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Chapter 7

Mathematical structure of the eccentric and epicyclic models in ancient Greece and India

Yukio Ôhashi

Abstract

The eccentric and epicyclic models were created in Ancient Greece, and further developed in India, Islamic World and Europe. The accuracy of their mathematical structures is examined in this paper. And also, special features of Indian models are discussed.

Keywords: Eccentric model, Epicyclic model, Greek astronomy, Indian astronomy, Geocentric system.

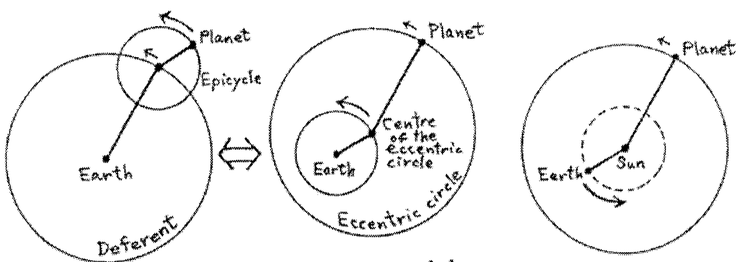
7.1 Introduction

The eccentric model (in the geocentric theory) is that the center of the orbit of a heavenly body is different from the earth. The epicyclic model is that a planet revolves along a small epicycle and that the center of the epicycle revolves along a large deferent. (This deferent can be an eccentric circle.)

The eccentric and epicyclic models were created in Ancient Greece sometime around the 3^{rd} century BC, and were introduced into India sometime around the 4^{th} century AD or so. Indian models were not simple imitation of Greek models, but have special features. I would like to discuss the mathematical structure of some of these models.

7.2 Planetary models in Ancient Greece, Islamic World, and Europe

7.2.1 Apollonius



Epicyclic model　　Eccentric model　　(Heliocentric model)

Figure 1: Geocentric Models and Heliocentric Model

We do not know who invented the eccentric and epicyclic models, but the first astronomer who mathematically treated these models seems to be Apollonius (ca. end of the 3^{rd} century BC). Ptolemy wrote in his *Almagest* (XII.1) that Apollonius explained the retrograde motion of planets by the epicyclic model as well as the eccentric model which is mathematically equivalent to the epicyclic model. (See Fig.1.)

7.2.2 Hipparchus

The equation of center of the sun and moon was explained by Hipparchus (2^{nd} century BC) using the eccentric model as well as the epicyclic model which is mathematically equivalent to the eccentric model. Ptolemy explained the method of Hipparchus in his *Almagest* (III∼IV). The model of Hipparchus is that the sun (or moon) revolves along an eccentric circle with a constant speed. Let this model be called "Simple eccentric model".

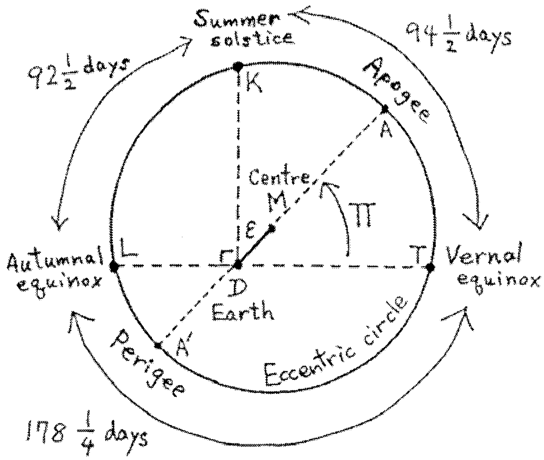


Figure 2: Simple Eccentric Model of the Solar Orbit

According to the *Almagest* (III.4), the solar theory of Hipparchus is as follows. (See Fig.2.) Hipparchus based on the observational data that the period from the vernal equinox to the summer solstice is 94½ days, and that the period from the summer solstice to the autumnal equinox is 92½ days, and determined the eccentric distance ε and the longitude of the apogee Π of the eccentric circle (radius = 1). If the length of a year is assumed to be 365¼ days, the period from the autumnal equinox to the vernal equinox becomes 178¼ days. From these data, ε and Π can be determined by plane geometry. In this model, the eccentric distance ε is double the modern eccentricity. (See Appendix 1.)

