TokyoTech (Tokyo Institute of Technology), HMA (History of Mathematics and Astronomy)

Lecture note 4: (2019)

(Mathematics and astronomy in traditional China)

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Rough Chronology of pre-modern China:

(A) Xia (夏) dynasty (legendary)

(B) Shang (商) (=Yin (殷)) dynasty (ca. 16th century BCE ~ ca. 11th century BCE)

(C) Western Zhou (西周) dynasty (ca. 11th century BCE ~ 770 BCE)

(D) Spring and autumn ("Chunqiu" (春秋)) period (770 BCE ~ 476 BCE)

(E) Warring states ("Zhanguo" (戦國)) period (475 BCE ~ 221 BCE)

(F) Qin (秦) dynasty (221 BCE ~ 206 BCE)

(G) Han (漢) dynasty (206 BCE ~ 220 CE) (Western (Former) Han (西漢 or 前漢) (206 BC ~ 8 CE) and Eastern (Later) Han (東漢 or 後漢) (25 CE ~ 220 CE)) (There was the Xin (新) dynasty from 9 CE to 23 CE between the Western Han and Eastern Han.)

(H) Three kingdoms ("Sanguo" (三國)) period (220 ~ 265) (Wei (魏) (220~265), Shu (蜀) (221~263) and Wu (呉) (222~280))

(I) Jin (晉) dynasty (265 ~ 420) (Western Jin (西晉) (265 ~ 316) and Eastern Jin (東晉) (317 ~ 420))

(J) Sixteen states ("Shiliuguo" (十六國)) period (304 ~ 439)

(K) Northern and southern dynasties ("Nanbeichao" (南北朝)) period (420 ~ 589) (Northern dynasties: Northern Wei (北魏), Eastern Wei (東魏), Western Wei (西魏), Northern Qi (北齊) and Northern Zhou (北周); and Southern dynasties: Song (宋) (Liu-Song (劉宋)), Qi (齊), Liang (梁) and Chen (陳))

(L) Sui (隋) dynasty (581 ~618)

(M) Tang (唐) dynasty (618 ~ 907)

(N) Five dynasties and ten states ("Wudai-shiguo" (五代十國)) period (907 ~ 979)

(O) Song (宋) dynasty (960 ~ 1279) (Northern Song (北宋) (960 ~ 1127) and Southern Song (南宋) (1127 ~ 1279))

(P) Liao (遼) dynasty (916 ~ 1125)

(Q) Jin (金) dynasty (1115 ~ 1234)

(R) Yuan (元) dynasty (1271 ~ 1368)

(S) Ming (明) dynasty (1368 ~ 1644)

(T) Qing (清) dynasty (1644 ~ 1911)

Mathematics and astronomy in traditional China

Shang (商) (=Yin (殷)) dynasty

(ca. 16th century BCE ~ ca. 11th century BCE)

商朝建立时，定都亳（今河南郑州），此后曾多次迁都。公元前1300年，盘庚迁都于殷（今河南安阳），商朝的统治区域不断扩展，其势力范围东到今山东西部，西达陕西西部，北到河北北部，南至长江流域一带，成为当时世界上的大国。

(From 中国历史地图及大事年表，北京，中国地图出版社，2013.)

Oracle bone script (甲骨文):

Numerals:

Their ideograms from one to ten in oracle bone script are roughly as follows:

Oracle:	一	二	三	亖	ㄨ	∩、∧	十	)(	九	丨	(Oracle bone script)
Modern:	一	二	三	四	五	六	七	八	九	十	(Modern Chinese)
~~Hindu:~~ Modern:	1	2	3	4	5	6	7	8	9	10	(Modern)

Hundred, thousand and ten thousand have the forms:

Oracle:	[illegible],	[illegible],	[illegible],	(Oracle bone script)
Modern:	百	千	萬(万)	(Modern Chinese)
	hundred	thousand	ten thousand	

Twenty, thirty, . . ., two hundred, three hundred, . . . and two thousand, thirty thousand, etc. are compound words made up of two characters, for example

二十	三十	四十	五十	六十	七十	八十
20	30	40	50	60	70	80
二百	三百	四百	五百	六百	八百	九百
200	300	400	500	600	800	900
二千	三千	四千	五千	八千	三萬	
2000	3000	4000	5000	8000	30 000	

For instance, 'two thousand six hundred and fifty-six' would be written in oracle bone script as [illegible].

(From Li and Du (1987), p.5, with my notes.))

Luni-solar calendar was already used in the Shang (=Yin) dynasty. Sexagenary cycle ("ganzhi" 干支) was used to express days.

Sexagenary cycle (“ganzhi” 干支):

10 “gan”s (十干)

文字 (“gan”s)	日本語 (Japanese pronunciation)	中国語 (Chinese pronunciation)
甲	コウ (きのえ)	jiǎ
乙	オツ (きのと)	yǐ
丙	ヘイ (ひのえ)	bǐng
丁	テイ (ひのと)	dīng
戊	ボ (つちのえ)	wù
己	キ (つちのと)	jǐ
庚	コウ (かのえ)	gēng
辛	シン (かのと)	xīn
壬	ジン (みずのえ)	rén
癸	キ (みずのと)	guǐ

12 “zhi”s (十二支)

文字 (“zhi”s)	日本語 (Japanese pronunciation)	中国語 (Chinese pronunciation)
子	シ (ね)	zǐ
丑	チュウ (うし)	chǒu
寅	イン (とら)	yín
卯	ボウ (う)	mǎo
辰	シン (たつ)	chén
巳	シ (み)	sì
午	ゴ (うま)	wǔ
未	ビ (ひつじ)	wèi
申	シン (さる)	shēn
酉	ユウ (とり)	yǒu
戌	ジュツ (いぬ)	xū
亥	ガイ (い)	hài

☆六十干支

Combination of “gan” and “zhi”

→ 十二支 (12 “zhi”s)

↓ 十干 (10 “gan”s)

	子	丑	寅	卯	辰	巳	午	未	申	酉	戌	亥
甲	1		51		41		31		21		11	
乙		2		52		42		32		22		12
丙	13		3		53		43		33		23	
丁		14		4		54		44		34		24
戊	25		15		5		55		45		35	
己		26		16		6		56		46		36
庚	37		27		17		7		57		47	
辛		38		28		18		8		58		48
壬	49		39		29		19		9		59	
癸		50		40		30		20		10		60

(From my lecture note for Hitotsubashi University and Hosei University.)

Western Zhou (西周) dynasty

(ca. 11th century BCE ~ 770 BCE)

(From 中国历史地图及大事年表, 北京, 中国地图出版社, 2013.)

In the bronze inscriptions (金文) of the Western Zhou period, the belief of “Mandate of Heaven” (天命) is mentioned.

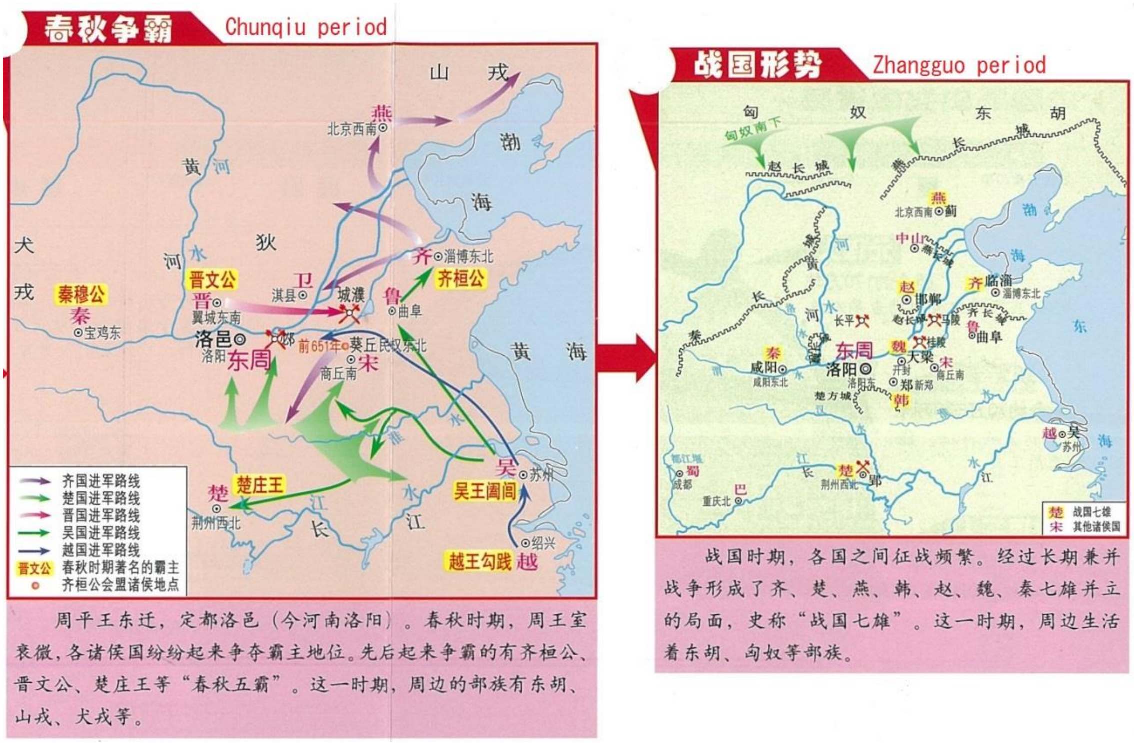
----- Importance of astronomical observation and accurate calendar.

Spring and autumn (“Chunqiu” (春秋)) period

(770 BCE ~ 476 BCE), and

Warring states (“Zhanguo” (戦國)) period

(475 BCE ~ 221 BCE)



(From 中国历史地图及大事年表，北京，中国地图出版社，2013.)

Luni-solar calendar:

By the end of the Warring states period, the 19-year cycle of intercalation, in which 7 intercalary months are added, was already in use.

And the length of a year was considered to be $365\frac{1}{4}$ days. This type of calendar is called *Sifen* calendar (四分暦) (“Quarter” calendar) named after the fraction of the length of a year.

Divisions of a year (“jieqi” 節氣):

And also, the divisions of a year, which finally became the 24 divisions of a year (二十四節氣) at (or before) the early Former Han dynasty, were being formed in this period.

24 Divisions of a year (二十四節気)

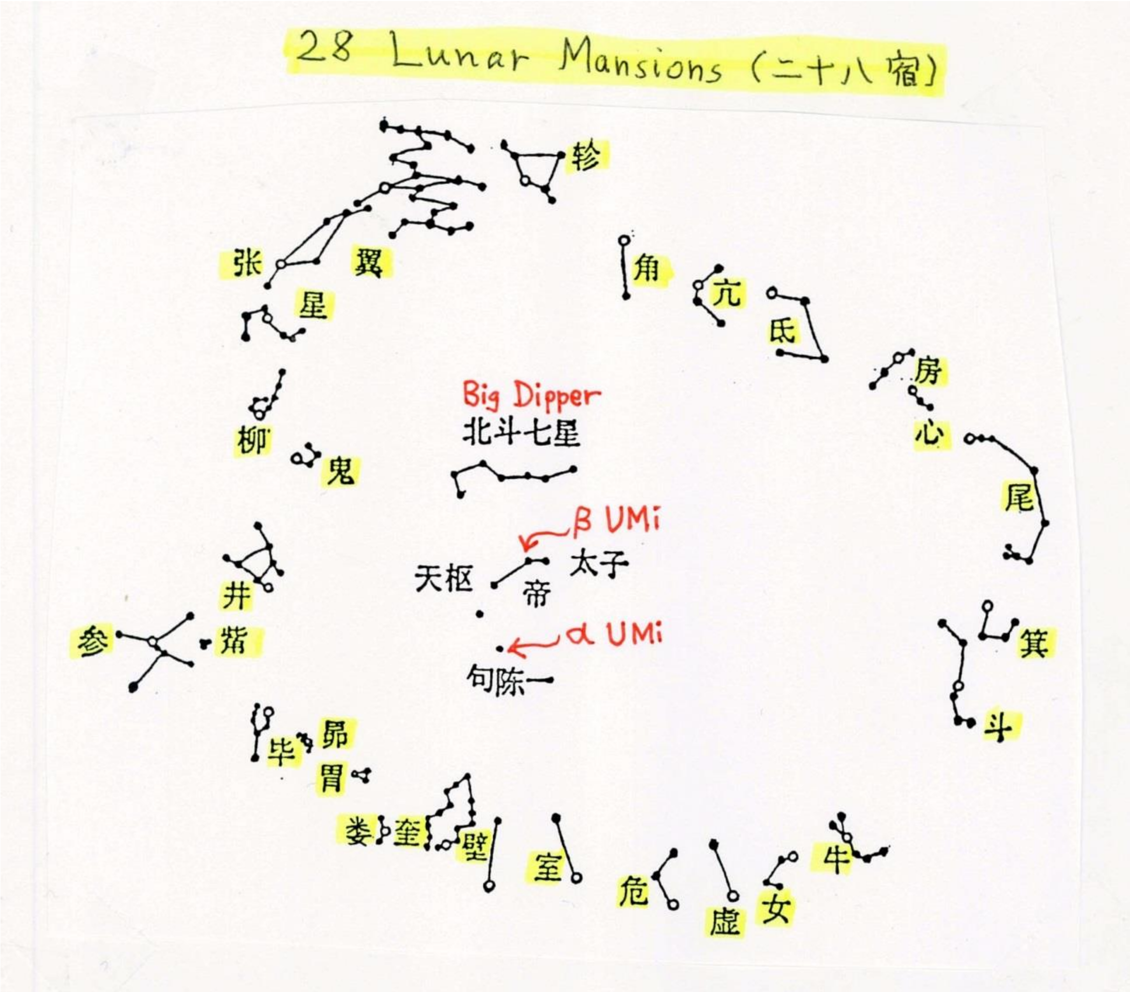
Japanese “sekki” 日本語	Chinese “jieqi” 中国語	Corresponding lunar months 旧暦との対応	Corresponding Gregorian dates 現在の西暦との対応
立春(りっしゅん)	立春Lìchūn	(正月節)	2月4日頃
雨水(うすい)	雨水Yǔshuǐ	(正月中)	2月18~19日頃
啓蟄(けいちつ)	惊蛰Jīngzhé	(二月節)	3月5~6日頃
春分(しゅんぶん)	春分Chūnfēn	(二月中)	3月20~21日頃
清明(せいめい)	清明Qīngmíng	(三月節)	4月4~5日頃
穀雨(こくう)	穀雨Gǔyǔ	(三月中)	4月20日頃
立夏(りっか)	立夏Lìxià	(四月節)	5月5~6日頃
小満(しょうまん)	小满Xiǎomǎn	(四月中)	5月21日頃
芒種(ぼうしゅ)	芒种Mángzhòng	(五月節)	6月5~6日頃
夏至(げし)	夏至Xiàzhì	(五月中)	6月21~22日頃
小暑(しょうしょ)	小暑Xiǎoshǔ	(六月節)	7月7日頃
大暑(たいしょ)	大暑Dàshǔ	(六月中)	7月22~23日頃
立秋(りっしゅう)	立秋Lìqiū	(七月節)	8月7~8日頃
処暑(しょしょ)	处暑Chǔshǔ	(七月中)	8月23日頃
白露(はくろ)	白露Báilù	(八月節)	9月7~8日頃
秋分(しゅうぶん)	秋分Qiūfēn	(八月中)	9月23日頃
寒露(かんろ)	寒露Hánlù	(九月節)	10月8~9日頃
霜降(そうこう)	霜降Shuāngjiàng	(九月中)	10月23~24日頃
立冬(りっとう)	立冬Lìdōng	(十月節)	11月7~8日頃
小雪(しょうせつ)	小雪Xiǎoxuě	(十月中)	11月22~23日頃
大雪(たいせつ)	大雪Dàxuě	(十一月節)	12月7日頃
冬至(とうじ)	冬至Dōngzhì	(十一月中)	12月21~22日頃
小寒(しょうかん)	小寒Xiǎohán	(十二月節)	1月5~6日頃
大寒(たいかん)	大寒Dàhán	(十二月中)	1月20~21日頃

((注) 旧暦との対応については、前漢の太初暦で確立された方法では、「X月中」というのが必ずその月に含まれ、「X月節」は、前月に含まれることもある。)

(From my lecture note for Hitotsubashi University and Hosei University.)

Lunar mansions (“xingxiu” 星宿):

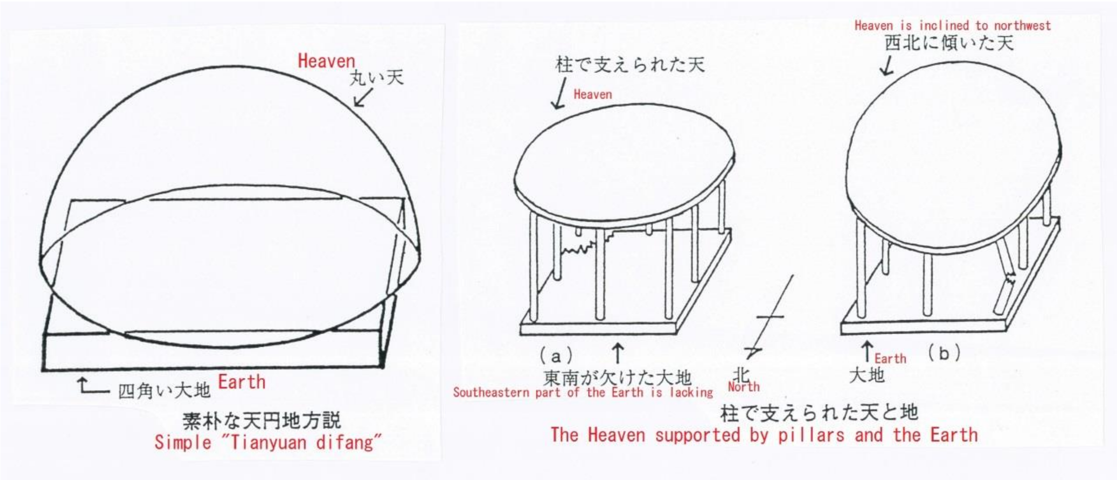
As regards the descriptive astronomy, already in the Warring states period, 28 lunar mansions (二十八宿) were established.



