6 Spaces of Constant Sectional Curvature.

Orthonormal Frame and Connection Forms. Let (M,g) be an orientable Riemannian m-manifold, and $(U;u^1,\dots,u^m)$ a local coordinate neighborhood.

Lemma 6.1. *There exists a m-tuple of vector fields $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_m\}$ on M which forms a positively-oriented orthonormal basis of $T_{\mathrm{P}}M$ for each $\mathrm{P}\in U$.*

Proof. The procedure of the Gram-Schmidt orthogonalization works for the m-tuple of vector fields $\{\partial/\partial u^j\}_{j=1}^m$ on U. □

We call such a m-tuple $\{\boldsymbol{e}_j\}_{j=1}^m$ a positively-oriented *orthonormal frame field*, or a *frame field* for short, on U.

Lemma 6.2. *Let $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_m\}$ be an orthonormal frame field on $U\subset M$. Then there exist C^∞-differential 1-forms ω_i^j $(i,j=1,\dots,m)$ satisfying*

$$\nabla_X\boldsymbol{e}_i=\sum_{j=1}^m\omega_i^j(X)\boldsymbol{e}_i \qquad (i=1,\dots,m), \tag{6.1}$$

$$\omega_i^j=-\omega_j^i \qquad (i,j=1,\dots,m) \tag{6.2}$$

for an arbitrary vector field X on U, where ∇ denotes the covariant derivative (4.12).

Proof. We set

$$\omega_i^j(X):=g\left(\nabla_X\boldsymbol{e}_i,\boldsymbol{e}_j\right)$$

for $X\in\mathfrak{X}(U)$. So by (4.15), $\omega_i^j(fX)=f\omega_i^j(X)$ holds for $f\in C^\infty(U)$. Hence $\omega_i^j(X)(\mathrm{P})$ depends only on X_{P} because of Lemma 5.3. Then each ω_i^j defines a 1-form on U. Smoothness of ω_i^j is obvious. Since $\{\boldsymbol{e}_j\}$ is an orthonormal basis, (6.1) follows.

Moreover, since $g(\boldsymbol{e}_i,\boldsymbol{e}_j)=\delta_{ij}$ is constant for each i and j, (4.18) implies

$$\begin{aligned}0=Xg(\boldsymbol{e}_i,\boldsymbol{e}_j)&=g\left(\nabla_X\boldsymbol{e}_i,\boldsymbol{e}_j\right)+g\left(\boldsymbol{e}_i,\nabla_X\boldsymbol{e}_j\right)\\&=g\left(\sum_{k=1}^m\omega_i^k(X)\boldsymbol{e}_k,\boldsymbol{e}_j\right)+g\left(\boldsymbol{e}_i,\sum_{k=1}^m\omega_j^k(X)\boldsymbol{e}_k\right)\\&=\sum_{k=1}^m\left(\omega_i^k(X)\delta_{kj}+\omega_j^k(X)\delta_{ik}\right)=\omega_i^j(X)+\omega_j^i(X).\end{aligned}$$

Hence (6.2) follows. □

We call $\{\omega_i^j\}$ in Lemma 6.2 the *connection forms* with respect to the frame $\{\boldsymbol{e}_j\}$.

By (6.2),

$$\boldsymbol{\omega}:=\begin{pmatrix}\omega_1^1&\dots&\omega_m^1\\\vdots&\ddots&\vdots\\\omega_1^m&\dots&\omega_m^m\end{pmatrix}\quad\text{satisfies}\quad\boldsymbol{\omega}+{}^t\boldsymbol{\omega}=O, \tag{6.3}$$

in other words, $\boldsymbol{\omega}$ is a skew-symmetric matrix-valued 1-form.