According to the *Almagest* (IV), Hipparchus determined the orbit of the moon (with certain error) from the observational data of three lunar eclipses. Hipparchus's method must have been similar to the method of Ptolemy as shown in Figure 3.

7.2.3 Ptolemy

Ptolemy (2^{nd} century AD), in his *Almagest*, also used the "Simple eccentric model" for the solar motion, but newly introduced the second inequality "evection" for the lunar motion, and also introduced the "Equant model" for the planetary motion.

7.2.4 Ptolemy's lunar theory (1) (equation of center)

According to the *Almagest* (IV.1), observational data of the lunar eclipses should be used to determine the equation of the center of the moon, because the moon is at the opposite direction to the sun, and its position can be determined without parallax from the (already known) solar position. The *Almagest* (IV.6) explains the method to determine the equation of center using the epicyclic model. For this purpose, the observational data of the lunar longitude at the time of maximum obscuration of three lunar eclipses as well as the (already known) length of the anomalistic month and that of the tropical month are used. From these data, the mean movement of the lunar apogee and that of the lunar longitude can be obtained.

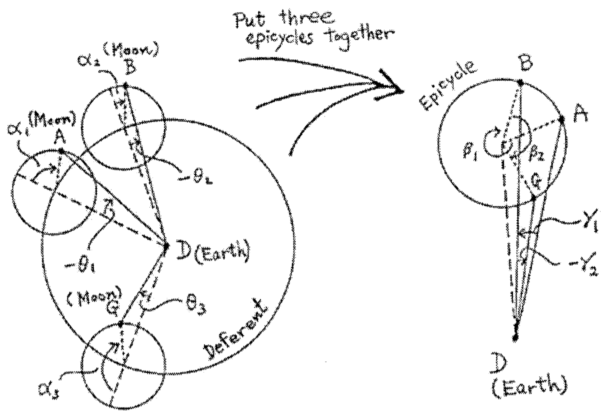


Figure 3: Determination of the Lunar Equation of Centre

The three points A, B, G in the left side of Figure 3 are the positions of the moon at the time of maximum obscuration of lunar eclipses. From the period between two eclipses and the length of the tropical month, the mean change of lunar longitude during this period is calculated. From this value and the actually observed change of lunar longitude during this period, the following values are estimated.

$$Y_1 = \theta_2 - \theta_1, \quad Y_2 = \theta_3 - \theta_2$$

From the mean movement of the lunar apogee during this period, which can be calculated from the length of the anomalistic month and that of the tropical month, the following values are estimated.

$$\beta_1 = \alpha_2 - \alpha_1, \quad \beta_2 = \alpha_3 - \alpha_2$$

Putting these three epicycles together like the right side of Figure 3, we can determine the position of the earth D from the above four values using plane geometry. By this method, the radius of the epicycle (with reference to the radius of the deferent) and the longitude of the lunar apogee at a certain point of time are obtained. Actually, the equation of center determined by this method is slightly smaller than the modern equation of center, due to the effect of evection at the time of lunar eclipse (i.e. full moon).

7.2.5 Ptolemy's lunar theory (2) (evection)

As the model as above does not coincide with the actual observations except for the new moon and full moon, Ptolemy introduced "evection" for the first time in his *Almagest* (V).

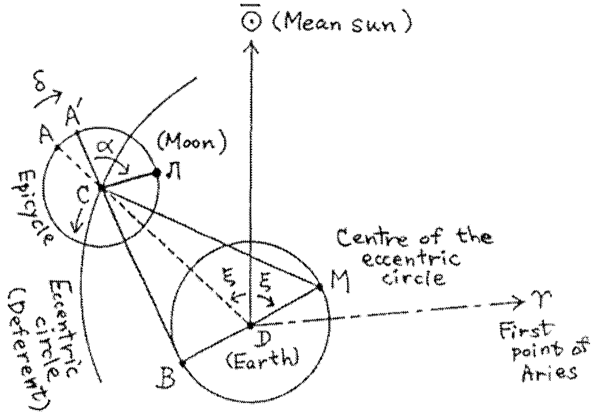


Figure 4: Ptolemy's Lunar Model with "Evection"

In Figure 4, the center M of the eccentric circle revolves along a small circle (MB). In the figure, ξ is the angular distance between the mean moon and the mean sun. By this method, the epicycle comes near to the earth at the time of half moon, and the eccentricity looks increased.