（中国天文学史整理研究小組・天文史話編写組《天文史話》(上海科学技術出版社、1981) p.24 より)

Cosmology:

The naïve cosmology in this period was the “*tian-yuan di-fang*” (天圓地方) theory, which means that the round heaven is over the square earth. This model developed into the *gaitian* theory (蓋天説) in the Former Han dynasty, in which the upper heaven and the lower earth are considered to be flat and parallel.



The "*tian-yuan di-fang*" (天圓地方) theory

(From Ôhashi (1998) with additional notes)

Early development of mathematics:

Nine-nines rhyme (九九表，九九歌)

The "nine-nines rhyme" for multiplication was used since Chunqiu-Zhanguo period or so.

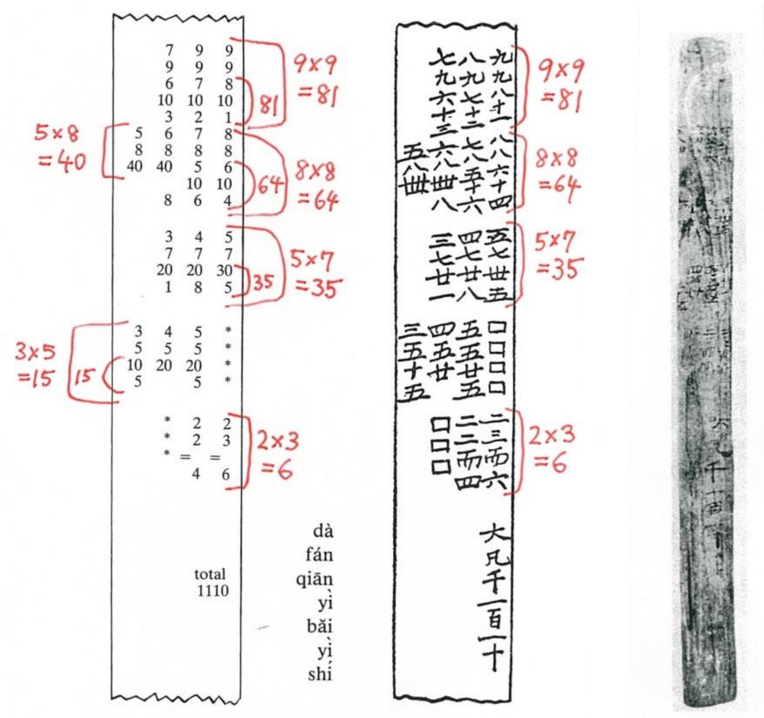
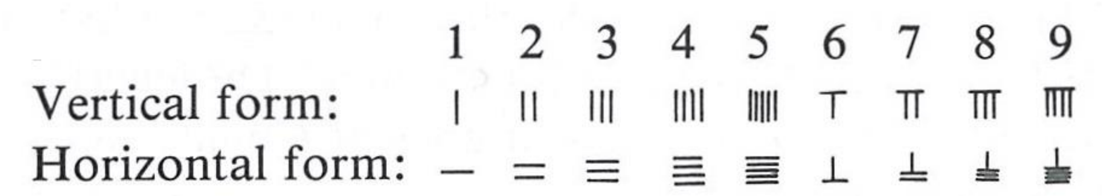


Fig. 1.3 Hàn bamboo strip with Nine-nines rhyme (Read from top down.)

(From Li and Du (1987), p.14 with my notes.)

Counting rods (算籌)

The "Counting rods" for multiplication was used since Chunqiu-Zhanguo period or so.



----- Decimal place value system.

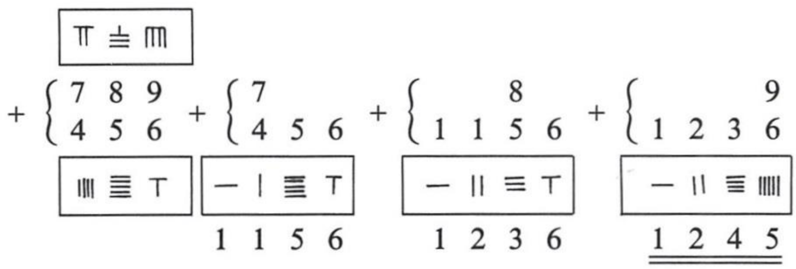
Vertical form: for the units, the hundreds, the ten thousands, ---.

Horizontal form: for the tens, thousands, ---.

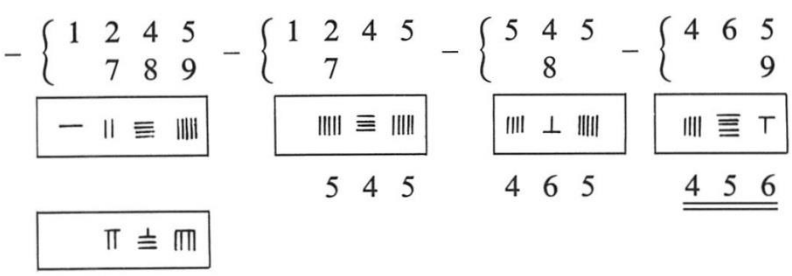
A blank space is used for zero.

Examples of calculations by counting rods:

Example: 456 + 789 using counting rods. First use counting rods to represent 456, then add 7 to the 4 in the hundreds' position. Second, add the numbers in the tens' and then in the units' position. So one starts from the highest place-value digit, calculating from left to right as follows:



Subtraction is similar. For instance, 1234 – 789. First lay down 1245 and subtract 7 from the hundreds' position. Second, subtract the numbers from the tens' and then the units' positions. Again this is carried out from left to right as below:



(From Li and Du (1987), pp.12-13.)

Acoustics (音律)

Around the Warring states period (戦國時代), the theory of musical tuning was established. One octave was divided into 12 pitches. These pitches were determined by the combination of 3:2 (perfect fifth) and 3:4 (perfect foueth) of the length of pitch pipe or chord. This method is called "*Sanfen-sunyi-fa*" (三分損益法). In the actual music, 5 or 7 pitches are used.

Five pitches (五音):

Gong (宮), Shang (商), Jue (角), Zhi (徵), Yu (羽).

Bamboo and wooden strips（竹簡、木簡）

It may be mentioned here that several bamboo and wooden strips of the Warring states period, Qin period and Han period, including mathematical and astronomical texts, are being excavated, and several new studies of the early development of mathematics and astronomy in China are being done. (For example, see Yokota (2012) and Ôno (2014) (in Japanese).)

Qin (秦) dynasty (221 BCE ~ 206 BCE)

秦定都咸阳（今陕西咸阳东北），疆域东至东海，西到陇西，北至长城一带，南达南海，曾是当时世界上最大的国家。秦朝实行郡县制，初并天下时，全国共设 36 郡；至秦末共有郡 40 余个，县 1 000 个左右。秦的周边部族有匈奴、东胡、乌孙、月氏及羌等。

(From 中国历史地图及大事年表, 北京, 中国地图出版社, 2013.)

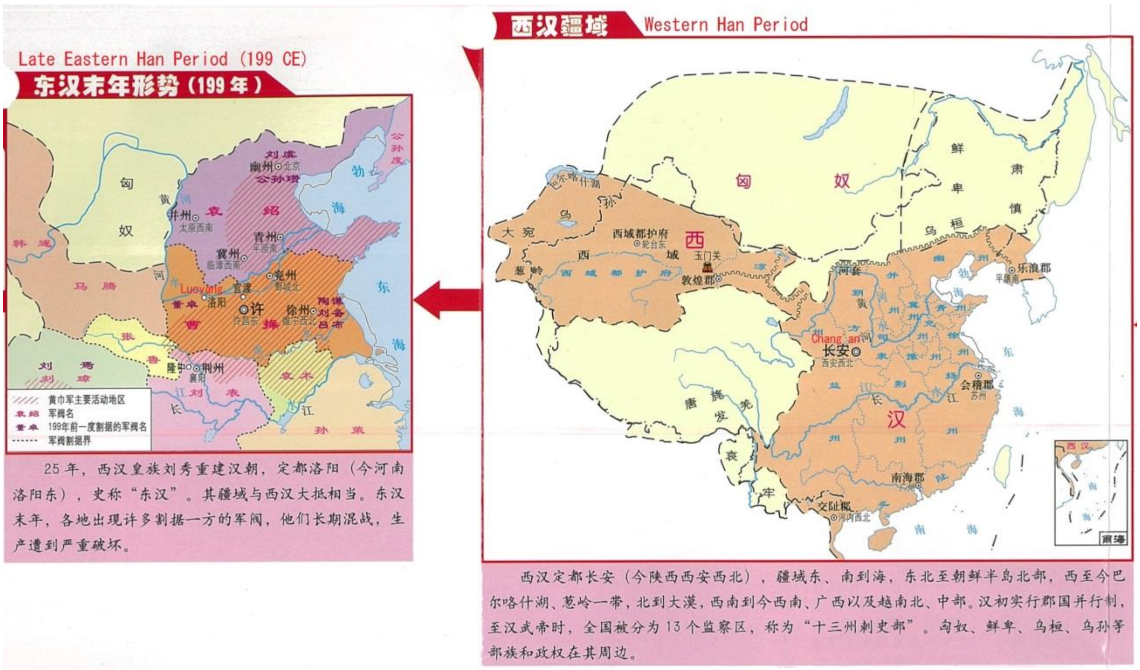
Qin Shi Huang (秦始皇, "The first emperor of Qin") united China, and standardized letters, weights and measures etc.

The *Zhuanxu* calendar (顓頊暦), a kind of *Sifen* calendar (四分暦), was used in the Qin dynasty. At that time, one year started from the 10th month of this calendar, and intercalary months were put at the end of years (i.e. after the 9th month).

Han (漢) dynasty (206 BCE ~ 220 CE)

--- (Western (Former) Han (西漢 or 前漢) (206 BC ~ 8 CE))

--- (Eastern (Later) Han (東漢 or 後漢) (25 CE ~ 220 CE))



(From 中国历史地图及大事年表, 北京, 中国地图出版社, 2013.)

It can be said that the foundation of the Chinese classical astronomy was formed in the Han period. The preceding Chunqiu-Zhanguo ("Spring and autumn" and "Warring states") period can be considered to be the period of the preparation of some fundamental ideas.

The earliest book catalogue in China, *Yiwen-zhi* (藝文志, Treatise of literature) in the *Hanshu* (漢書, Official History of the Former Ham Dynasty), which is based on a catalogue made in the late Western Han period, classifies the books into 6 divisions, namely:

Confucianist classics (六藝),

Philosophical works (諸子),

Poetries (詩賦),

Military works (兵書),

Astronomical (including mathematical) and divinatory works (數術),

Medical works and works concerning human body (方技).

Here, we can see that astronomy and mathematics were already established learnings in this period.

Development of the calendrical science in the Han (漢) period

Taichu calendar (太初暦)

At the beginning of the Former (Western) Han dynasty (206 BCE – 8 CE), the *Zhuanxu* calendar (顓頊暦), a kind of *Sifen* calendar (四分暦), of the previous Qin dynasty (221-206 BCE) was still used. In this calendar, an intercalary month was put at the end of the relevant year.

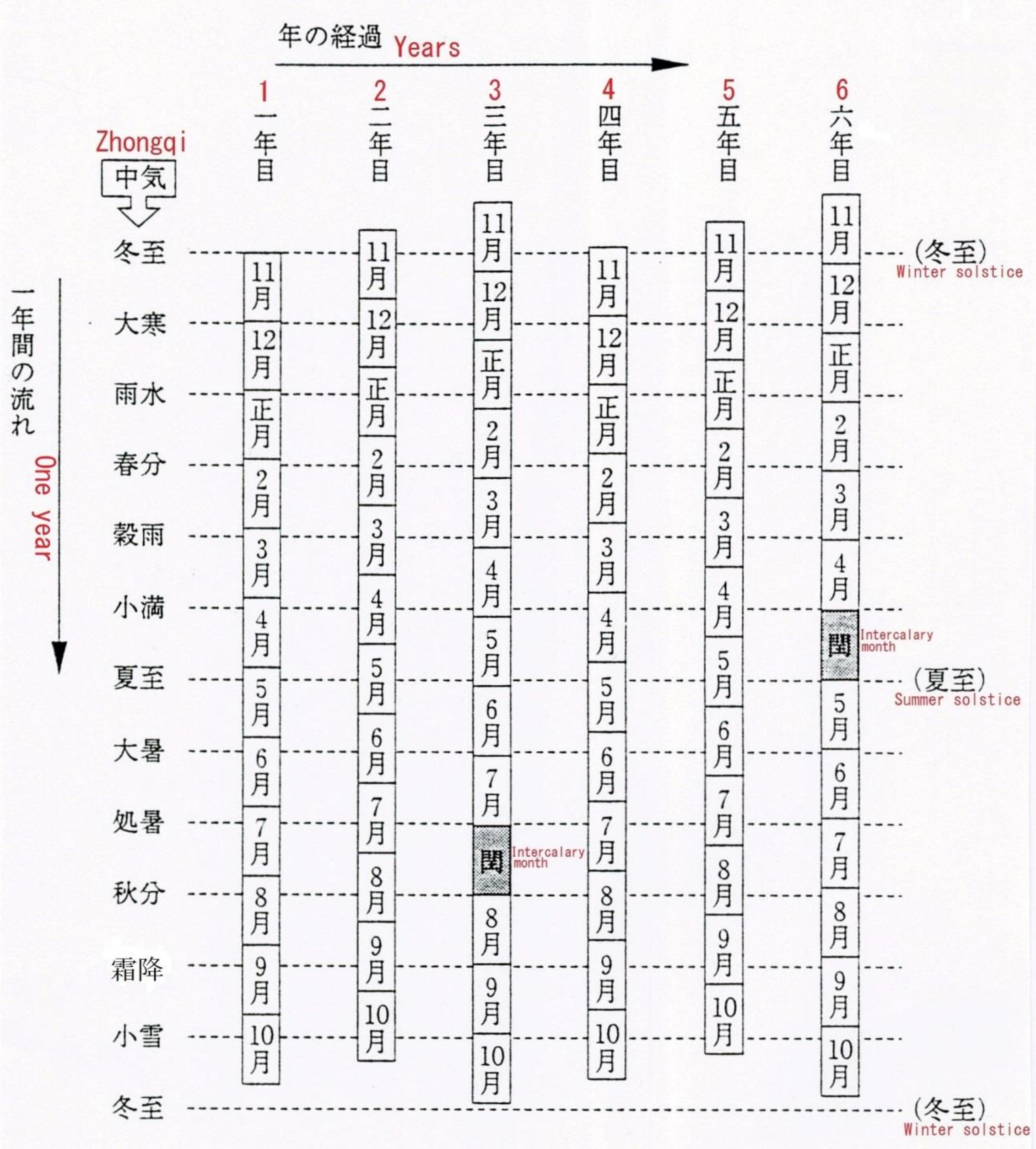
As the exact calendar was considered to be the symbol of the dynasty's authority, calendar reform was proposed in 104 BCE under the reign of Wu-di (武帝, Emperor Wu) (reign 141-87 BCE). The emperor ordered to make this year the first year of the new era *"Taichu"* (太初), and several intellectuals discussed about calendar reformation.

After the proposal of several calendars, the calendar made by DENG Ping (鄧平), which was the same as the calendar made by LUOXIA Hong (落下閎), was finally adopted. It was used from the fifth month of the first year of *"Taichu"* (104 BCE as the *Taichu* calendar. At that time, the celebrated historian SIMA Qian (司馬遷) was the "director of the Institute of chronology (and astronomy)" (*taishi-ling*), and DENG Ping was appointed to be the deputy director.

In the *Taichu* calendar, the 19-year cycle of intercalation was used as before, but the length a year was changed into 365 $\frac{385}{1539}$ days, and that of a synodic month 29 $\frac{43}{81}$ days. Its accuracy is almost the same as that of the *Sifen* calendar. Just denominators of fractions are different. (At that time, decimal fraction was not used.)

Sifen calendar: 1 year = 365.25 days, 1 month ≈ 29.53085 days.
Taichu calendar: 1 year ≈ 365.2502 days, 1 month ≈ 29.53086 days.

One merit of the *Taichu* calendar is the new method of intercalation. By the beginning of the Former Han dynasty, one year from the winter solstice to the next winter solstice was divided into 24 equal periods, and 24 points of time called *jieqi* (節気) were established. In the *Taichu* calendar, alternative 12 points called *zhongqi* (中気) were selected from the 24 *jieqi*, and the name (serial number) of a month was determined by the *zhongqi* which was included in the month. As the length of a synodic month is a little shorter than the interval of the *zhongqi*, sometimes a month without *zhongqi* is produced, and this month becomes an intercalary month. This method of intercalation was followed by later Chinese classical calendars.



(From Ôhashi (1998) with additional notes)

Santong calendar (三統曆)

At the end of the Former Han dynasty, LIU Xin (劉歆) (d.23 CE) added a kind of method of the prediction of lunar eclipses, a method to calculate the position of five planets, and the concept of grand epoch etc. This enlarged calendar is known as *Santong* calendar.

Development of the calendrical astronomy in the Later Han dynasty:

Chinese classical astronomy further developed in the Later (Eastern) Han dynasty (25 AD – 220 AD). A new calendar *Hou-Han Sifen* calendar (後漢四分暦) was made in 85 AD.

And also the armillary sphere was further developed. Previously, the armillary sphere in the Former Han dynasty was only used to measure equatorial coordinates. At the beginning of the Later Han dynasty, the observation of the sun and moon along the ecliptic was started, and it was discovered that the movement of the moon is not uniform even if it is measured along the ecliptic. It was also discovered that the lunar orbit is slightly inclined to the ecliptic.

LIU Hong (劉洪) made the *Qianxiang* calendar (乾象暦) in 206 CE, which was used in Wu kingdom of the Three Kingdoms period.

Cosmology in the Han (漢) period

At the time of Han dynasty, there were three theories of cosmology, namely:

the *gaitian* theory (蓋天説) (where the heaven and earth are flat),
the *huntian* theory (渾天説) (where the heaven is spherical), and
the *xuanye* theory (宣夜説) (where the heaven is infinite).

Among them, the *huntian* theory became the orthodox theory.

