Arabic Culture*, Routledge, 1998, (Japanese translation: ディミトリ・グタス（山本啓二訳）『ギリシア思想とアラビア文化，初期アッバース朝の翻訳運動』、勁草書房, 2002).

Yajima: 矢島祐利『アラビア科学史序説』、岩波書店, 1977.

References of the history of mathematics and astronomy in the Islamic World

Berggren, L.L.: *Episodes in the Mathematics of Medieval Islam*, New York, Springer-Verlag, 1986.

Rashed, Roshdi: *Entre Arismétique et Algèbre: Recherches sue l'Histoire des mathématiques arabes*, Paris, Siciété d'Edition Les Bells Lettres, 1984, (Japanese translation: ロシュディー・ラーシェド（三村太郎訳）『アラビア数学の展開』、東京大学出版会, 2004).

Kennedy, E.S.: *Studies in the Islamic Exact Sciences*, Beirut, American University of Beirut, 1983.

Saliba, George: *A History of Arabic Astronomy*, New York and London, New York University Press, 1994.

Saliba, George: *Islamic Science and the Making of European Renaissance*, Cambridge, Mass., The MIT Press, 2011.

Mimura: 三村太郎『天文学の誕生，イスラーム文化の役割』、岩波科学ライブラリー173、岩波書店, 2010.

References of the history of science in Medieval Europe

Crombie, A.C.: Augustine to Galileo, Cambridge, Mass., Harvard University Press, 1953, (Japanese translation: クロムビー（渡辺正雄、青木靖三訳）『中世から近世への科学史』（上）（下）、コロナ社, 1962 – 1968).

Grant, Edward: *Physical Science in the Middle Ages*, 1971; reprinted: Cambridge, Cambridge UniversityPress, 1977, (Japanese translation: E. グラント（横山雅彦訳）『中世の自然学』、みすず書房, 1982).

Dales, Richard C.: *The Scientific Achievement of the Middle Ages*, University of Pennsylvania Press, 1973.

Grant, Edward: *A Source Book in Medieval Science*, Cambridge, Mass., Harvard University Press, 1974.

Pederson, Olaf: *Early Physics and Astronomy*, 1974; revised edition: Cambridge, Cambridge University Press, 1993.

Lindberg, David C. (ed.): *Science in the Middle Ages*, Chicago, University of Chicago Press, 1978.

Lindberg, David C.: *The Beginnings of Western Science*, Chicago, University of Chicago Press, 1992, (Japanese translation: Lindberg（高橋憲一訳）『近代科学の源をたどる』、朝倉書店, 2011).

Grant, Edward: *The Foundations of Modern Science in the Middle Ages*, Cambridge University Press, 1996, (Japanese translation: E. グラント（小林剛訳）『中世における科学の基礎づけ』、知泉書館, 2007).

References of some topics in Medieval Europe

Haskins, Charles Homer: *The Renaissance of the Twelfth Century*, Harvard University Press, 1971, (Japanese translation: ハスキンズ（野口訳）『十二世紀ルネサンス』、創文社, 1985； C.H. ハスキンズ（別宮・朝倉訳）『十二世紀ルネサンス』、みすず書房, 1989、(『十二世紀のルネサンス』、講談社学術文庫, 2017）).

Haskins, Charles Homer: *Rise of Universities*, Ithaca, Cornel university Press, (C.H. ハスキンズ（青木靖三・三浦常司訳）『大学の起原』、法律文化社, 1970).

三浦伸夫『フィボナッチ［アラビア数学から西洋中世数学へ］』、現代数学社, 2016.

Clagett, Marshall: *The Science of Mechanics in the Middle Ages*, Madison, The University of Wisconsin Press, 1959.

Apel, Willi: *The Notation of Polyphonic Music 900-1600*, reprinted: Oxford City Press, 2010.

Yokoyama: 横山正（解説、翻訳、製図）『アール・ヴィヴァン叢書、空間の発見①、ヴィアトールの透視図法 1505』、リブロポート, 1981.

Crosby, Alfred W.: *The Measure of Reality, Quantification and Western Society, 1250 – 1600*, Cambridge University Press, 1997, (Japanese translation: アルフレッド・W.クロスビー（小沢千重子訳）『数量化革命』、紀伊国屋書店, 2003).

References of the history of mathematics and science including mathematics and science in the Islamic World and Medieval Europe:

Van Brummelen, Glen: *The Mathematics of the Heavens and the Earth: The Early History of Trigonometry*, Princeton, Princeton University Press, 2009.

Van Brummelen, Glen: *Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry*, Princeton, Princeton University Press, 2013.

Ito: 伊東俊太郎『中世の数学』、共立出版, 1987.

Ito: 伊東俊太郎『近代科学の源流』、originally published by 中央公論社 in 1978、中公文庫, 2007.

Miura: 三浦伸夫『数学の歴史』、放送大学教育振興会, 2013.

Freiesleben, Hans-Christian: *Geschichte der Navigation*, Wiesbaden, Franz (There is a Japanese translation of this book: フライエスレーベン（坂本賢三訳）『航海術の歴史』、岩波書店, 1983

Additional readings

(Japanese mini lecture 1): 大橋由紀夫「科学史ミニ講義(1), 離心円と周転円の話」、『科学史通信』No.394, 2009.

(Japanese mini lecture 2): 大橋由紀夫「科学史ミニ講義(2), 蓋天説と渾天説の話」、『科学史通信』No.395, 2010.

(Japanese mini lecture 3): 大橋由紀夫「科学史ミニ講義(3), アストロラーベの構造」、『科学史通信』No.396, 2010.

(Japanese mini lecture 4): 大橋由紀夫「科学史ミニ講義(4), 二重周転円とコペルニクス」、『科学史通信』No.397, 2010.