There is another effect also. Let B be the opposite point of M, and A^1 be a cross point of the epicycle and the straight line BC. The moon revolves once in an anomalistic month from A^1 (not from A). In the figure, α is the mean anomaly. Due to the effect of δ (=$\angle$AC A^1), the longitude of the apogee looks changed except for the time of new moon, full moon, and half moon. (See Appendix 3.)

7.2.6 Ptolemy's planetary theory

For planets, Ptolemy used the epicyclic model, where an epicycle is revolving along a deferent. The deferent is an eccentric circle. He determined the distance of the deferent of Venus (from the earth) by the observation of the greatest elongation which indicates the apparent radius of the epicycle. From this distance, eccentricity is estimated. Then, from the difference between the actual observation and the prediction according to the "Simple eccentric model" of the deferent when Venus is at the right-angled point from the apogee, he introduced a point which was later called "equant". He also used the "Equant model" for other planets. (See Appendix 2.)

Ptolemy used a very complicated model for Mercury, which I shall not discuss here.

7.2.7 Islamic World

In the Islamic world, some geometrical models which give similar results to the "Equant model" were devised. One of them was the "Double epicycle model" of ash-Shātir (14^{th} century AD). (See Appendix 4.)

7.2.8 Europe

Copernicus (1473 – 1543) also used a model which is equivalent to the "Double epicycle model" for planets. It is not known whether he was directly influenced by the model of ash-Shātir or not. Copernicus, however, still used a model which is basically similar to the "Simple eccentric model" for the earth. He mentioned something like double epicycle (or its variation) for the earth in his *De Revolutionibus* (III), but it was for the secular variation of the eccentricity and the longitude of perihelion, and was not for the effect due to the "Equant model". In the Copernican model, the earth was still different from other planets.

Tycho Brahe (1546 – 1601) also used a model which is equivalent to the "Double epicycle model" for planets, but used the "Simple eccentric model" for the sun.

Kepler (1571 – 1630) used elliptic orbit for all planets including the earth, and the earth became an ordinary planet truly.

7.3 Planetary models in traditional India

7.3.1 The manda-correction and the śīghra-correction

In the Hindu Classical Astronomy, geocentric epicyclic and eccentric systems are used. The *Mahābhāskarīya* of Bhāskara I (7^{th} century) treats the epicyclic and eccentric systems as mathematically equivalent models for both of the *manda*-correction and the *śīghra*-correction.

Firstly, mean (*madhya*) planet, which is supposed to rotate constantly around the earth, is calculated, and then, corrections are applied to the mean planet in order to obtain the true (*sphuṭa*) planet. One correction is the *manda*-correction, which corresponds to our equation of center. The other is the *śīghra*-correction, which corresponds to the annual parallax in the case of outer planets, and the planet's own revolution in the case of inner planets. Firstly, the *manda*-correction is applied to the mean planet, which corresponds to the planet's own mean revolution in the case of outer planets, and the sun's mean revolution in the case of inner planets. The result is called "*manda-sphuṭa* planet", which is the mean planet corrected by the equation of center only. Then, the *śīghra*-correction is applied to the "*manda-sphuṭa* planet", and the true planet is obtained. In the actual calculation, some special methods are used in the classical texts.

7.3.2 The size of the epicycles in the Āryabhaṭīya

One interesting feature of the *Āryabhaṭīya* (AD499) of Āryabhaṭa is that the size of the epicycles of the planets changes in different anomalistic quadrants. This is quite different from the simple geometrical model. The modern *Surya-siddhānta* (ca. 10 ~ 11^{th} century AD) etc. also use a similar method.

According to the *Āryabhaṭīya*, the *manda*-epicycles of Mars, Jupiter and Saturn are small in the 1^{st} and 3^{rd} quadrants, and are large in the 2^{nd} and 4^{th} quadrants. The *manda*-epicycles of the Mercury and Venus, and the *śīghra*-epicycles of the five planets are large in the 1^{st} and 3^{rd} quadrants, and are small in the 2^{nd} and 4^{th} quadrants.

According to the interpretation of Bhāskara I, their size given in the *Āryabhaṭīya* is the value at the beginning of each quadrant, and the size changes linearly. However, there are other Hindu astronomers who interpret that the size is the value at the end of each quadrant.