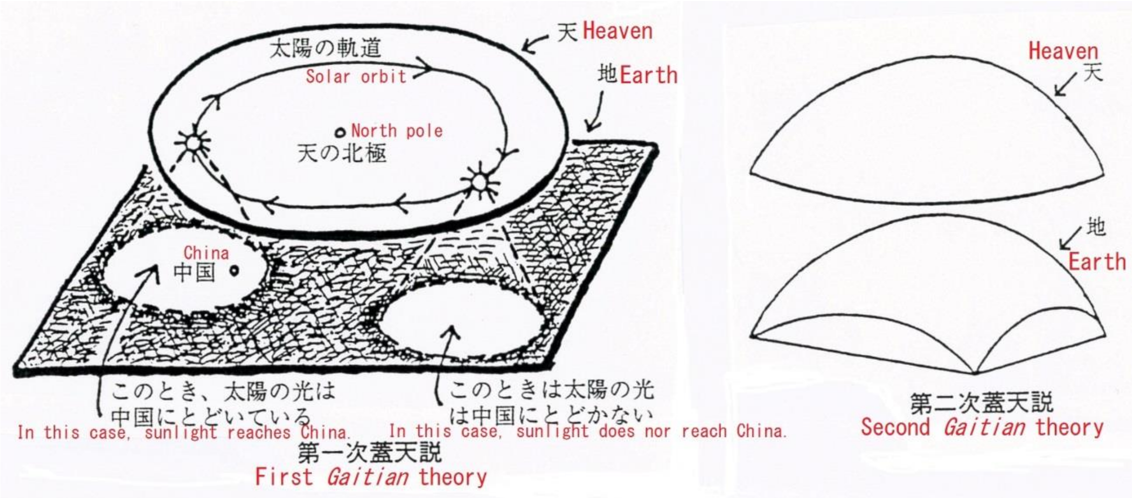
The *gaitian* theory (蓋天説):

The *gaitian* theory (蓋天説) (where the heaven and earth are flat) is explained in the *Zhoubi suanjing* (周髀算経), which is a work on mathematical cosmology.

The so-called "Pythagorean theorem" was utilized there. It was known in Ancient Mesopotamia, Ancient India and Ancient China most probably independently.

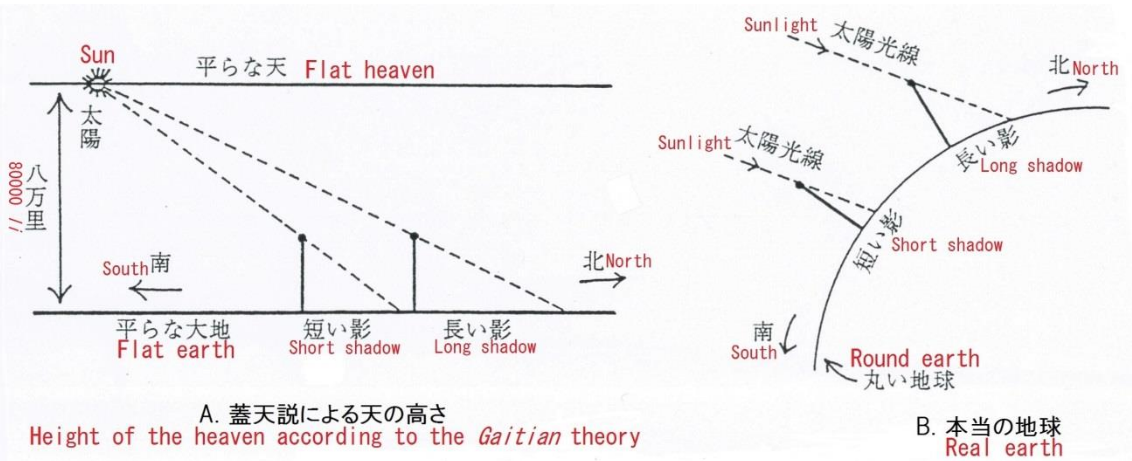
[For its detail, see Cullen (1996) and/or Hashimoto's Japanese translation of the text included in Yabuuti (1980).]

Two kinds of the *Gaitian* theory in the *Zhoubi suanjing* (周髀算経):



(From Ôhashi (1998) with additional notes)

The height of the heaven according to the observations of the midday shadow on the summer solstitial day in the North (long) and the South (short):

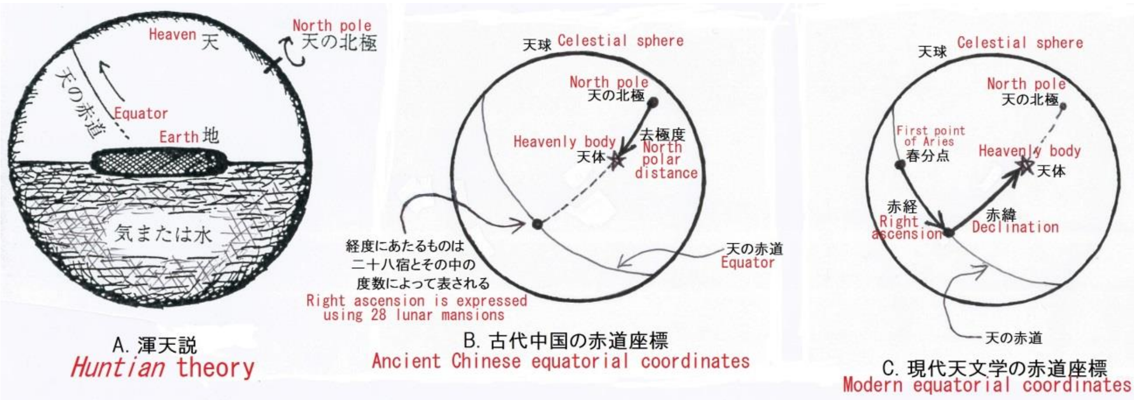


(From Ôhashi (1998) with additional notes)

The difference of the length of shadow is actually due to the fact that the earth is round, but in ancient China, the earth was believed to be flat. If the earth is flat, the method of the Gaitian theory is geographically correct.

The *huntian* theory (渾天説):

The *huntian* theory (渾天説) (where the heaven is spherical) became the most standard cosmology in ancient China. By this model, spherical coordinate system could be established on the heaven.



(From Ôhashi (1998) with additions)

ZHANG Heng (張衡), a famous astronomer in the Later (Eastern) Han dynasty, fully developed the *huntian* theory (渾天説).

He composed cosmological works, the *Lingxian* (霊憲) (Delicate law) and the *Hunyi* (渾儀) (Armillary sphere). The latter is sometimes called *Huntianyi-zhu* (渾天儀注) etc. In his *Hunyi*, ZHANG Heng wrote that the heaven is like the shell of a hen's egg, and the earth is at its centre like the egg's yolk.

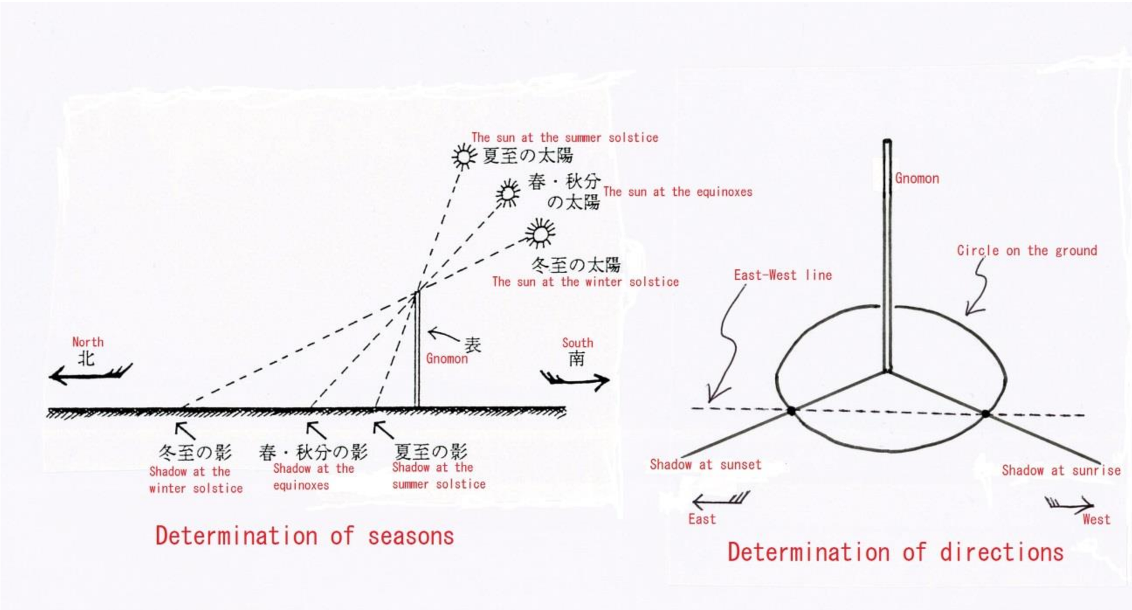
[For their Japanese translation, see Hashimoto's translation in Yabuuti (1980).]

The *xuanye* theory (宣夜説) (according to which the heaven is infinite) did not develop much.

Astronomical instruments

Gnomon ("*biao*", 表):

The gnomon was used since ancient period. It was used for determination of seasons, determination of directions etc. And also, it was very important in the *gaitian* theory as we have seen above.

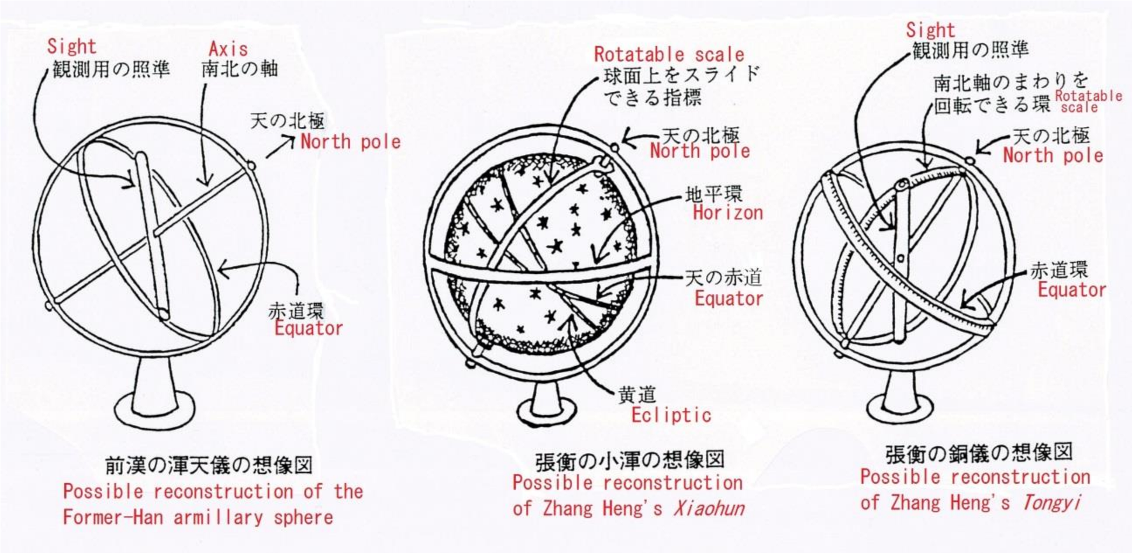


(From Ôhashi (1998) with additions)

Armillary sphere ("*huntianyi*", 渾天儀, or "*hunyi*", 渾儀):

The armillary sphere was used from the Former Han period. At the beginning, only the right ascension, based on the 28 lunar mansions, was observed. Later, the north polar distance was laso observed. In the Former Han period, only equatorial coordinate system was used. In the later Han period, the ecliptic was also considered.

According to his *Hunyi*, ZHANG Heng (張衡) constructed an armillary sphere called *"tongyi"* (銅儀) (bronze instrument) for observation, and a celestial globe called *"xiaohun"* (小渾) (small sphere) for demonstration and graphic calculation. According to a historical record (*Jin-shu*, 晉書), ZHANG Heng's celestial globe was rotated by waterpower in a room, and coincided with the actual sky precisely. Its construction is not recorded, but it is evidently the beginning of the water-driven celestial globe in China.

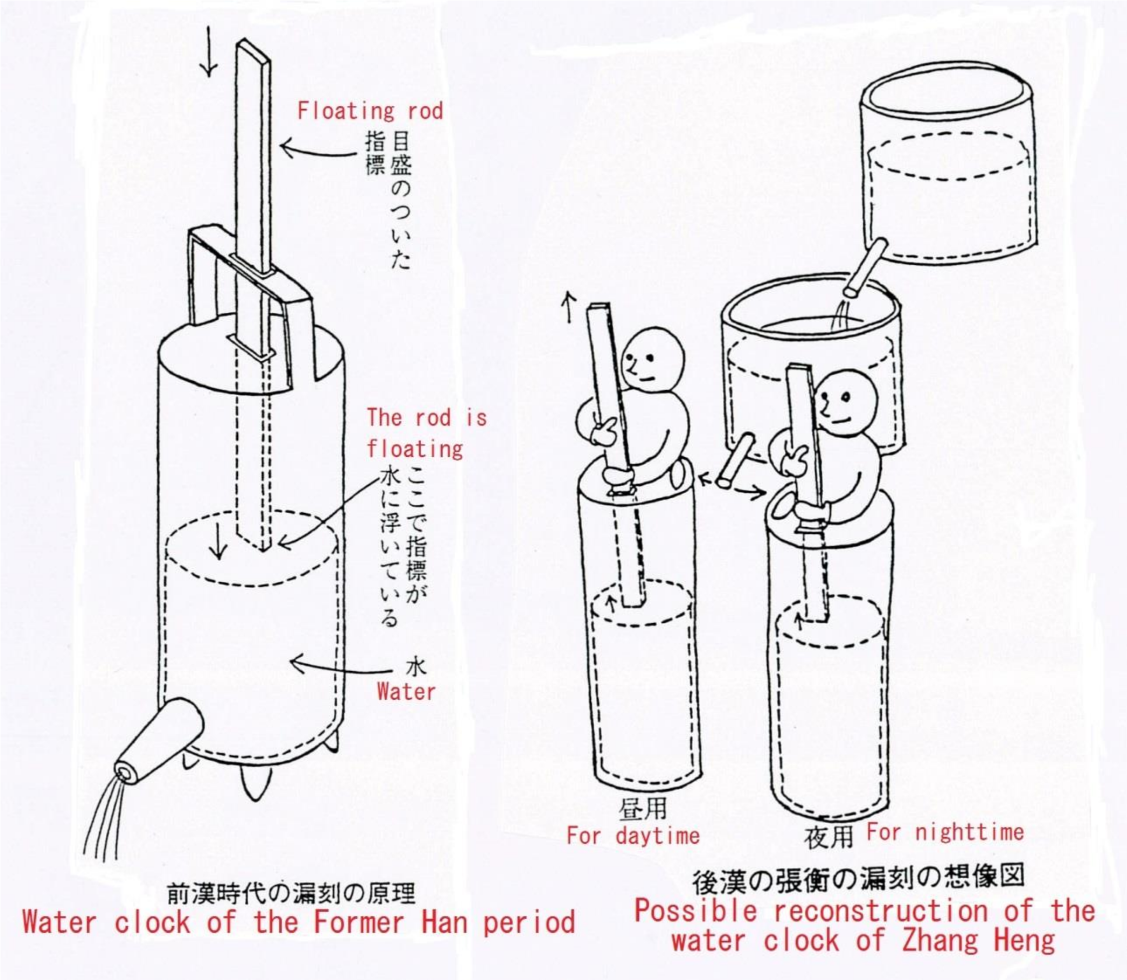


(From Ôhashi (1998) with additional notes)

Water clock (“louke”, 漏刻):

The water clock is said to have already been used in Chunqiu-Zhanguo period (770-221 BC), but its construction is not recorded. The extant water clocks date back to the Former Han dynasty, which are simple-outflow type water clock.

(The Water clock of the Former Han Period in the National Museum of Chinese History, Beijing)



(From Ôhashi (1998) with additional notes)

In the Later Han period, ZHANG Heng (張衡) constructed an inflow-type water clock with double reservoir.

The double reservoir is to make the water-flow constant. As water is supplied by the upper reservoir, the water level and water-flow of the lower reservoir do not decrease much. This is the first attempt to make the water-flow constant in China. Due to the difference of the length of daytime and nighttime, two acceptors were used for daytime and nighttime respectively.

Constellations in the Han (漢) period

Several constellations over the visible sky are described is the treatise of constellations (天官書) in the *Shiji* (史記, Record of history) (ca.91 BCE) of Sima Qian (司馬遷). Here, a little more than 90 constellations or a little more than 500 stars are described.

The *Jiuzhang suanshu* (九章算術) (Han (漢) Period)

The *Nine Chapters on the Mathematical Art* (九章算術, *Jiuzhang suanshu*) is a practical mathematical work, and is a fundamental mathematical work in ancient China.

The beginning of the *Jiuzhang suanshu*:

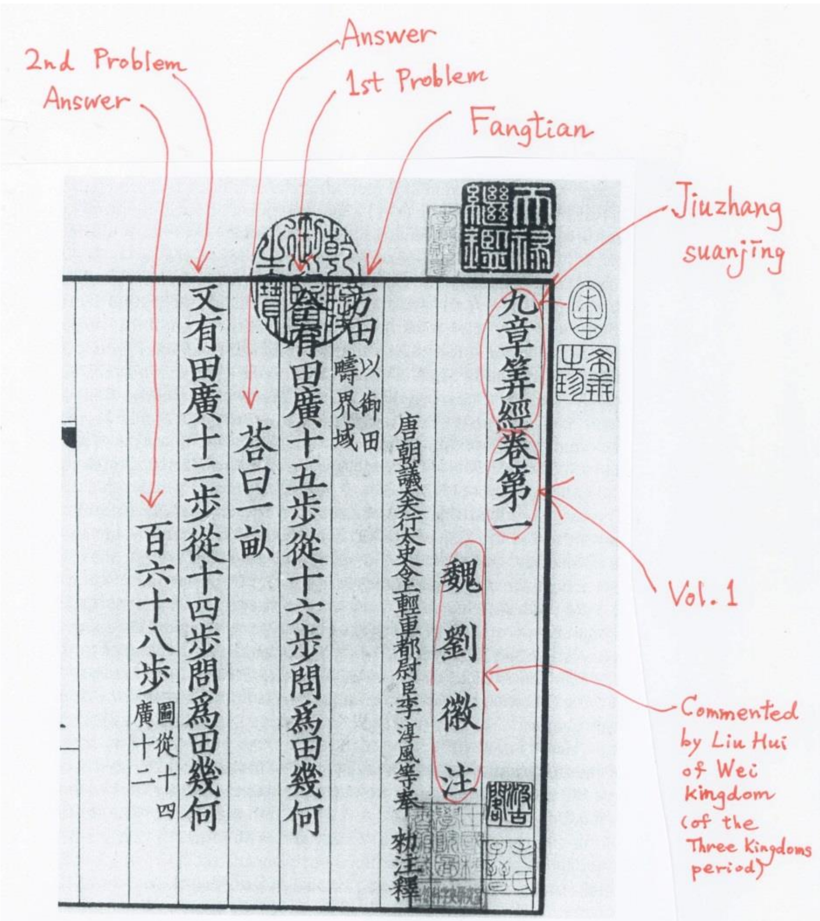


Fig. 2.5 The opening of Chapter 1 of the *Nine Chapters on the Mathematical Art* (Southern Sòng edition, in the Shànghǎi library).