In the case of the *manda*-epicycles of the Mars, Jupiter and Saturn, if the interpretation of Bhāskara I is correct, it can be said that the change of the distance of the planets becomes somewhat similar to that of the "equant" model. However, the change of the *śighra*-epicycles is not understandable. Further researches are necessary.

7.3.3 Indian method of the manda-correction

The *manda*-correction is not based on a simple geometrical model, but a special modification is applied. Its result is that the equation of center in this method becomes a simple sine function of anomaly. Its accuracy is slightly less than the simple eccentric model, but their errors (in the opposite direction) are not so different.

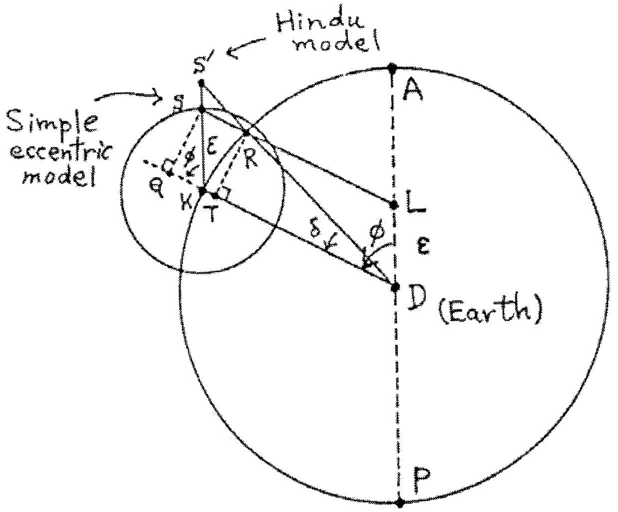


Figure 5: Equation of Centre in the Hindu Classical Astronomy

In Figure 5, the point S corresponds to the heavenly body in the simple eccentric model (In the figure, $SK||AP$, where AS corresponds to the apogee, and P corresponds to the perigee.). In the Figure 5, S^1 corresponds to the heavenly body in the Hindu model. Here, $KS = DL = \varepsilon = 2e$ (e is eccentricity in modern sense), $SL||KD$, and the point S^1 is at the cross point of the straight lines KS and DR. Now, $RT = SQ = \varepsilon \cdot \sin\phi$. Therefore, $\delta \approx \sin\delta = RT = \varepsilon \cdot \sin\phi$. So, we can say that the Hindu equation of center is a simple sine function of anomaly. This Hindu method is well explained in the *Mahābhāskarīya* (IV) of Bhāskara I (7^{th} century).

As I have shown in Appendix 1, the equation of center of the simple eccentric model and that of the true Kepler motion are:

$\theta \approx 2e \sin\phi + 2e^2 \sin 2\phi$ (simple eccentric model)

$\theta \approx 2e \sin\varphi + \frac{5}{4}e^2 \sin 2\varphi$ (true Kepler motion)

It is clear from the above equations that the second term of the simple eccentric model is too large. The Indian method can be expressed as $\theta \approx 2e \sin\phi$, and its inaccuracy is not so different from that of the simple eccentric model (in the opposite direction).

This Indian method is quite different from the Greek method.

7.3.4 The method in the Brāhma-sphuṭa-siddhānta

The method in the *Brāhma-sphuṭa-siddhānta* (AD 628) of Brahmagupta is very interesting. Its process, except for Mars, is as follows. Firstly, the amount of the *manda*-correction is calculated from the mean planet, and is applied to the mean planet. The result is the *manda-sphuṭa* planet. Secondly, the amount of the *śīghra*-correction is calculated from the *manda-sphuṭa* planet, and is applied to the *manda-sphuṭa* planet. The result is the true planet after the first approximation. From the result, the amount of the *manda*-correction is calculated, and applied to the original mean planet. From this result, the amount of the *śīghra*-correction is calculated and applied. The result is the true planet after the second approximation. This process is repeated until a constant value is obtained.

In the case of Mars, the above mentioned process is not used. The method for Mars is as follows. Firstly, a half of the amount of the *manda*-correction and a half of the *śīghra*-correction are applied, and the once corrected Mars is obtained. From the result, the amount of the *manda*-correction is calculated, and its whole amount is applied to the original mean Mars. From this result, the amount of the *śīghra*-correction is calculated, and its whole amount is applied. The result is the true Mars. This method is the ordinary method used by other Indian astronomers in those days for five planets.