(From Li and Du (1987), p.36, with my notes.)

(For the *Jiuzhang suanshu*, see Shen et. al. (1999) in English, and/or Japanese translation by Kawahara in Yabuuti (1980).

Contents of the *Nine Chapters on the Mathematical Art*:

The Nine Chapters on the Mathematical Art (*Jiŭzhāng suànshù*, 九章算術)

The *Nine Chapters on the Mathematical Art* has nine chapters and each chapter has a specific title. Each chapter gives the method of calculation for one or more types of particular example.

Chapter I is called 'Field measurement' (方田, Fāng tián). The central theme is the calculation of the areas of cultivated land. The character 方 (fāng) is the unit for measuring areas; it means 'square unit', 田 (tián) means 'field' and 方田 (fāng tián) means 'to calculate how many square units a field contains'. In addition this chapter contains a detailed discussion of computations with fractions.

Chapter II is called 'Cereals' (粟米, Sù mǐ, literally millet and rice). It discusses various problems to do with proportions and in particular it is concerned with proportions for the exchange of cereals.

Chapter III is called 'Distribution by proportion' (衰分, Cuī fèn). 衰 Cuī means 'by proportion', 分 (fèn) means 'to distribute'. What is discussed are problems on proportional distribution.

Chapter IV is called 'What width?' (少廣, Shǎo guǎng). 少 (shǎo) means 'how much' and 廣 (guǎng) means 'width'. Shǎo guǎng means, given the area or volume, to find the length of a side. In this chapter the methods for finding square and cube roots are also explained.

Chapter V is 'Construction consultations' (商功, Shāng gōng). 商 (shāng) means to discuss or negotiate, 功 (gōng) means construction. This chapter deals with various kinds of calculations for constructions. It is mainly about the calculation of the volumes of various shapes of solid.

Chapter VI, 'Fair taxes' (均輸, Jūn shū) is about calculating how to distribute grain and corvée labour (see p. 272) in the best way according to the size of the population and the distances between places.

Chapter VII, 'Excess and deficiency' (盈不足, Ying bù zú), is about the use of the method of false position for solving some difficult problems. Let us take the first problem in chapter VII as an example. The original text says: 'Consider a group of people purchasing. Each person contributes 8, and 3 are left over; 7 are contributed, 4 is the deficit.' It asks: 'How many people and what is the price?' There are two hypotheses. If p is the price and n is the number of people, the first hypothesis gives $8n = p + 3$, the second gives $7n = p - 4$. This is a problem typical of the 'Excess and deficiency' chapter.

Chapter VIII, 'Rectangular Arrays' (方程, Fāng chéng), is about equations. It discusses problems on simultaneous linear equations and it also discusses the concepts of positive and negative numbers and the methods of addition and subtraction of positive and negative numbers.

Chapter IX is on 'Gōugŭ (勾股). It discusses the Gōugŭ theorem and problems on similar right-angled triangles. In this chapter general methods of solving quadratic equations are also introduced.

The contents of the *Nine Chapters on the Mathematical Art* are comprehensive and interesting and at the same time they are closely connected with practical life. The topics that are closely connected with real life reflect the collective wisdom and abilities of the people of ancient China. But who is the author of this outstanding work, and when was this material assembled into a book? Even at the present time we are unable to give a precise and definitive answer, but according to the material presently available we can conclude that this work dates, at the latest, from about the first century AD (in the middle of the Eastern Hàn Dynasty).

(From Li and Du (1987), pp.33-34)

Three kingdoms ("Sanguo" (三國)) period (220 ~ 265)

(From 中国历史地图及大事年表, 北京, 中国地图出版社, 2013.)

Jingchu calendar (景初曆)

The *Jingchu* calendar (景初曆) of YANG Wei (楊偉) in the Sanguo (Three kingdoms) period was officially used from 237 CE by the Wei (魏) dynasty of the Sanguo period. Its treatise is recorded in the calendrical chapter of the *Jinshu* (晉書). It officially started to predict solar and lunar eclipses using the node distance.

It uses two ecliptic limits. One is the limit of definite eclipses, and the other is the limit of small eclipses. YANG Wei must have found these limits experimentally, and these limits are enough good.

CHEN Zhuo (陳卓)'s constellations

CHEN Zhuo (陳卓) (fl. ca.265–317 CE), an astronomer of the Wu (呉) dynasty of the Sanguo (Three kingdoms) period and the subsequent Western Jin dynasty, made a comprehensive survey of constellations, and recorded 283 constellations or 1465 (or 1464) stars.

Constellations other than lunar mansions were divided into three groups, and were attributed to three ancient legendary astronomers, GAN De (甘徳) and SHI Shen (石申) of the "Warring states" period and WU Xian (巫咸) of Shang (= Yin) dynasty, respectively.

Although CHEN Zhuo's own work is not extant, his system of constellations has become the standard system of East Asian traditional constellations, and used in pre-modern Korea, Japan etc. also..

LIU Hui (劉徽)'s mathematics

LIU Hui (劉徽) (fl.263) was a mathematician in the Wei (魏) kingdom of the Three Kingdoms period. He wrote a commentary on the *Nine Chapters on the Mathematical Art* (九章算術, *Jiuzhang suanshu*), and wrote the *Sea Island Mathematical Manual* (海島算經, *Haidao suanjing*), which is a kind of supplement to the *Nine Chapters on the Mathematical Art*.

LIU Hui, using inscribed regular polygons, calculated that π is 3927/1250 (= 3.1416).

The *Sea Island Mathematical Manual* (海島算經, *Haidao suanjing*) is a work on surveying.

[For an English translation of the *Sea Island Mathematical Manual*, see Swetz (1992).]

Let us see an example from the *Sea Island Mathematical Manual* (海島算經):

The Sea Island Mathematical Manual (海島算經 , *Hǎidǎo suànjīng*)

The contents of the *Sea Island Mathematical Manual* concern the 'method of double differences' (重差 , chóng chā), a method used in surveying.

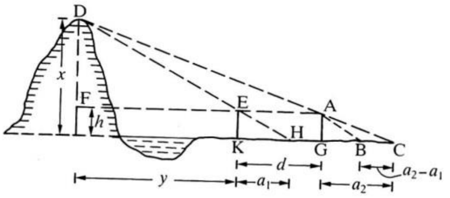


Fig. 3.11 Calculating the height and distance of a sea island.

Why is it called the 'double differences method'? We can explain this by summarizing Problem 1 in the *Sea Island Mathematical Manual*.

The content of Problem 1 is as follows: Observe a sea island whose height and distance are unknown. Erect two poles (表, biǎo) AG and EK, as in Fig. 3.11. The height of the poles is b feet and the distance between the two poles is d paces, while the two poles and the island are lined up in the same vertical plane. Step back a_1 paces from the front pole (EK) so as to observe the top of the pole and the top of the island in the same straight line when the eye is at ground level. Again, step back a_2 paces from the rear pole and observe the top of the pole and the top of the island in the same straight line with the eye at ground level. It is required to find the height of the island (x) and the distance between the front pole and the island (y).

The solution in the original text of the *Sea Island Mathematical Manual* is:

Use the pole height [h] multiplied by distance between poles as numerator, the differences [$a_2 - a_1$] as denominator. The result obtained added to the pole height gives the height of the island. To find the distance from the front pole to the island [y], use the distance walked forward from the front pole [a_1] multiplied by the pole distance [d] as numerator, the differences [$a_2 - a_1$] as denominator to get the number of miles from island to pole.

Using modern algebraic notation the above explanation yields the formulae:

$$x = \frac{d}{a_2 - a_1} \cdot h + h \,,$$

$$y = \frac{d}{a_2 - a_1} \cdot a_1 \,.$$

These formulae can be proved using Fig. 3.11. Let D be the top of the island. From A construct AB//DE. Then knowing ΔABC is similar to ΔADE and also ΔACG is similar to ΔADF we have

$$\frac{AE}{BC} = \frac{d}{a_2 - a_1} = \frac{AD}{AC} = \frac{DF}{AG} = \frac{DF}{h} \,,$$

$$\text{and hence } x = DF + h = \frac{d}{a_2 - a_1} \cdot h + h \quad \text{QED.}$$

Again because Δ EKH is similar to Δ DFE,

$$\frac{y}{a_1} = \frac{EF}{KH} = \frac{DF}{EK} = \frac{DF}{h} = \frac{\frac{d}{a_2 - a_1} \cdot h}{h} = \frac{d}{a_2 - a_1} \,,$$

$$\text{hence } y = \frac{d}{a_2 - a_1} \cdot a_1 \quad \text{QED.}$$

(From Li and Du (1987), pp.75-77)

Introduction of Indian astronomy

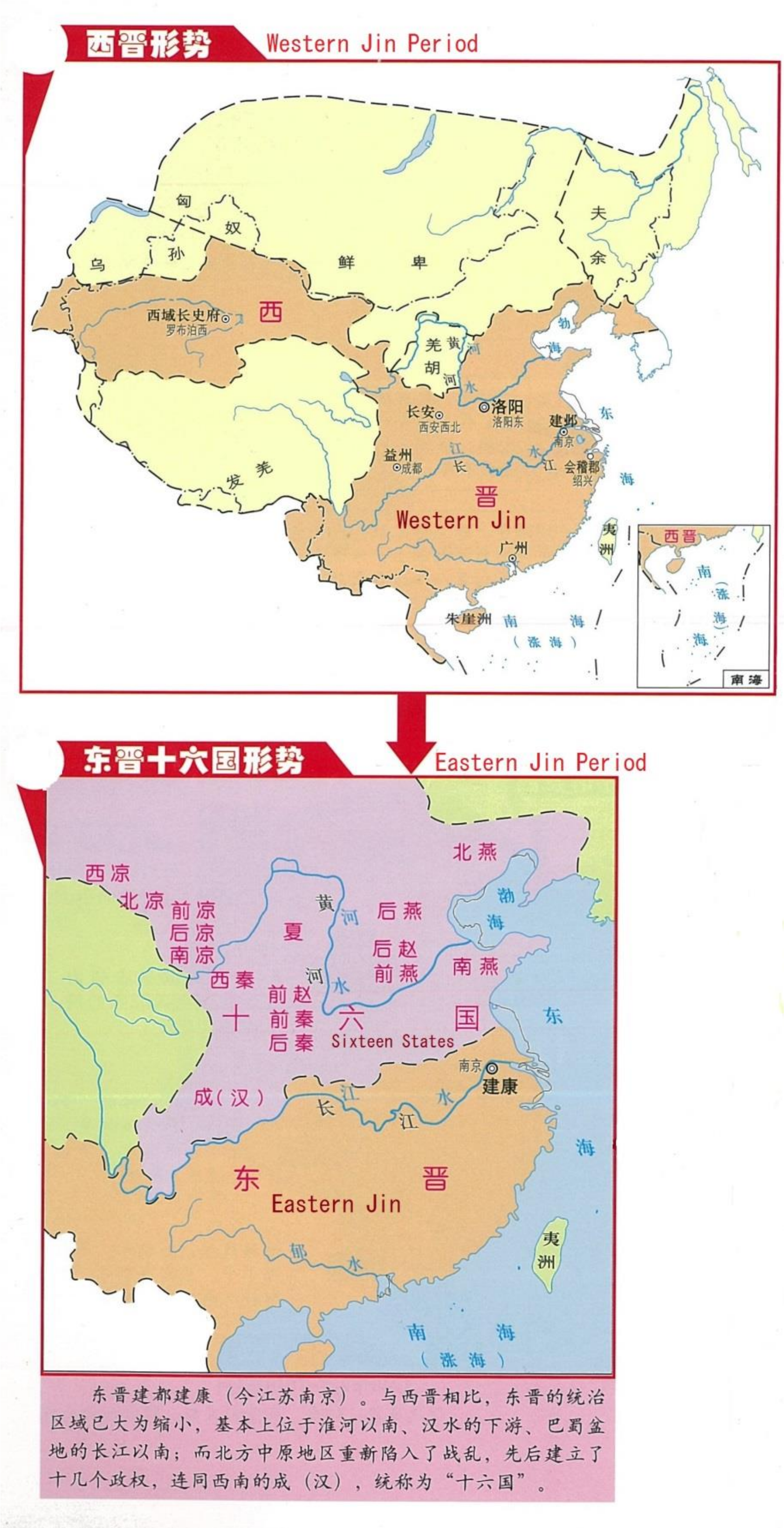
At the time of Sanguo (Three Kingdoms) period (the mid 3rd century CE), a Buddhist text called *Śārdūlakarṇa-avadāna* in Sanskrit was translated into Chinese by ZHU Lüyan (竺律炎) and ZHI Qian (支謙) as the *Madengqie-jing* (摩登伽經). This is the first Chinese text where Indian astronomy and astrology are explicitly mentioned. This text explains the lunar mansions and astrology based on them at length, and also mentions some calendrical information.

The astronomical system of the *Śārdūlakarṇa-avadāna* belongs to the stage of the *Vedāṅga* astronomy, which is one of the six branches of the auxiliary learning for the *Veda*. According to my study, the *Vedāṅga* astronomy was produced in North India sometime during the 6th and the 4th centuries BCE. The description of astrology in the *Śārdūlakarṇa-avadāna* is also based on Indian traditional system.

The original Sanskrit version of the *Śārdūlakarṇa-avadāna* has the description of the annual variation of the gnomon-shadow, which is similar to that of the *Vedāṅga* astronomy. The Chinese version *Madengqie-jing* also has the description of the annual variation of the gnomon-shadow, but it is different from the Sanskrit original. SHINJŌ Shinzō (新城新藏), a pioneer of the study of the history of Eastern astronomy in Japan, pointed out that the description of the *Madengqie-jing* is based on the data around 43°N, and that the data might have been incorporated in Central Asia.

The *Śārdūlakarṇa-avadāna* was also translated into Chinese as the *Shetoujian-taizi ershiba-xiu jing* (舎頭諫太子二十八宿経) by ZHU Fahu (竺法護) at the time of the Western Jin (西晋) dynasty (265 – 316).

Jin (晋) dynasty (265 ~ 420)



(From 中国历史地图及大事年表，北京，中国地图出版社，2013.)

PEI Xiu (裴秀)'s cartography

PEI Xiu (裴秀) (224 – 271) was a cartographer from the Wei (魏) kingdom of the Three Kingdoms period to the Western Jin (西晋) dynasty. He established a theory to make maps. The map made by him is not extant.

YU Xi (虞喜)'s discovery of the precession of equinoxes (歳差)

At the time of the Eastern Jin dynasty (317 ~ 420), YU Xi (虞喜) (281 ~ 356) discovered the precession of equinoxes. The precession had already been discovered by a Greek astronomer Hipparchus (2nd century BCE), but YU Xi's discovery must have been independent of Hipparchus.

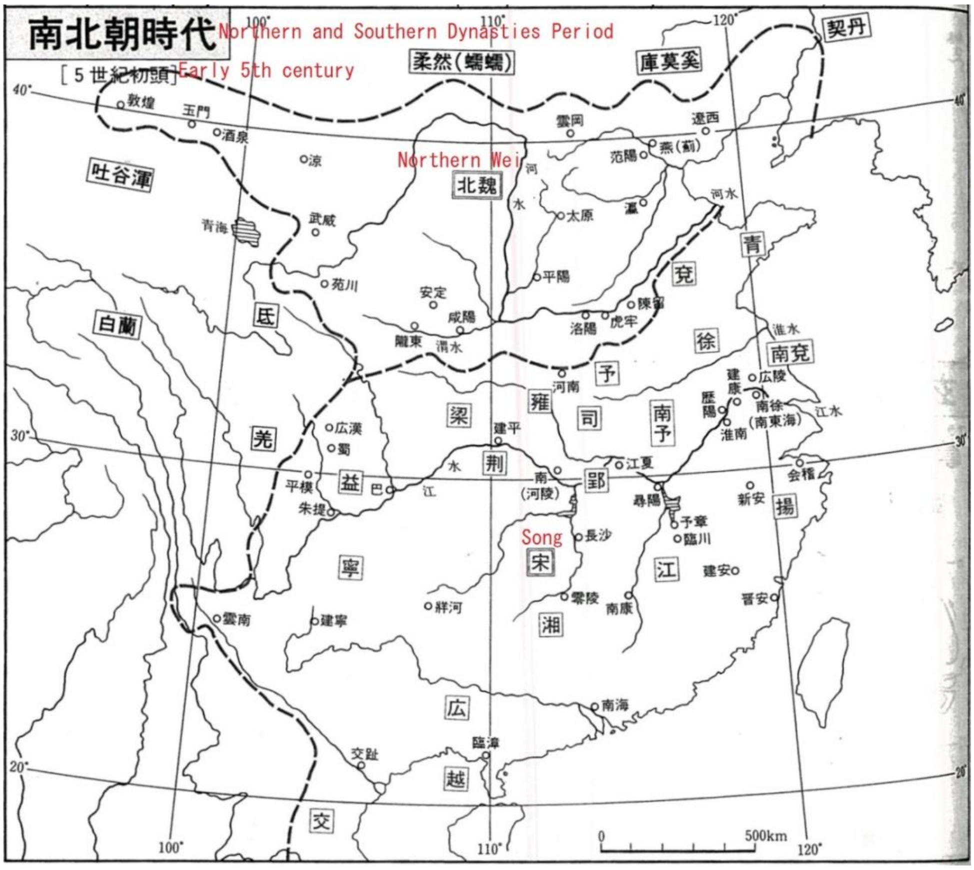
Yixing (一行) (683 ~ 727), a celebrated monk astronomer of the Tang dynasty, described the discovery of YU Xi as follows in his discourse *Dayan-liyi* (大衍暦) (Discourse on the *Dayan* calendar) (article 7) (recorded in the *Xin-Tangshu* (新唐書) (New official history of the Tang dynasty), *Lizhi* (暦書) (Chapter of calendar) (III-1)). I would like to note one thing before translating the discourse that one "Chinese degree" was the angular distance (on the celestial sphere) which was traversed by the (mean) sun in one day, and "[the degrees of] the circumference of the celestial sphere" in the following quotation is the same as the number of days in a sidereal year. Therefore, one Chinese degree is slightly smaller than one modern degree. Yixing wrote:

> "In the old calendars, [the movement of] the sun was uniform, and [the degrees of] the circumference of the celestial sphere was the same as the length of a [tropical] year. Therefore, the positions of the stars were fixed to the divisions of season. This theory looks true but is not so actually, and errors increase in a long term. YU Xi noticed this fact, and differentiated the circumference of the celestial sphere and the [tropical] year. He investigated the difference and traced its effect, and concluded that [the position of the sun at certain season] retrogrades 1 degree (Chinese degree) in 50 years." (Translated by me from the *Xin-Tangshu*, *Lizhi* (III-1).)