The above mentioned process of the *Brāhma-sphuṭa-siddhānta*, except for the case of Mars, means that the amount of the equation of center is a function of the true anomaly of the planet, and not of the mean anomaly. This fact shows that the Indian model of planetary motion is not a simple imitation of the Greek geometrical model, and further investigation of the Indian model is needed. At present, I suspect that Indian astronomers in those days considered that the inequality of the *manda*-correction is produced by a kind of physical force originated to the apogee, and this force is equilibrated with the displacement of planetary position due to the inequality. If so, the amount of the inequality should be a function of the actual position of the true planet (instead of the mean position like the Greek model).

It may be mentioned here that Brahmagupta did not use the above mentioned successive approximation for the calculation of true planets in his *Khaṇḍa-khādyaka* (AD 665), but used the ordinary method, like the case of Mars in his earlier work, for five planets.

7.3.5 The lunar theory in the Laghu-mānasa

Mañjula was an astronomer of the 10^{th} century AD. His name is sometimes spelled Muñjala, but Kripa Shankar Shukla pointed out that Mañjula is his real name.

Mañjula composed the *Laghu-mānasa* (932 AD). This is a *karaṇa* work (handy practical work of astronomy) of mathematical astronomy. It is a small but very important work. It contains the second correction for the moon. According to Yallaya's commentary (1482 AD) on the *Laghu-mānasa*, this correction originated from Vaṭeśvara's work (early 10^{th} century). The statement of Yallaya has not been confirmed by Vaṭeśvara's extant works.

According to K. S. Shukla's commentary in his edition of the *Laghu-mānasa*, Mañjula's second correction for the moon can be expressed as follows:

$$144'26'' \quad \cos(S-U)\ \sin(M-S) \tag{A}$$

where S, M, and U are the true longitudes of the sun, moon, and the moon's apogee (*mandocca*) respectively. Shukla pointed out that this is a combination of the "deficit of the equation of center" and the "evection".

We can understand this expression as follows. From the equation (8) in my Appendix 3, the eccentricity looks $E \approx e - \frac{15}{8}me.\ \cos 2\xi$, and if it is obtained from the observation of lunar eclipses (like Hipparchus etc.), when ξ is 180°, the eccentricity appears as $E \approx e - \frac{15}{8}me$, and the equation of center appear as $2(e - \frac{15}{8}me)\sin\phi$. Here, $2\times\frac{15}{8}me.$ $\sin\phi$ corresponds to the "deficit of the equation of center". Then:

$$\text{"deficit of the equation of center"} = \tfrac{15}{4}\ me\ \sin\phi \tag{B}$$

$$\text{"evection"} = \frac{15}{4}\ me\ \sin(2\xi - \phi) \tag{C}$$

Mañjula's second correction corresponds to the sum of (B) and (C).

$$\begin{aligned}(B)+(C) &= \tfrac{15}{4}me\{\sin\phi + \sin(2\xi-\phi)\} \\ &= \frac{15}{4}me \times 2\{\sin\xi.\cos(\phi-\xi)\}\end{aligned}$$

or

$$(B) + (C) = \frac{15}{2} me\{\sin(\Lambda - \Xi).\cos(\Xi - A)\}. \tag{D}$$

If we use the notation of the expression (A), the expression (D) can be expressed as:

$$(B) + (C) = \frac{15}{2} me\{sin(M - S).\cos(S - U)\}.$$

This expression corresponds to the expression (A). From this fact, we know that Mañjula's second correction is justified. Mañjula's way of thinking might have been different from Ptolemy, and we should investigate his method further.

7.3.6 The Bīja-upanaya controversy

There is a small work *Bīja-upanaya* (= *Bījopanaya*), where the lunar inequality corresponding to the "variation" is mentioned and attributed to Bhāskara II, but its authorship is controversial. Ghosh maintained that it is Bhāskara II's own work, and published the following edition:

Ghosh, Ekendranath (ed.): *Bhaskariya-Bījopanaya*, Motilal Banarsidass, Lahore, 1926.

This work mentions two corrections, an improvement of the equation of center in Hindu Classical Astronomy, and an inequality corresponding to the variation. Although the text itself mentions Bhāskara as its author, Kuppanna Sastry, criticizing Ghosh, pointed out several reasons that it cannot be the work of Bhāskara II, some of which are as follows. The first correction produces an error which is unusual for the otherwise accurate system of Bhāskara II. The *Bīja-upanaya* was not known to astronomers who followed Bhāskara II. In this work, the corrections are given in the form of tabular values, although this kind of calculation is usually given in the form of equations. And also, the style of this work is not Bhāskara II's.

These arguments of Kuppanna Sastry are understandable, and the *Bīja-upanaya* will not be a work of Bhāskara II, but the real author of this *Bīja-upanaya* is still to be investigated.

7.4 Conclusion

We have seen that eccentric and epicyclic models in Indian Classical Astronomy have several origins. Greek models are evidently geometrical, but Indian models may not be purely geometrical. More research on the Indian models is needed.

Appendix

Appendix 1: The accuracy of the simple eccentric model

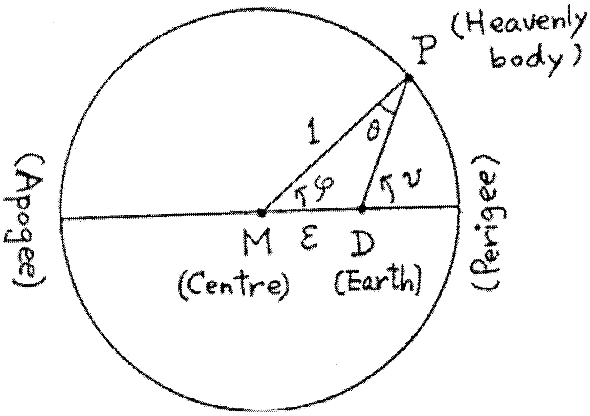


Figure 6: Simple Eccentric Model

Figure 6 shows an eccentric circle whose radius is 1, and a heavenly body P moves along the circle with a constant speed. Let this model be called "Simple eccentric model". In this figure, let

ϕ : Mean anomaly,

v : True anomaly,

θ (equation of center) $\equiv v - \phi$

$MD \equiv \varepsilon$ (eccentric distance).

From Figure 7,

$\sin\theta = \varepsilon \sin v = \varepsilon \sin(\phi + \theta) = \varepsilon sin\phi \ \cos\theta + \varepsilon \cos\phi \sin\theta$

Therefore,

$\sin\theta(1 - \varepsilon\cos\phi) = \varepsilon \sin\phi \cos\theta$

$\tan\theta = \frac{\varepsilon \sin\varphi}{1-\varepsilon\cos\varphi} \approx \varepsilon \sin\phi(1 + \varepsilon\cos\phi)$

$= \varepsilon \sin\phi + \varepsilon^2 \sin\phi\cos\phi = \varepsilon\sin\phi + \frac{1}{2}\varepsilon^2 \sin 2\phi$

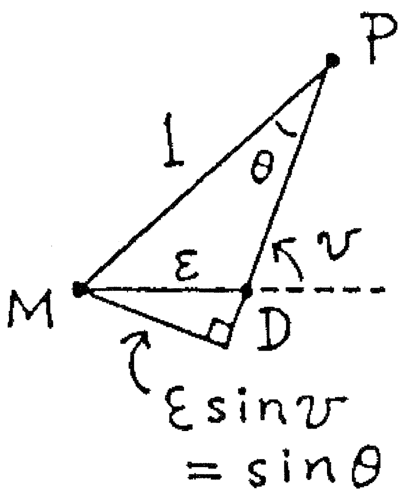


Figure 7: A part of the Figure 6

Now, let $\varepsilon = 2e$. As θ is small, $\tan\theta \approx \theta$ approximately. Therefore, we get the following expression as the **equation of center according to the "Simple eccentric model"**.

$$\theta \approx 2e\sin\phi e^2 \sin 2\phi \tag{1}$$

Here, e corresponds to the eccentricity in the modern sense.

In the **true Kepler motion** (according to modern astronomy), the equation of center is expressed as:

$$\theta \approx 2e\sin\varphi + \frac{5}{4}e^2\sin 2\varphi \tag{2}$$

Therefore, if $\varepsilon = 2e$, the first terms of the equations (1) and (2) are the same, and their second terms differs slightly. It means that these equations give the same result at the apogee, perigee, and the points which are at the distance of 90° from them, but give slightly different results at other points.