According to the above quotation, it is clear that YU Xi understood the precession of equinoxes and the difference between the sidereal year and the tropical year correctly. The exact value of the precession is 1° per about 71.6 years, and the value of YU Xi was slightly larger.

At the time of YU Xi, Greek influence is not found in Chinese astronomy, and the discovery of YU Xi must be independent of Hipparchus (2nd century BCE).

Northern and southern dynasties ("Nanbeichao" (南北朝)) period (420 ~ 589)



(From 『大修館 現代漢和辞典』、大修館、1996.)

HE Chengtian (何承天)

In the Song (宋) (Liu-Song) dynasty (420-479), the first dynasty of the Southern dynasties, an astronomer HE Chengtian (何承天) (370-447) made an excellent calendar called *Yuanjia* calendar (元嘉曆). It was used in Japan also.

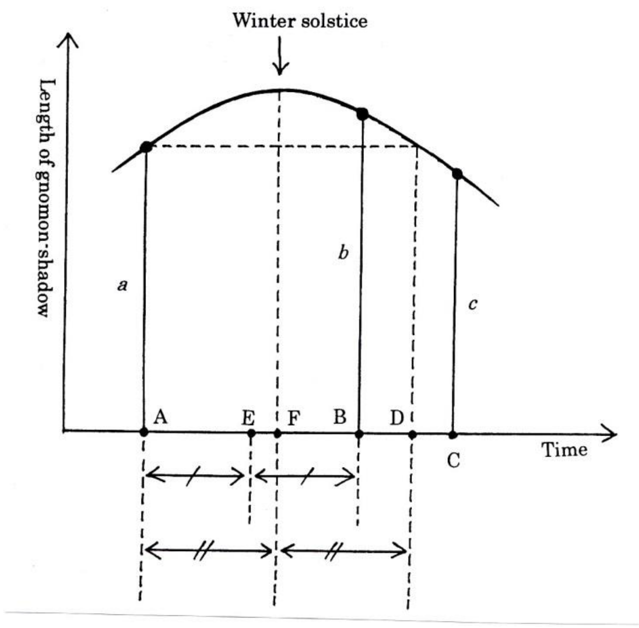
ZU Chongzhi (祖沖之)

ZU Chongzhi (祖沖之) is a Chinese mathematician and astronomer in the Southern dynasties in the Northern and southern dynasties period.

ZU Chongzhi was born in 429, and died in 500 AD. ZU Chongzhi worked as a government officer of Song (宋) (Liu-Song) dynasty and Qi (斉) dynasty. He made a new calendar entitled *Daming* calendar (大明暦), and requested to use it officially in 462 AD, but was severely opposed, and the calendar was not accepted.

ZU Chongzhi's son ZU Gengzhi (or ZU Geng) was also a mathematician and astronomer. Thanks to ZU Gengzhi's effort, the *Daming* calendar was officially used since 510 CE.

ZU Chongzhi devised a method to determine the exact time of winter solstice from the observations of the midday gnomon-shadow. ZU Chongzhi's explanation is recorded in the *Song-shu* (宋書), *Lizhi* (暦志) (III).



For the determination, three observations (A, B, and C in the figure) of the midday gnomon-shadow (a, b, and c) are used. Here, $b > a > c$, and the period BC is one day. In ancient China, one day was divided into 100 *"ke"* (刻). Let an imaginary gnomon-length at D (between B and C) be equal to a. The point E is the midpoint of AB, and F the midpoint of AD, that is the time of winter solstice. Then, EF is a half of BD. Now, by linear interpolation,

$$\text{BD} = \frac{100 \times (b-a)}{b-c} \ \textit{ke}.$$

Therefore,

$$\text{EF} = \frac{100 \times (b-a)}{2 \times (b-c)} \ \textit{ke}.$$

As the time E is already known, the time F of winter solstice is obtained from this equation.

ZU Chongzhi (祖沖之) was also a great mathematician. He calculated that the value of π lies between 3.1415926 and 3.1415927. He is said to have composed a high-grade mathematical work *Zhuishu* (綴術), which is not extant by now.

ZHANG Zixin (張子信)

ZHANG Zixin (張子信) discovered the inequality corresponding to the equation of centre of the sun in the 6th century AD at the time of the Northern dynasties.

According to the *Suishu* (隋書) (Official history of the Sui dynasty), *Tianwenzhi* (天文志) (Chapter of astronomy) (III), ZHANG Zixin renounced the world and observed heavenly bodies for about 30 years, and found that the movement of the sun is slow after the vernal equinox, and is fast after the autumnal equinox.

By this time, the equation of centre of the sun was already known in the ancient Mediterranean world and also in India, but ZHANG Zixin' discovery must be independent of the ancient Mediterranean and Indian astronomies.

Buddhist astronomy in the Northern and Suthern Dynasties Period

In the Buddhist text *Yuecang-fen* (月蔵分), translated into Chinese at the end of Northern Qi (北斉) (566 CE) and included in an anthology of Buddhist texts *Daji-jing* (大集經), zodiacal signs are mentioned.

Sui (隋) dynasty (581 ~618), and Tang (唐) dynasty (618 ~ 907)

隋朝定都长安（今陕西西安）。其疆域东到东海，南到南海，西到今新疆东部、广西和越南北部，北到内蒙古北部，东北至辽河。隋时期，周边地区生活着突厥、契丹、靺鞨等部族和政权。

唐朝定都长安（今陕西西安），前期的疆域广大，东到东海，西达咸海，南及越南中部，北至西伯利亚，大大超过了西汉鼎盛时期。周边其他部族，如东北地区的室韦、靺鞨诸部，西南地区的吐蕃等，与唐友好往来。

(From 中国历史地图及大事年表，北京，中国地图出版社，2013.)

Development of calendars in the Sui and Tang periods

In the Sui (隋) (581 – 618) and Tang (唐) (618 – 907) dynasties, several good calendars were made. A rough history of calendars before Yixing in this period is as follows. The *Huangji* calendar (皇極暦) (600 CE) of LIU Zhuo (劉焯) (544 – 610) was not officially used, but was an excellent calendar where the inequalities corresponding to the equation of centre of the sun and of the moon and the precession of the equinoxes were considered, and also the second order interpolation was used for the first time in China. The *Linde* calendar (麟徳暦) (665 CE) of LI Chunfeng (李淳風)(602 – 670) is also a famous calendar. LI Chunfeng is also famous for his armillary sphere. The *Linde* calendar was used in Korea and Japan also.

Then, Yixing (一行) composed his *Dayan* calendar (大衍暦)(727 CE).

The *Dayan* calendar (727 CE) of Yixing is one of the best calendars of the Tang dynasty. After Yixing, the *Xuanming* calendar (宣明暦)(822 CE) of XU Ang (徐昂) is also famous, and the method of the prediction of eclipses was improved. And also, the *Chongxuan* calendar (崇玄暦)(892 CE) of BIAN Gang (邊岡) also contains several devices. The *Dayan* calendar and the *Xuanming* calendar were used in Japan also. The *Xuanming* calendar was also used in Korea.

Introduction of Indian astronomy in the Sui and Tang periods

In the Sui dynasty, some Indian works on Indian mathematics and astronomy were introduced into China, but they are not extant.

In the Tang dynasty, a detailed work of Indian mathematical astronomy, the *Jiuzhi* calendar (九執暦) (718 CE), was composed in Chinese by an Indian astronomer (resident in China since his grandfather) Qutan Xida (瞿曇悉達) (Chinese transliteration of Gotama-siddha in Sanskrit), and was included in his ((*Da*-)*Tang*-) *Kaiyuan-zhanjing* (((大)唐)開元占經).

Yixing had certain knowledge of Indian astronomy, but made his *Dayan* calendar in Chinese traditional style. This fact should not be forgotten.

In the 8th century, a Chinese version of Indian astrology, the *Xiuyao-jing* (宿曜經), was composed in Chinese by an Indian monk Bukong (不空) (whose Sanskrit name was Amoghavajra)(705-774). Amoghavajra was a disciple of Vajrabodhi, from whom Yixing also studied.

Yixing (一行) (683 – 717):

Yixing (一行)(683 – 727)[2)] was a Chinese Buddhist monk and astronomer in the Tang (唐) dynasty (618 – 907) of China. "Yixing" is his Buddhist name, and his secular name is ZHANG Sui (張遂). He is sometimes called "Seng Yixing" (僧一行) (Monk Yixing) or "Yixing chanshi" (一行禪師) (Zen master Yixing).

Yixing was born in 683, and died in 727 CE. In 717 CE, Yixing received a call of the Emperor Xuanzong(玄宗), and moved to Chang'an(長安) (present-day Xi'an(西安)) the then capital. After that, Yixing studied Esoteric Buddhism from Indian monks Śubhakarasiṁha (whose Chinese name is Shan-wuwei (善無畏)) and Vajrabodhi (whose Chinese name is Jingang-zhi (金剛智)).

In AD 721, Yixing started a project to make a new calendar at the Emperor's request. Yixing made an armillary sphere with his colleague LIANG Lingzan (梁令瓚) in around AD 724, and observed stars. From AD 724, Yixing conducted astronomical observations at several places all over China with his colleague NANGONG Yue (南宮說), and determined the altitude of the celestial north pole (which corresponds to the altitude) of those places. In AD 725, Yixing made a water-driven celestial globe with Liang Lingzan.

After these preparations, Yixing started to compile a new calendar, and completed the draft of the new *Dayan* calendar (大衍曆) in AD 727.

As Yixing expired in this year, ZHANG Shui (張說) and CHEN Xuanjing (陳玄景) edited Yixing's draft, and the *Dayan* calendar was officially used since AD 729.

The *Dayan* calendar (727 CE) of Yixing is one of the best calendars of the Tang dynasty, and was used in Japan also.

The Ten Books of Mathematical Manuals

Tem mathematical books were used as text books of mathematics since mid-7th century, namely:

Zhoubi suanjing (周髀算經), a book on mathematical cosmology of the Han period.

Jiuzhang suanshu (九章算術)), "Nine chapters on the Mathematical Art" of the Han period.

Haidao suanjing(海島算經), "Sea Island Mathematical Manual" of of Liu Hui (劉徽) in the Wei state of the Three Kingdoms period.

Sunzi suanjing (孫子算經), "Master Sun's Mathematical Manual" (4th – 5th century).

Xiahou Yang suanjing (夏侯陽算經), "Xiahou Yang's Mathematical Manual" (4th – 5th century).

Zhang Qiujian suanjing (張邱建算經), "Zhang Qiujian's Mathematical Manual" (4th – 5th century).

Zhuishu (綴術) of Zu Chongzhi (祖沖之) and Zu Gengzhi (祖暅之) (now lost).

Wucao suanjing (五曹算經), "Mathematical Manual of the Five Government Department" of Zhen Luan (甄鸞) of the Northern Zhou state of the Northern and Southern Dynasties period.

Wujing suanjing (五經算經), "Arithmetic in the Five Classics" of Zhen Luan (甄鸞) of the Northern Zhou state of the Northern and Southern Dynasties period.

Xugu suanjing (輯古算經), "Continuation of Ancient Mathematics", of Wang Xiaotong (王孝通) of the Tang period.

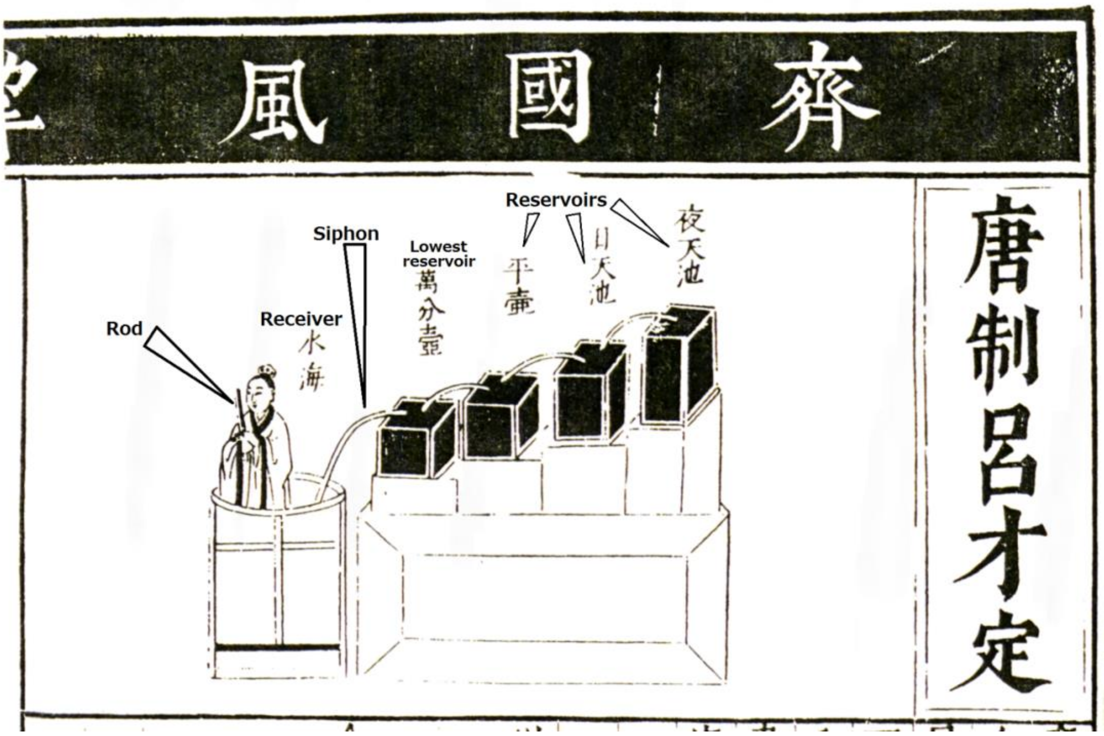
Among them, the *Zhuishu* (綴術) has been lost, and later the *Shushu jiyi* (数術記遺) "Memoir on some Traditions of Mathematical Art" of Zhen Luan (甄鸞) was supplied, and they are now known as *Suanjing shishu* (算經十書) "Ten Mathematical Manuals".

Song (宋) dynasty (960 ~ 1279)
（Northern Song（北宋）(960 ~ 1127）and Southern Song (南宋) (1127 ~ 1279))

(From 中国历史地图及大事年表, 北京, 中国地图出版社, 2013.)

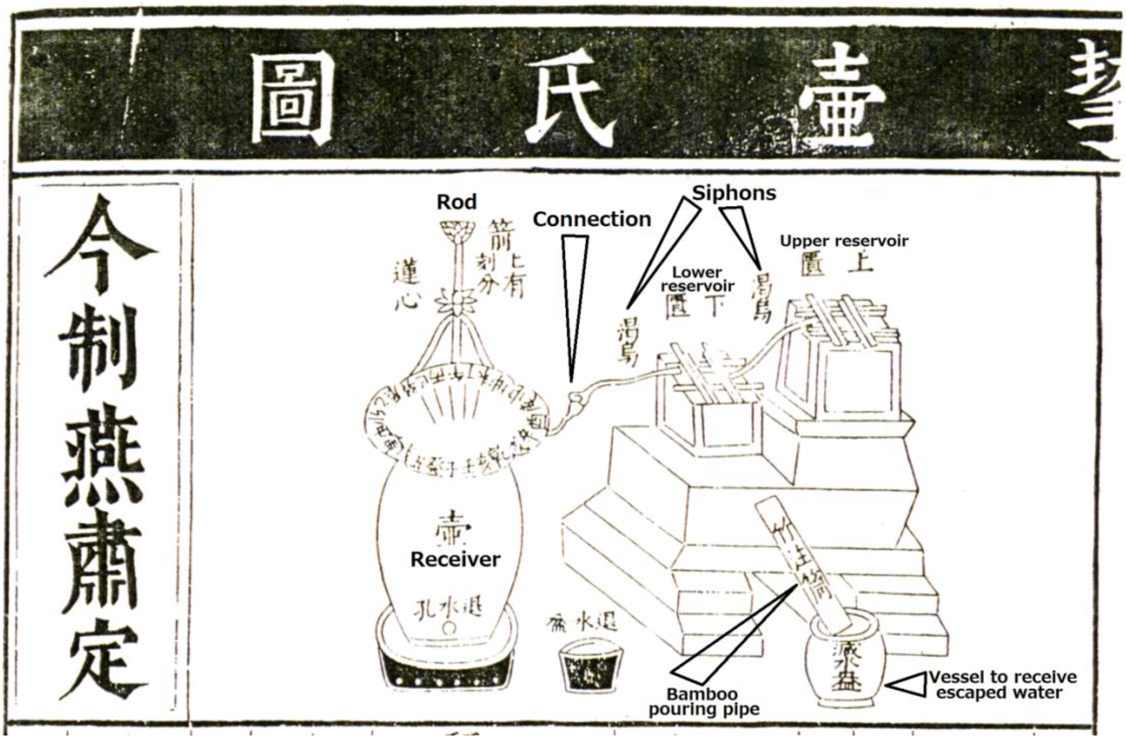
Development of water clock in the Song (宋) period

In the Tang (唐) dynasty (618-907), Lü Cai (呂才) made an inflow type of water clock with fourfold reservoir, where upper three reservoirs are to supply water, in the 7th century. In this clock, siphons were used in order to supply water.