Appendix 2: The accuracy of the "Equant" model

Figure 8 shows an eccentric circle whose radius is 1, and a heavenly body P revolves around a point E ("equant") with a constant angular velocity. Let this model be called "Equant model".

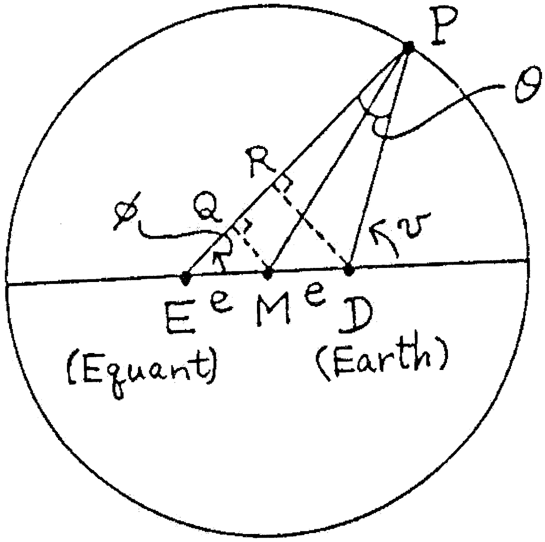


Figure 8: Equant Model

Let $MD = ME = e$. Then, as $QM = e\sin\phi$, and $PM = 1$, we get

$$PQ = \sqrt{1 - e^2\sin^2\varphi}.$$

And also, from this equation and $RQ = QE = e\cos\phi$,

$$PR = PQ - RQ = \sqrt{1 - e^2\sin^2\varphi} - e\cos\varphi.$$

And also, as $RD = 2e\sin\phi$, we have

$$\tan\theta = \frac{RD}{PR} = \frac{2e\sin\varphi}{\sqrt{1 - e^2\sin^2\varphi} - e\cos\varphi}.$$

Multiply its denominator and numerator by $\sqrt{1 - e^2\sin^2\varphi} + e\cos\varphi$, and also, use Taylor series $(1+x)^{1/2} \approx 1 + {}^1\!/_2 x$, where x is small. Then, we get:

$$\tan\theta \approx \frac{2e\sin\varphi(1 - \frac{1}{2}e^2\sin^2\varphi + e\cos\varphi)}{1 - e^2}.$$

Here, if terms higher than e^3 is ignored, we get:

$$\tan\theta \approx 2e\sin\phi + 2e^2\sin\phi\cos\phi$$

Then, as θ is small, using $\tan\theta \approx \theta$, we get the following expression as the **equation of center according to the "Equant model"**.

$$\theta \approx 2e\sin\phi + e^2\sin 2\phi \tag{3}$$

From this equation, we know that the second term of the "Equant model" is closer to that of the true Kepler motion (equation (2)) than that of the "Simple eccentric model" (equation (1)). And also, it is evident that the distance to the heavenly body in the "Equant model" is close to the actual distance, but that of the "Simple eccentric model" is not so.

Appendix 3: The composition of the equation of center and the evection of the moon

In this section, I shall show that Ptolemy's model (regarding the longitude of the moon) is justified with respect to the composition of the modern equation of center and the evection of the moon. (Of course, the distance of the moon in Ptolemy's model is not justified.) (I have consulted Araki (1956) p.259-260 and Araki (1980) p.259-260.)

Let:

Λ : lunar mean longitude

Ξ : solar mean longitude

A: mean longitude of lunar apogee (which revolves once in about 9 years to the direct direction).

And also:

$\Phi \equiv \Lambda - A$ (lunar mean anomaly)

$\xi \equiv \Lambda - \Xi$

(mean angular distance between the moon and the sun).

If we neglect higher term than e^2 (e is eccentricity), the modern equation of center and evection can be expressed as follows:

$$\text{Equation of center} = 2e\sin\phi \tag{1}$$

$$\text{Evection} = \frac{15}{4} me \sin(2\xi - \phi) \tag{2}$$

Here, $m \equiv n'/n$;

(n' is the sun's mean angular velocity, n is the moon's mean angular velocity, and e and m are small values at the order of 10^{-2} or so.)

Let us express the composition of the equation of center and evection as follows:

$$\text{Equation of center + evection} \approx 2E\sin(\phi + \delta). \tag{3}$$

We can define E and δ as follows:

$$E\sin\delta \equiv \tfrac{15}{8}\ me\ \ \sin 2\xi. \tag{4}$$

$$E\cos\delta \equiv e - \frac{15}{8}\ me\ \ \cos 2\xi. \tag{5}$$

The reason why we can define like this is that we can obtain the following equation from the equation (3):

$$\text{Equation of center + evection} \approx 2E(\sin\phi\cos\delta + \cos\phi\sin\delta).$$

If we substitute the equations (4) and (5) here, the result is the same as the sum of the equations (1) and (2).

The sum of the squares of the equations (4) and (5) is:

$$\begin{aligned} E^2 &= (e - \frac{15}{8} me\cos 2\xi)^2 + (\tfrac{15}{8} me\sin 2\xi)^2 \\ &= e^2 - \frac{15}{4} me^2\cos 2\xi + \frac{15}{8} m^2e^2. \end{aligned}$$

Neglecting higher terms of m and e, and using Taylor series $(1+x)^{1/2} \approx 1+{}^1\!/\!_2 x$, we get:

$$E \approx e - \tfrac{15}{8} me\cos 2\xi. \tag{6}$$

Dividing the equation (4) by the equation (5), we get :

$$\tan\delta = \frac{\frac{15}{8} me\sin 2\xi}{e - \frac{15}{8} me\cos 2\xi} \approx \frac{15}{8} m\sin 2\xi.$$

As δ is small, using $\tan\delta \approx \delta$, we get:

$$\delta \approx \frac{15}{8} m \sin 2\xi. \tag{7}$$

From the equations (3), (6) and (7), we get:

$$\textbf{Equation of center + evection} \approx 2E\sin(\phi+\delta); \tag{8}$$

$$\text{here, } E \approx e - \frac{15}{8} me\cos 2\xi,\ \ \delta \approx \tfrac{15}{8} m \sin 2\xi.$$

From the equation (8), we know that: E is smallest at the time of new moon and full moon, and is largest at the time of half moon; δ is positive in the 1st and 3rd quadrants, and the perigee looks as if going back, and is negative in the 2^{nd} and 4th quadrants, and the perigee looks as if moving ahead. These results agree with Ptolemy's model.

We should note that Ptolemy's model is only good for the lunar longitude. As regards the distance of the moon from the earth (or apparent diameter of the moon), Ptolemy's model is far from the truth.

Appendix 4: Double epicycle model

The "Double epicycle model" is a device to give an almost similar result to the "Equant model" by a combination of uniform circular motion only. It will be clear from Fig. 9.

In the "Double epicycle model", the first (larger) epicycle is on the deferent, and is revolving eastwards. There is a second (smaller) epicycle on the first epicycle, and the second epicycle is revolving westwards with respect to the line between the center of the deferent and the center of the first epicycle, and so the second epicycle is always at the direction of the apogee on the first epicycle. The heavenly body is on the second epicycle, and revolves eastwards. The heavenly body's speed is double the speed of the center of the epicycles, and is in the direction of the perigee at the apogee and perigee.

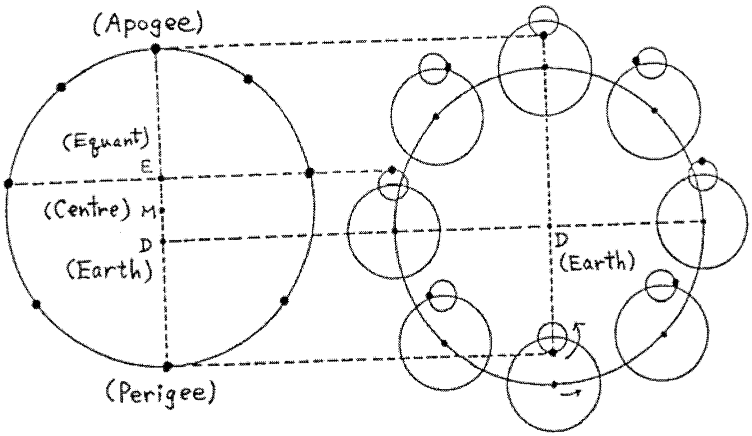


Figure 9: Equant Model and Double Epicycle Model

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