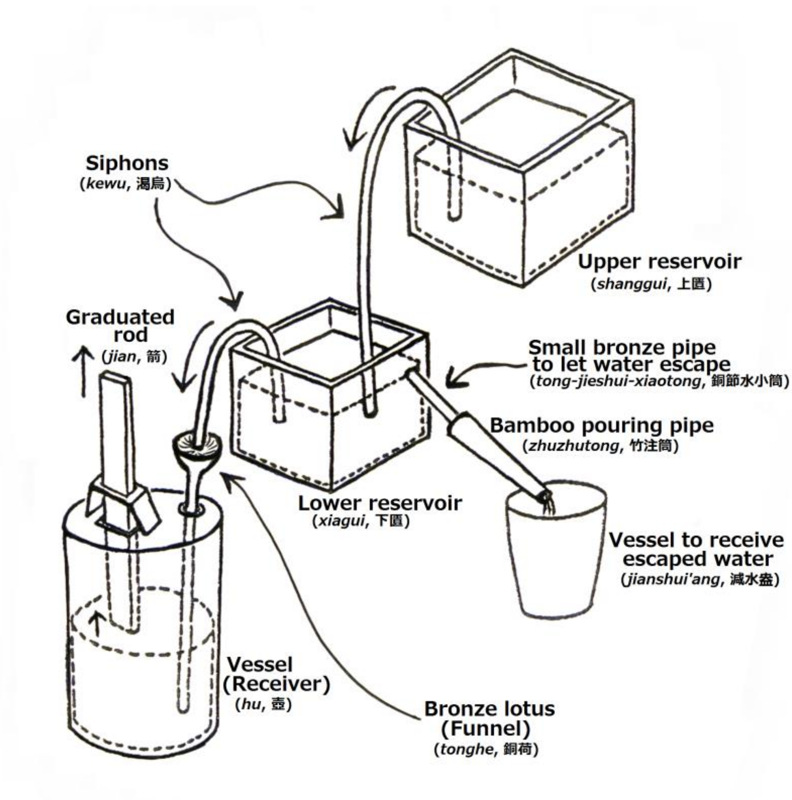


Lü Cai's water clock (from the *Liujingtu* (六經圖), with my notes)

In the Northern Song (北宋) dynasty (960-1127), Yan Su (燕肅) made a kind of ultimate water clock in 1030 CE. In this instrument, water is exceedingly supplied by the upper reservoir to the lower reservoir, and water overflows through a tube attached to the lower reservoir so that the water level of the lower reservoir is at the height of the tube forever. This type of water clock was improved in around 1050 CE.



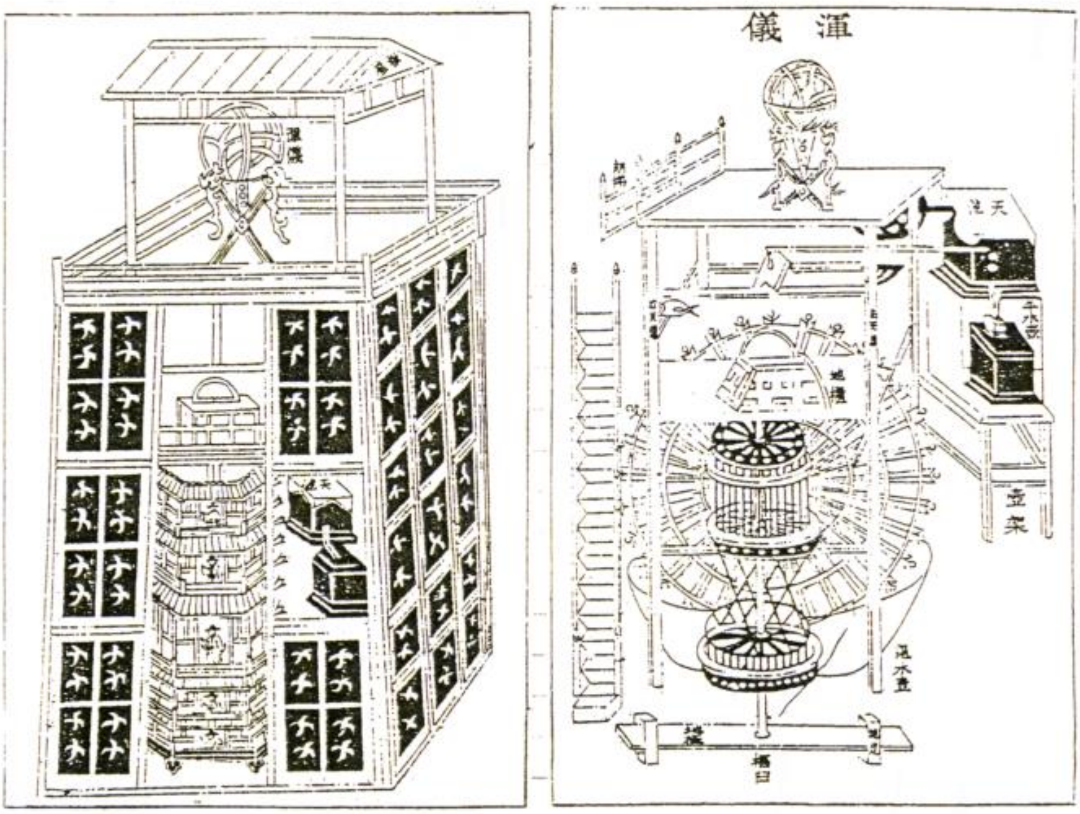
Yan Su's water clock (from the *Liujingtu*, with my notes)



Yan Su's water clock (reconstruction)

The device of overflow type of the water clock was also utilized by Su Song (蘇頌) (1020-1101 AD) of the Northern Song（北宋）dynasty, who made a water driven mechanical clock（水運儀象台), and described its construction in his *Xin-yixiang-fayao* (新儀象法要) at the end of the 11th century AD.

(For its detailed construction, see Needham et al. (1960, 1986) in English, and Yamada and Tsuchiya (1997) in Japanese.)



Su Song's Shuiyun-yixiang-tai (from the *Xin-yixiang-fayao*)

Star maps and maps in the Song (宋) period

Methods of the projection of star maps:

(1), Circular star map:

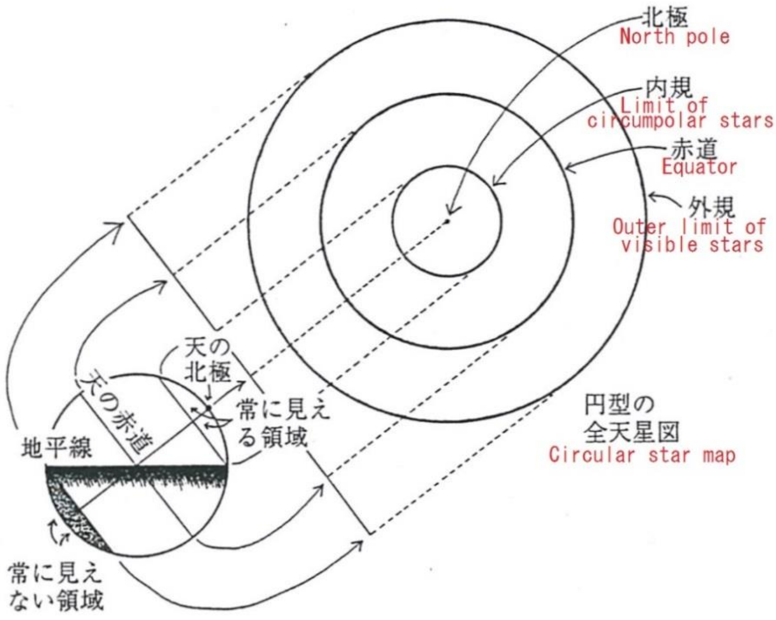


図18　星図の表現法（その１）

(2), Rectangular and circular star maps:

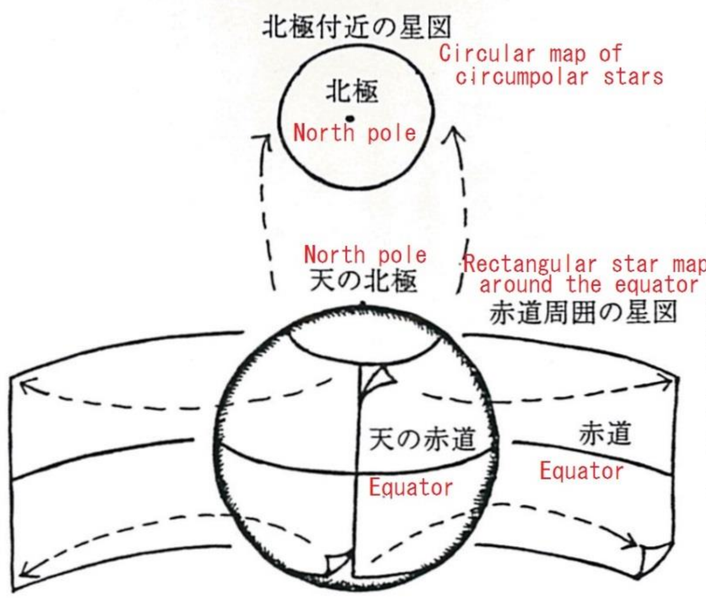


図19　星図の表現法（その２）

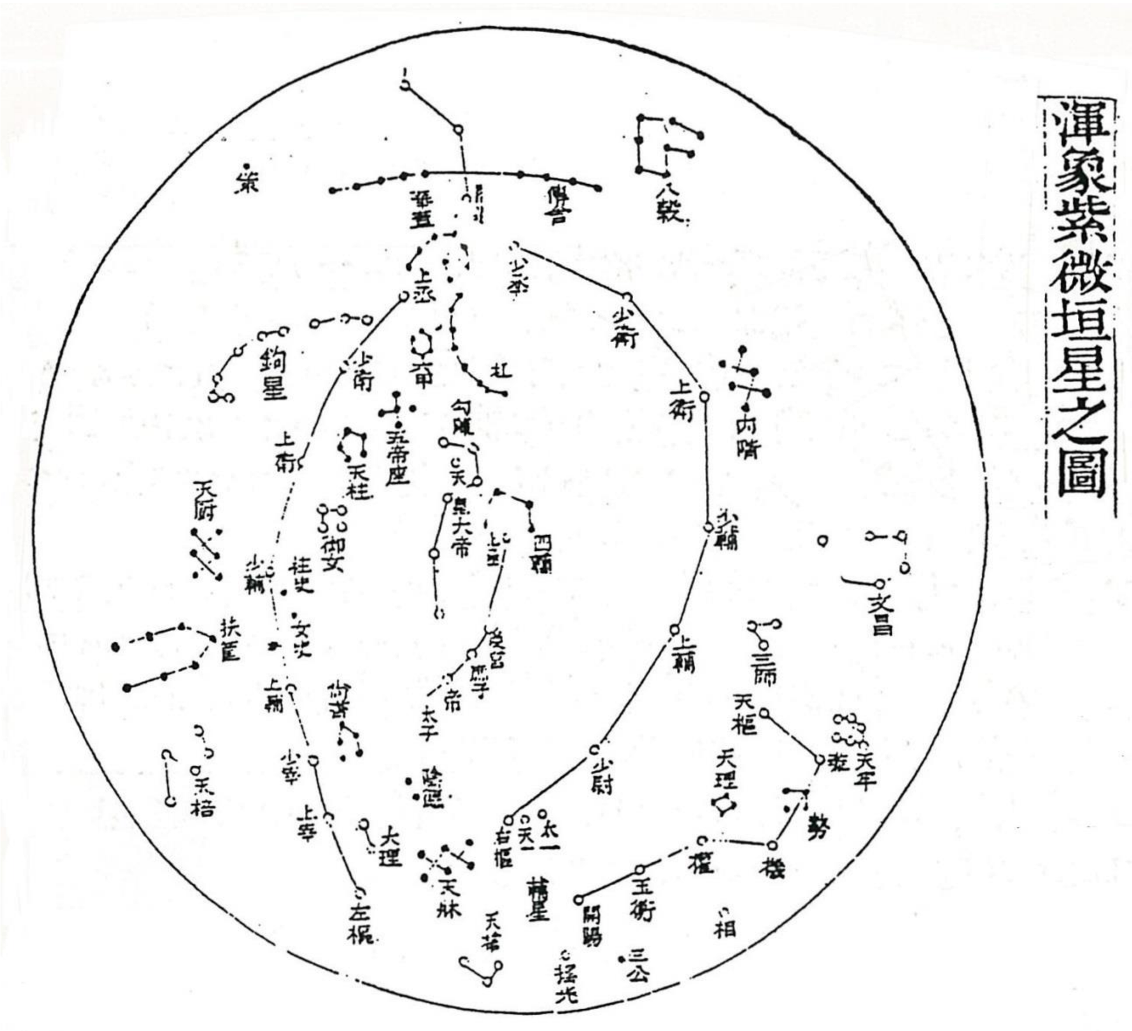
(From Ôhashi (1998) with additions)

Star maps in the *Xin-yixiang-fayao* (新儀象法要):

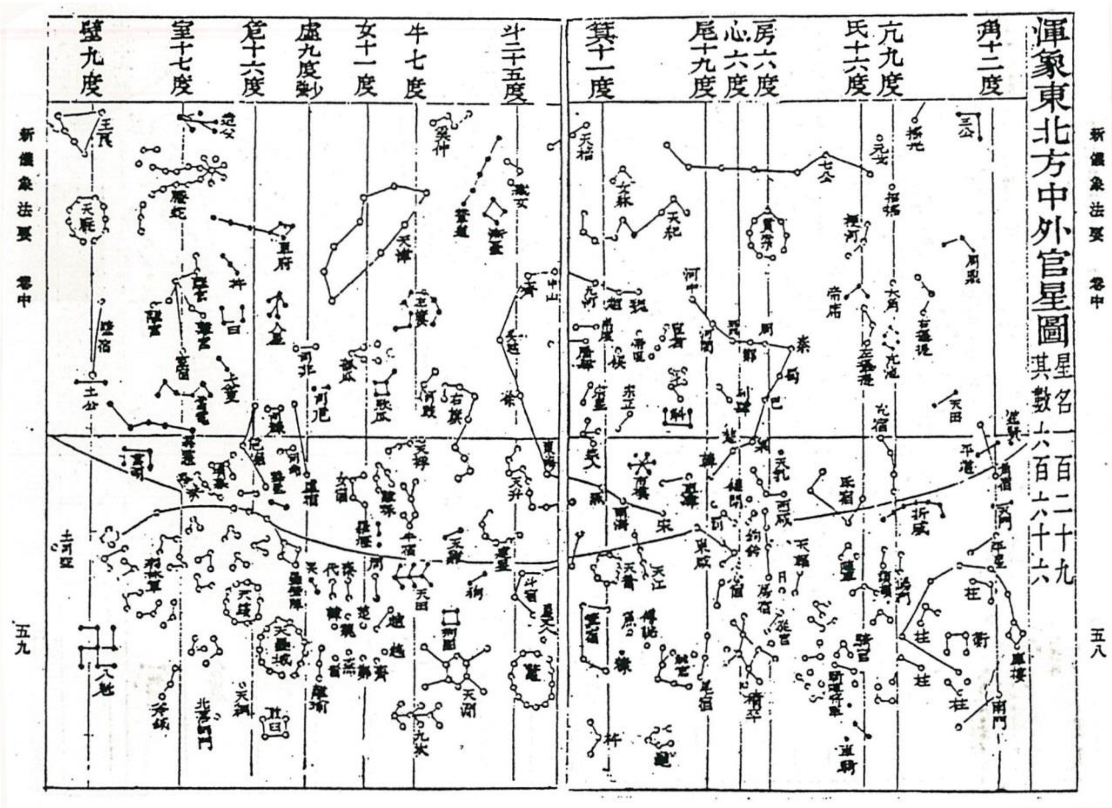
(the end of the 11th century):

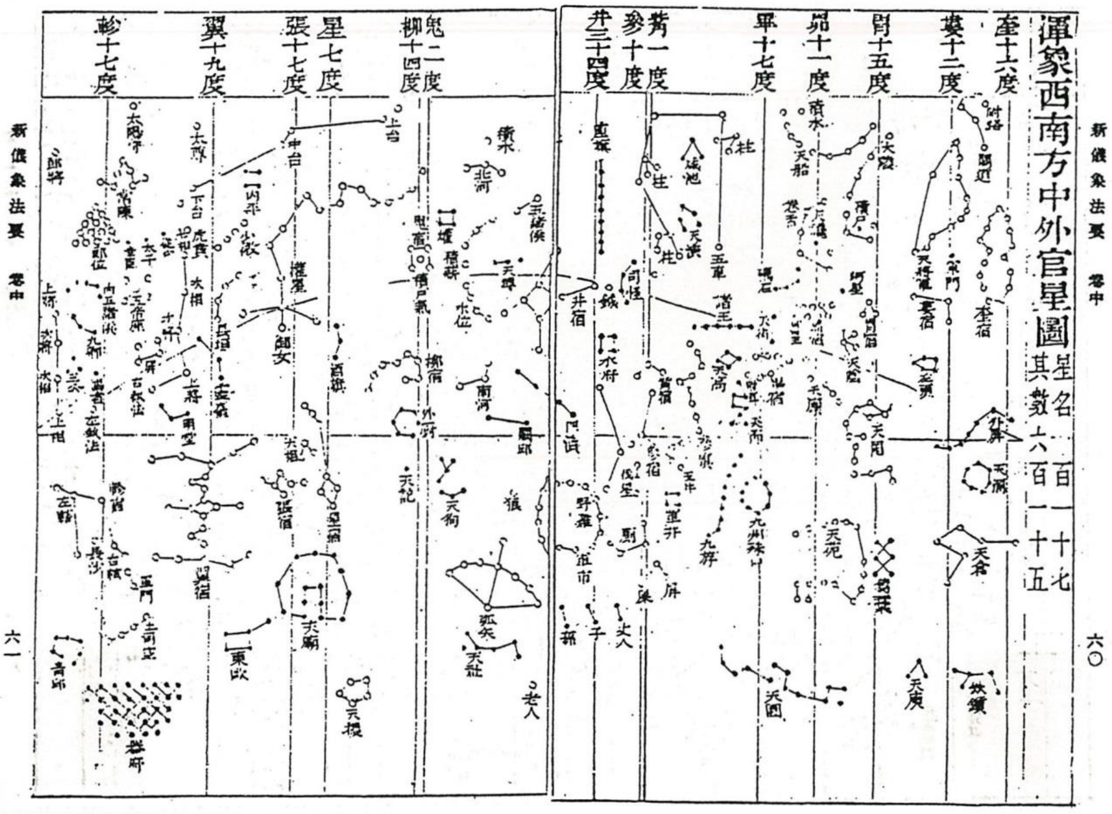
In the *Xin-yixiang-fayao* (新儀象法要), circular star maps and rectangular star maps are given. They are for making the celestial globe in the water driven mechanical clock (水運儀象台)

Circumpolar stars:

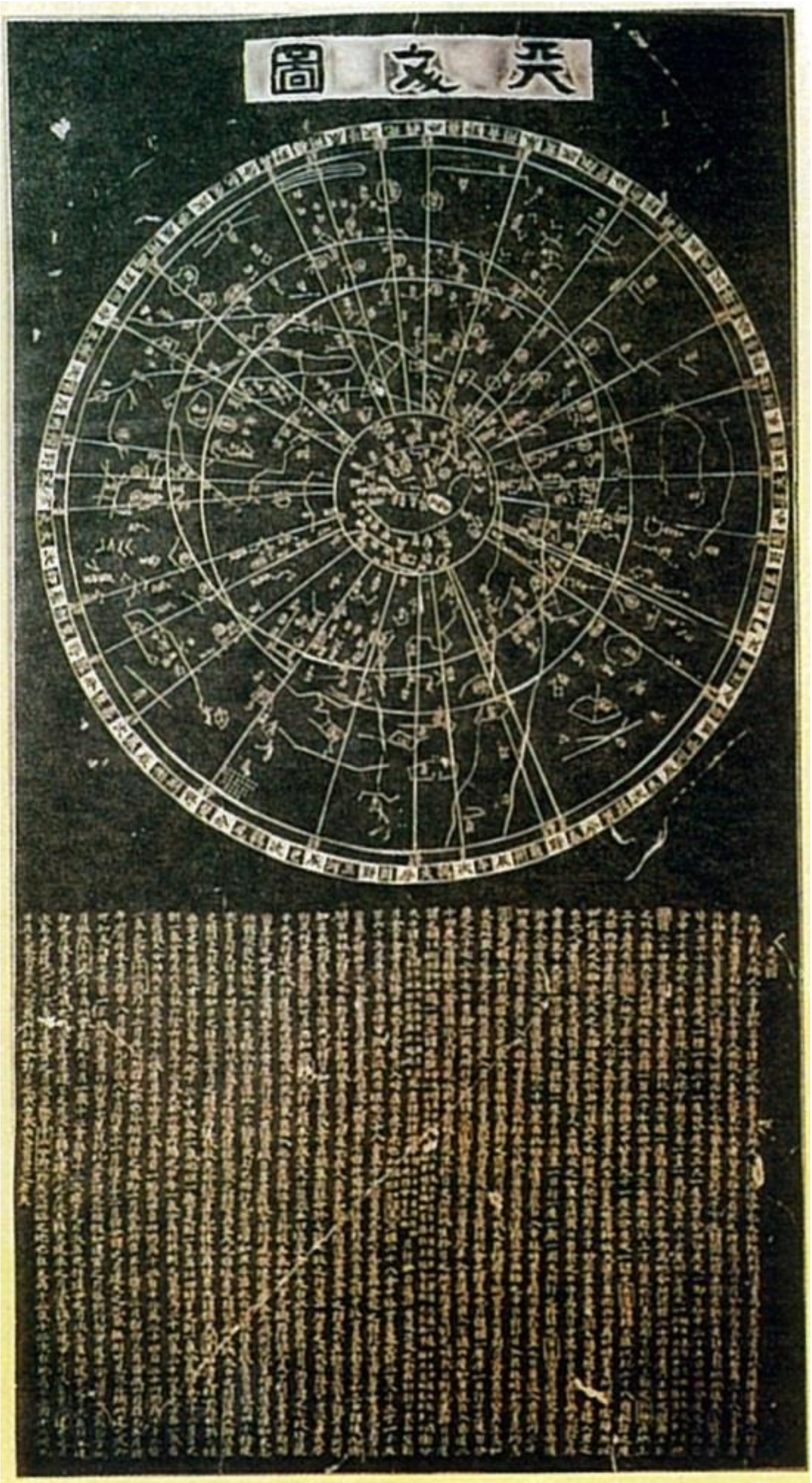


Rectangular star maps (around the equator):





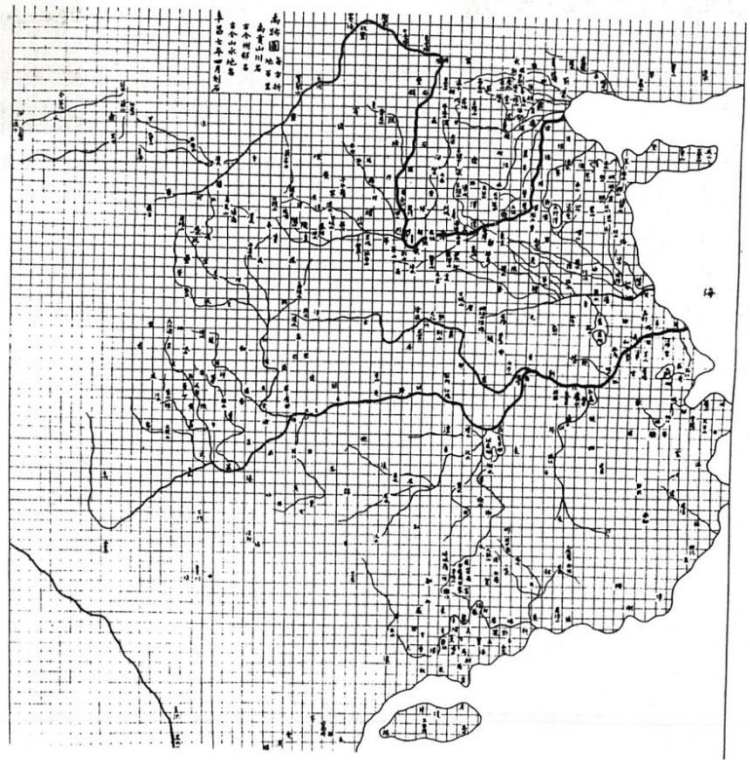
A star map (1247 CE) in Suzhou (蘇州) (stone inscription):.

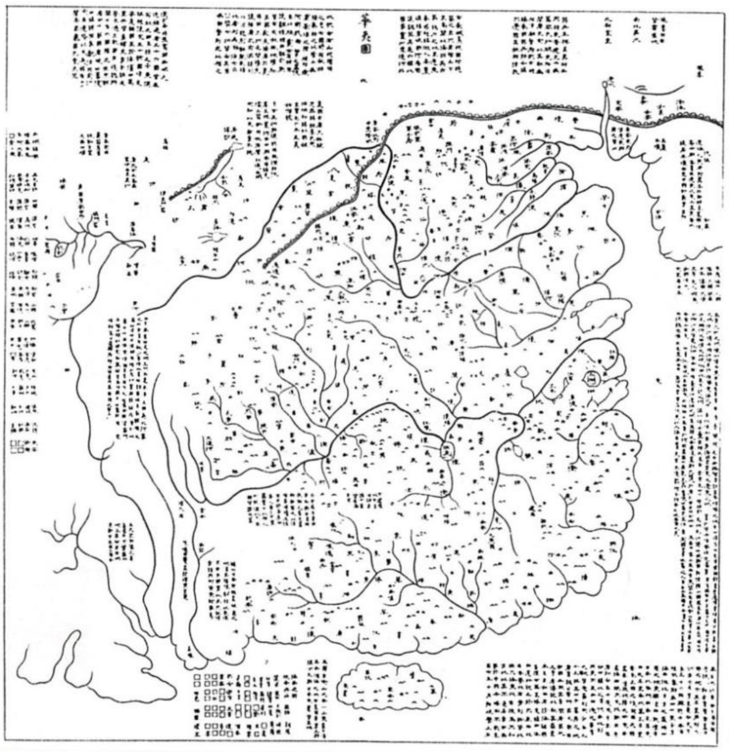


Maps in the Song (宋) period:

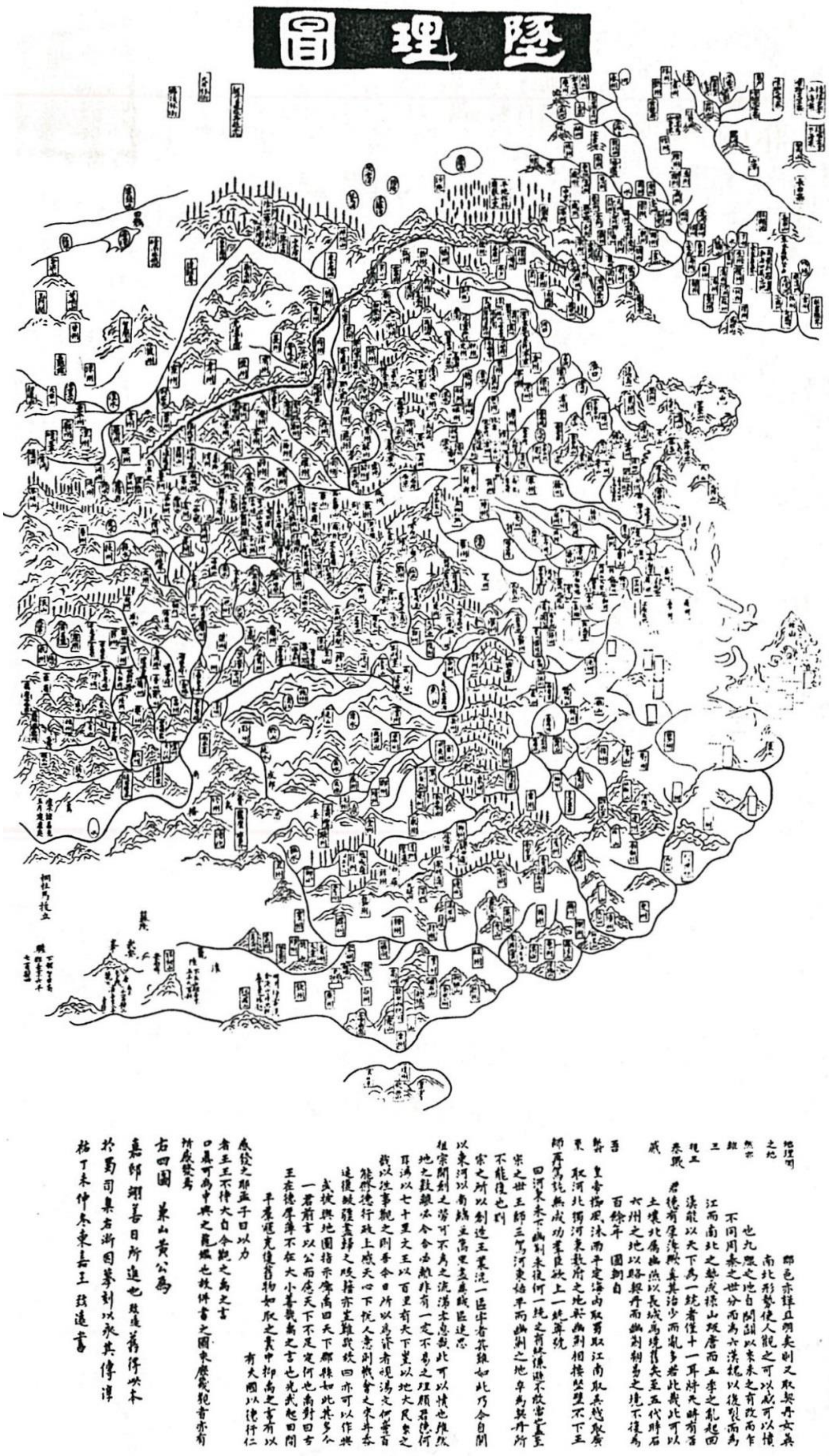
(The following traced maps are from 中国古代地图集(战国一元), 文物出版社, 1990.)

Two maps in Xi’an (西安) (1136 CE) (stone inscriptions):





A map (1247 CE) in Suzhou (蘇州) (stone inscription):.



Development of mathematics in the Song (宋) period

In the Southern (南宋) period, Qin Jiushao (秦九韶) wrote the *Mathematical Treatise in Nine Chapters* (數書九章, *Shushu jiuzhang*) (1247 CE). Here, the method to solve equations of higher degree, the indeterminate analysis etc. are discussed.

(For its detail, see Libbrecht (1973).)

Also in the Southern Song period, Yang Hui (楊輝) (the 13th century), wrote the *Yang Hui's Methods of Computation* (楊輝算法, *Yang Hui suanfa*) etc. Here, besides the method to solve equations, magic squares are discussed.

(For its detail, see Lam (1977).)

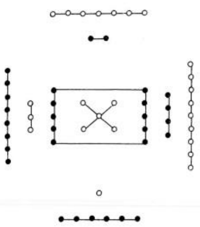
Some examples of magic square:

From the *Yang Hui suanfa* (Lam (1977), pp.145 – 146.)

CHAPTER ONE[1]

Magic squares

Ho T'u[2] 河圖 [lit. the River Chart]



Odd numbers (*t'ien shu* 天數) [lit. heavenly numbers]: 1, 3, 5, 7, 9.

Even numbers (*ti shu* 地數) [lit. earthly numbers]: 2, 4, 6, 8, 10.

Total (*chi* 積): 55.

Method of finding the total: Add the top and bottom numbers <to give the sum 11> and multiply it by the highest number <10 to obtain 110.> Halve this <to obtain 55> which is the sum of the odd and even numbers.

Lo shu[3] 洛書 [lit. Lo River Writing]

4	9	2
3	5	7
8	1	6

[1] This Chapter is found in *CTST* and not in *ICTTS* and *MC*.

[2] In the text this diagram is called *lo shu* and the next one *ho t'u*. The translator has interchanged their names to follow the present usage. This interchange of names was due to Chu Hsi of the 12th century. See *Shih Chia Chai Yang Hsin Lu*, Chapter 1, pp. 6 and 7.

[3] See footnote 2 above.

Arrange the nine numbers [in three rows] slanting downwards to the right so that the top and bottom numbers are opposite each other, the left and right numbers face each other, and these four cardinal points are projected outwards.

1
4 2
7 5 3
8 6
9

9 is worn on the head and 1 is the shoe, 3 is on the left and 7 on the right. 2 and 4 form the shoulders and 6 and 8 form the feet.

9
4 2
3 5 7
8 6
1

Magic square of order four (*hua shih lu t'u* 花十六圖) [lit. diagram of sixteen flowers].

<The vertical, horizontal [and diagonal] sums are 34.>

2	16	13	3
11	5	8	10
7	9	12	6
14	4	1	15

The *yin* magic square (*yin t'u* 陰圖) [lit. the female diagram].

<The total sum is 136.>

4	9	5	16
14	7	11	2
15	6	10	3
1	12	8	13

Method of interchange (*huan i* 換易) to form the above magic square:

13	9	5	1
14	10	6	2
15	11	7	3
16	12	8	4

Arrange the sixteen numbers in four successive columns. First interchange the numbers in the four corners; <interchange 1 and 16, 4 and 13>. Similarly interchange the numbers in the four inner corners <interchange 6 and 11, 7 and 10.> The horizontal, vertical and diagonal sums are all 34. The small numbers are thus balanced by this interchange. This can also be regarded as a general method.

Method of finding the total sum: Add the top and bottom numbers <the top number is 1, the bottom number is 16 and the sum is 17>. Multiply this sum by the highest number <16> and halve the product <to obtain 136>. Divide this by the number of rows or columns <4, to obtain 34 which is the sum of each row and column.>

Method of finding equal sums (*ch'iu têng shu* 求等術) Divide the numbers into two columns <1, 16; 2, 15; 3, 14; 4, 13; 5, 12; 6, 11; 7, 10; 8, 9> so that all pairs of numbers have equal sums <17>. First arrange these numbers into four columns

Jin (金) dynasty (1115 ~ 1234), and Yuan (元) dynasty (1271 ~ 1368)

元朝疆域空前辽阔，其东、东南、东北到海，西南包括西藏、云南，北面包括西伯利亚大部。元朝确立了行省制度，加强对辽阔版图的管理，为后代所沿用。元时期，西藏和台湾成为中央王朝密不可分的一部分，民族融合进一步发展。

At the time of Pre-Yuan Mongol (1206-1271) and Yuan (元) (1271-1368) dynasty, huge area was ruled by Mongols, and Islamic astronomy was introduced into China. A Khitan politician and astronomer Yelu Chucai (耶律楚材) (1190-1244) was an early contributor to the introduction of Islamic astronomy. And also, seven "Western (Islamic) astronomical instruments" were made in China by a Persian astronomer Jamālud-Dīn (扎馬魯丁) in 1267 CE. In 1271, Huíhuí-sītiān-tái (回回司天台) (Islamic astronomical observatory) was established at Shàngdū (上都) (in Inner Mongolia), and Jamālud-Dīn was appointed to be its director.

Guo Shoujing (郭守敬) and Shoushi calendar (授時暦)

Guo Shoujing (郭守敬) (1281 – 1316) was a Chinese water conservancy engineer and astronomer in the Yuan (元) dynasty.

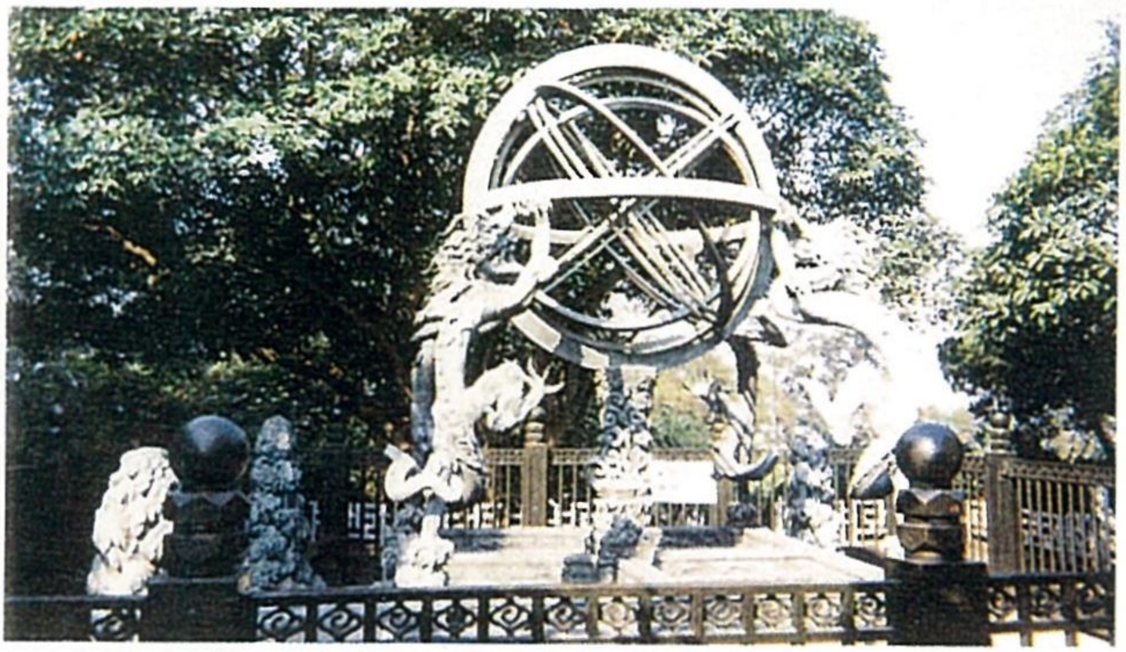
In 1276, Khubilai Khan ordered to make a new calendar. At that time, the Revised Daming calendar (大明暦) of the previous Jin (金) dynasty was still used, but its error had grown up, and more accurate calendar for the new Yuan dynasty was needed. Although the Yuan dynasty already had a national observatory "Sitian-tai" (司天台), a new department for the compilation of a new calendar was established, and Wang Xun (王恂), Guo Shoujing etc. took in charge. Wang Xun was in charge of calculation, and Guo Shoujing was in charge of observation. In 1278 or 1279, the department was developed into the "Taishi-yuan" (太史院) (Institute of chronology (and astronomy)). The institute was constructed in Dadu (大都) (now Beijing), and Wang Xun was appointed to be its director, and Guo Shoujing its deputy director. In 1280, the *Shoushi* calendar (授時暦) was established by them, and was officially used since 1281.
[For the *Shoushi* calendar, see Sivin (2009).]

Astronomical instruments of Guo Shoujing

Guo shoujing created 17 new astronomical instruments. Among them, 13 instruments are for the Institute of chronology (and astronomy), and 4 are for traveling observers.

Among the instruments for the institute, the most important ones are the "jianyi" (簡儀) (simplified instrument) and the "gaobyao" (高表) (high gnomon) along with the "jingfu" (or possibly pronounced as "yingfu") (景符) (tally for shadow).

Traditional Chinese armillary sphere (渾儀):



Traditional Chinese armillary sphere made at the time of Ming dynasty (now in Purple Mountain Observatory, Nanjing)

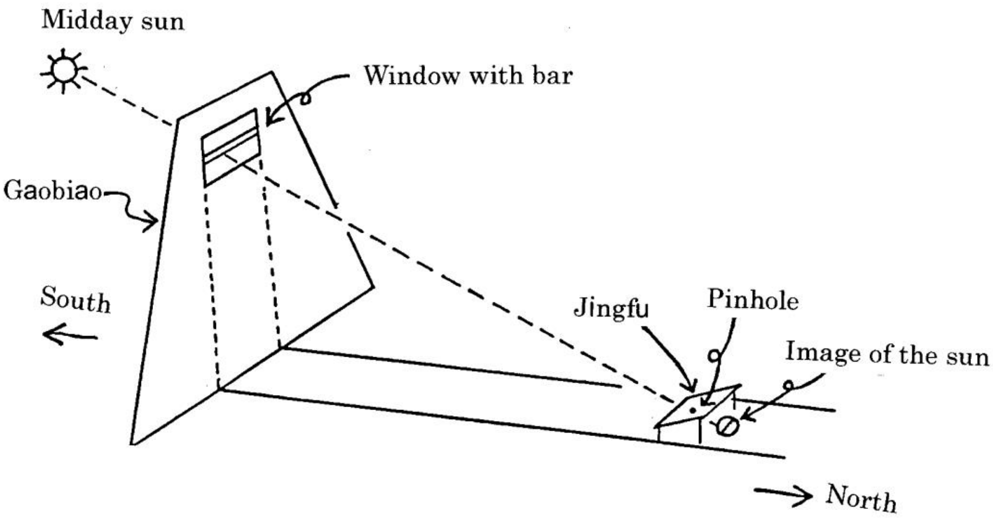
The Jianyi (簡儀):

The "jianyi" designed by Guo Shoujin (Reconstructed at the time of Ming dynasty (now in Purple Mountain Observatory, Nanjing))

The Gaobiao (高表):

The “gaobiao” in the Guanxingtai (觀星台) Dengfeng (登封) in Henan (河南) province

The Gaobiao and Jingfu (景符):



The principle of “jingfu”.

The Shoushi Calendar (授時曆)

1 year = 365.2425 days.

----- Very accurate!

For the determination of the length of a year, the observations by the Gaobiao and Jingfu were used.

The time of winter solstice, when the length of the gnomon-shadow is longest, was observed.

The observation is possible at noon only.

Why they could determine the fraction of year-length?

----- Zu Chongzhi's method was used.

Development of mathematics in Mongolia and Yuan periods

----- Tianyuan-shu (天元術) and Siyuan-shu (四元術)

Tianyuan-shu (天元術)

----- A kind of algebra using counting rods.

Li Ye (李冶) (or Li Zhi (李治)) explained the "tianyuan-shu" in his *Sea Mirror of Circle Measurements* (測圓海鏡, Ceyuan haijing) (1248 CE) and *New Steps in Computation* (益古演段, Yigu yanduan) (1259 CE).

The following is an example of the "tianyuan-shu".

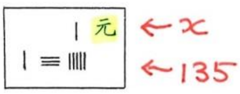
(From Li and Du (1987), pp.135 – 138.)

The origin and development of the 'technique of the celestial element' (天元術, tiān yuán shù)

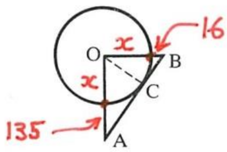
'Celestial element' means the unknown number in the problem, 'establish the celestial element as such and such' means 'let x be such and such'. In the 'technique of the celestial element' a polynomial or polynomial equation is usually indicated by using the character 元 (yuán, element) at the first degree term or the character 太 (tài) at the constant term.

Below we present Problem 2 of Chapter 7 of Lǐ Yě's *Sea Mirror of Circle Measurements* as an example in order to explain the general procedure in setting up equations in the 'technique of the celestial element'. The original text of the problem read: '[Assume there is a circular fort of unknown diameter and circumference,] person A walks out of the south gate 135 steps and person B walks out of the east gate 16 steps and then they see one another. [What is the diameter?]' Lǐ Yě gave five different solutions for this problem. Below we give the second solution putting the original text (in translation) on the left and using modern mathematical notation on the right to explain the procedures involved.

'Briefly put: Let one celestial element be the radius of the fort, lay it down and first add to it the southward steps getting the gǔ (股).

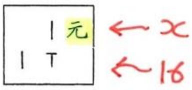


[Explanation.] Let x be the radius of the circular fort then side OA $= x + 135$, side OB $= x + 16$.

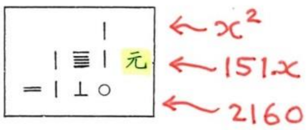


Finding the diameter of a circular fort.

'Secondly put down the easterly steps getting the gōu (勾).

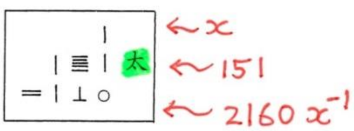


'Multiply the gōu and gǔ together, getting



OA × OB $= (x + 135)(x + 16)$
$= x^2 + 151x + 2160$,

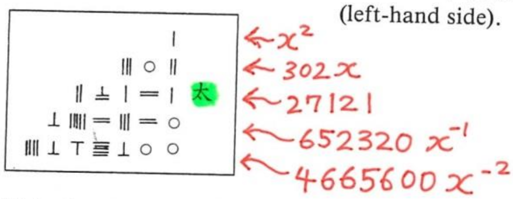
'Divide by the celestial element, getting the hypotenuse



Divide by x getting hypotenuse $=$ $x + 151 + 2160x^{-1}$
($\because$ AB.OC $=$ OA.OB $= 2\triangle$ABO)

'Multiply this by itself, getting the square of the hypotenuse and place it on the left.

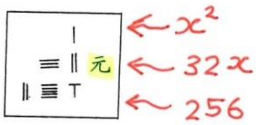
Multiply this by itself, getting $(\text{hyp})^2 = x^2 + 302x + 27121 + 652320x^{-1} + 4665600x^{-2}$ (left-hand side).



'Multiply the gōu by itself, getting

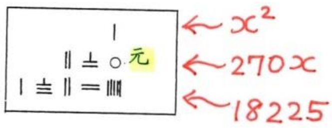
Again

$$OB^2 = (x + 16)^2 = x^2 + 32x + 256,$$



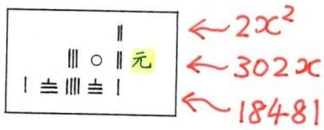
and again multiply the gŭ by itself, getting

$$OA^2 = (x + 135)^2 = x^2 + 270x + 18225.$$



'The two configurations added give

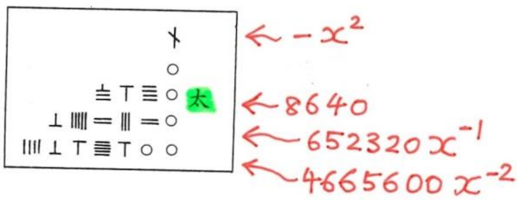
$OB^2 + OA^2 = 2x^2 + 302x + 18481 = (\text{hyp})^2$



which is the same value [as obtained before for the hypotenuse squared]. Cancel it with the [hypotenuse squared],

Equate to the left-hand side and simplify getting

$$-x^2 + 8640 + 652320x^{-1} + 4665600x^{-2} = 0,$$



which is a quartic equation giving 120 steps as the radius of the fort.'

rationalizing the equation we get $-x^4 + 8640x^2 + 652320x + 4665600 = 0$.

Solving, we get $x = 120$ (steps) as the radius of the circular fort.

On comparing the left- and right-hand sides of the above it is clear that, as a method of finding an equation, the 'technique of the celestial element' is roughly similar to the method used in present-day textbooks in algebra. In Lǐ Yě's book the derivation of the equation in the above example by cancellation after getting the two forms for the square on the hypotenuse is called 'cancelling the same number' or 'cancelling like results'.

Siyuan-shu (四元術)

----- A kind of algebra to solve equations with four unknowns.

Zhu Shijie (朱世傑) wrote the Introduction to the Mathematical Studies (算学启蒙, Suanxue qimeng) (ca.1299 CE) on the "tianyuan-shu" tec., and Precious Mirror of the Four Elements (四元玉鑑, Siyuan yujian) (ca.1303 CE) on the "siyuan-shu".

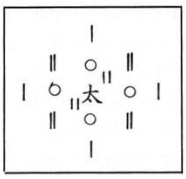
56

The method of the four unknowns (x, y z and u) using counting rods is to put the constant term (太 "tai") in the centre, the various coefficients of x below it, the various coefficients of y on the left, the various coefficients of z on the right, and the various coefficients of u above it. The coefficients of the products involving two unknowns (such as xy^2, z^3u^4, . . . etc.) are recorded at the corresponding points of intersection of two lines. The products of two non-adjacent unknowns as recorded in the corresponding holes of the grid as in the diagram below:

⋮	⋮	⋮	⋮	⋮	⋮	⋮
....y^3u^2	y^2u^2	yu^2	u^2	zu^2	z^2u^2	z^3u^2....
....y^3u	y^2u	yu	u	zu	z^2u	z^3u....
				yz		
....y^3	y^2	y		z	z^2	z^3....
			xu			
....xy^3	xy^2	xy	x	xz	xz^2	xz^3....
....x^2y^3	x^2y^2	x^2y	x^2	x^2z	x^2z^2	x^2z^3....
....x^3y^3	x^3y^2	x^3y	x^3	x^3z	x^3z^2	x^3z^3....
⋮	⋮	⋮	⋮	⋮	⋮	⋮

So, for example, $x + y + z + u$ is recorded as

and $(x+y+z+u)^2 = x^2 + y^2 + z^2 + u^2 + 2xy + 2xz + 2xu + 2yz + 2yu + 2zu$ is recorded as



Addition and subtraction of polynomials in four unknowns requires matching constant term with constant term and the coefficients of other terms with corresponding coefficients and then adding and subtracting the corresponding coefficients. .....

(From Li and Du (1987), p.142 with additions.)

Ming (明) dynasty (1368 ~ 1644)

(From 中国历史地图及大事年表, 北京, 中国地图出版社, 2013.)

Navigation of the fleet of Zheng He (鄭和)

Zhong He (鄭和) conducted a fleet and sailed to the Indian Ocean etc. from 1405 to 1433. Magnetic compass and astronomical observations were used for navigation.

Equal temperament of Zhu Zaiyu (朱載堉)

Zhu Zaiyu (朱載堉) (1536 - 1611) described the mathematical theory of the equal temperament. It is almost contemporaneous with the theory of equal temperament of Simon Stevin (1548 – 1620) of Nederland. I think that they are independent.

Abacus calculation "zhusuan" (珠算)

Chinese abacus was widely used from this period until recent time. Cheng Dawei (程大位) wrote the popular manual *Suanfa tongzong* (算法統宗) (1592 CE).

A page from the Suanfa tongzong (算法統宗) of Cheng Dawei

Cheng Dawei (程 大位)

The *Tiangong kaiwu* (天工開物) of Song Yingxing (宋應星)

Song Yingxing (宋應星) wrote the *Tiangong kaiwu* (天工開物) which is a detailed work on technology.

Late Ming (明) ~ Earle Qing (清)

Introduction of Western Sciences into China,

----- Jesuit Missionaries

Matteo Ricci (1552 – 1610) (利瑪竇) arrived at Beijing in 1601, and introduced Western sciences into China.

Since then, Jesuit missionaries visited China, and introduced Western mathematics, astronomy, other science and technology etc., and they made astronomical instruments and placed them in an observatory in Beijing.

[For the introduction of Western mathematics in China, see Jami (2012).]

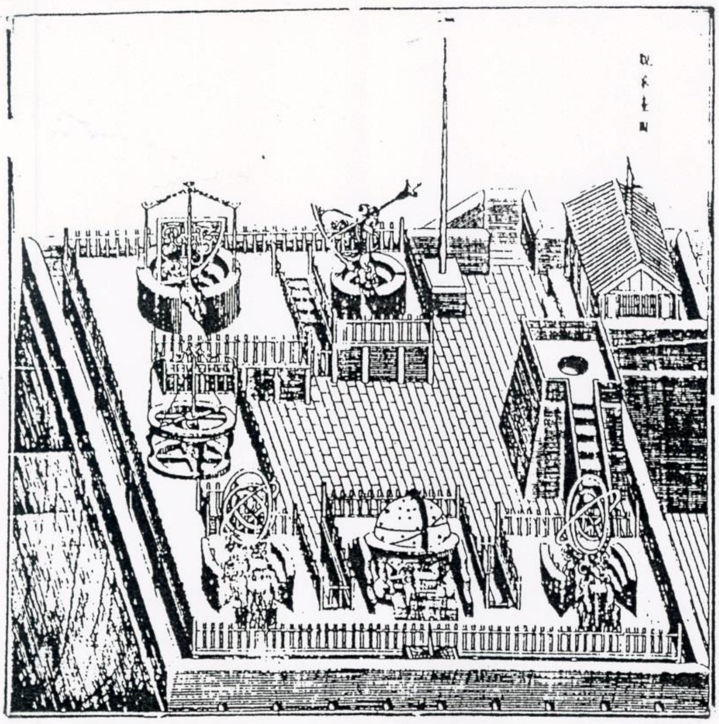
Chinese Astronomical Works based on Western Astronomy:

***Chongzhen-lishu* (Calendrical treatise in the name of Emperor Chongzhen) (崇禎暦書) (1631 – 34) ----- Simple eccentric model for solar orbit.**

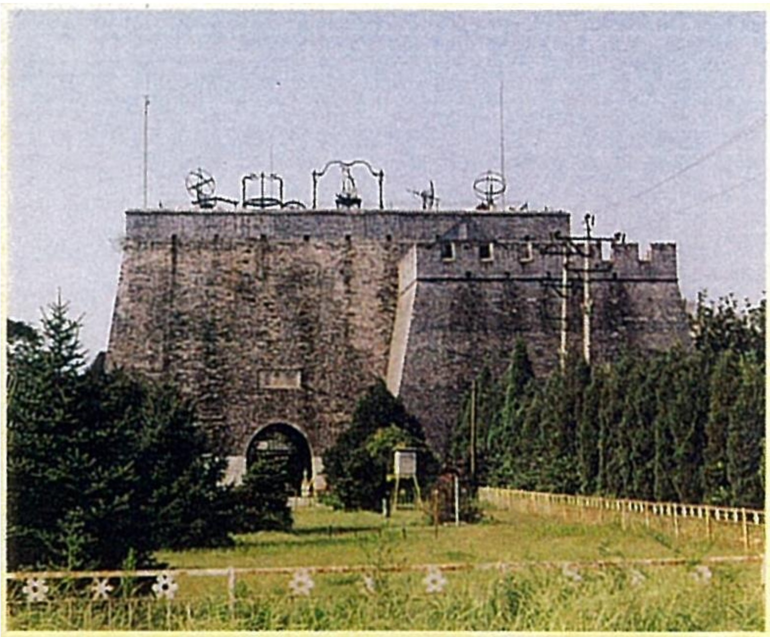
***Lixiang-kaocheng* (Treatise of calendrical phenomena) (暦象考成) (1722) ----- Double epicycle model for solar orbit.**

***Lixiang-kaocheng-houbian* (Second part of the treatise of calendrical phenomena) (暦象考成後編) (1742) ----- Kepler's elliptic orbit.**

Guanxiangtai
観象台 (Observatory) in Beijing



From the Lingtai yixiangzhi (霊台儀象志) (1674)
of Nan Huairen (南懐仁) (= Verbiest)



Beijing Ancient Observatory (北京，古観象台)

Qing (清) dynasty (1644 ~ 1911)

(From 中国历史地图及大事年表，北京，中国地图出版社，2013.)

The study of Chinese classics highly developed in the Qing period, and the classics of mathematics and astronomy were also studied.

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