Gauge transformations and the Curvature Form. Let $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_m\}$ and $\{\boldsymbol{f}_1,\dots,\boldsymbol{f}_m\}$ be two positively-oriented or-

23. July, 2019.

thonormal frames on $U \subset M$. Then there exists a smooth map $G = (G_{ij})\colon U \to \mathrm{SO}(m)$ such that[8]

$$(6.4)\quad (\boldsymbol{e}_1, \dots, \boldsymbol{e}_m) = (\boldsymbol{f}_1, \dots, \boldsymbol{f}_m)G = \left(\sum_{a=1}^m G_{1a}\boldsymbol{f}_a, \dots, \sum_{a=1}^m G_{ma}\boldsymbol{f}_a\right).$$

Let $\boldsymbol{\omega} = (\omega_i^j)$ (resp. $\widetilde{\boldsymbol{\omega}} = (\tilde{\omega}_a^b)$) be the connection forms with respect to the orthonormal frame $\{\boldsymbol{e}_j\}$ (resp. $\{\boldsymbol{f}_a\}$). The

Lemma 6.3. *Under the situation above, it holds that*

$$(6.5)\qquad \widetilde{\boldsymbol{\omega}} = G^{-1}dG + G^{-1}\boldsymbol{\omega}G.$$

Proof. By definition,

$$\nabla(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m) = (\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)\boldsymbol{\omega}, \qquad \text{and}$$
$$\nabla(\boldsymbol{f}_1, \dots, \boldsymbol{f}_m) = (\boldsymbol{f}_1, \dots, \boldsymbol{f}_m)\widetilde{\boldsymbol{\omega}}$$

hold. Hence, by (4.16), it holds that

$$\begin{aligned}(\boldsymbol{f}_1, \dots, \boldsymbol{f}_m)\widetilde{\boldsymbol{\omega}} &= (\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)G\widetilde{\boldsymbol{\omega}}\\ (\boldsymbol{f}_1, \dots, \boldsymbol{f}_m)\widetilde{\boldsymbol{\omega}} &= \nabla\big((\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)G\big)\\ &= (\nabla(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m))\,G + (\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)dG\end{aligned}$$

[8]As defined in Section 1, $\mathrm{SO}(m) = \{A \in \mathrm{M}_m(\mathbb{R})\,;\, {}^tAA = A^tA = \mathrm{id}, \det A = 1\}$ denotes the special orthogonal group. A map $G\colon U \to \mathrm{SO}(m)$ is said to be smooth (of class C^∞) if it is of class C^∞ as a map into $\mathrm{M}_m(\mathbb{R})$, the set of $m \times m$-real matrices, which is identified with $\mathbb{R}^{n^2}$.

$$\begin{aligned}&= (\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)\boldsymbol{\omega}G + (\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)dG\\ &= (\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)\,(\boldsymbol{\omega}G + dG)\,,\end{aligned}$$

where $\nabla\boldsymbol{v}$ means a 1-form $X \mapsto \nabla_X\boldsymbol{v}$. Since $\boldsymbol{e}_1, \dots, \boldsymbol{e}_m$ are linearly independent, the conclusion follows. □

The formula (6.5) is called the *Gauge transformation* of the connection forms.

Definition 6.4. The *curvature form* with respect to the frame field $\{\boldsymbol{e}_j\}$ is a skew-symmetric matrix-valued 2-form

$$(6.6)\qquad \boldsymbol{\Omega} := d\boldsymbol{\omega} + \boldsymbol{\omega}\wedge\boldsymbol{\omega} = \left(d\omega_i^j + \sum_{k=1}^m \omega_i^k\omega_k^j\right)_{i,j=1,\dots,m}.$$

Lemma 6.5. *Under the transformation as in* (6.4)*, the curvature form $\boldsymbol{\Omega}$ and $\widetilde{\boldsymbol{\Omega}}$ with respect to the frame field $\{\boldsymbol{e}_j\}$ and $\{\boldsymbol{f}_a\}$, respectively, satisfy*

$$\widetilde{\boldsymbol{\Omega}} = G^{-1}\boldsymbol{\Omega}G.$$

Proof. Problem 6-1.

Lemma 6.6. *The curvature form $\boldsymbol{\Omega} = (\Omega_i^j)$ with respect to the frame field $\{\boldsymbol{e}_j\}$ satisfies*

$$\Omega_i^j(X, Y) = R(X, Y, \boldsymbol{e}_i, \boldsymbol{e}_j),$$

where R is the Riemann-Christoffel curvature tensor, and X, Y are vector fields.

Proof. Since $\{\boldsymbol{e}_j\}$ is an orthonormal basis, it holds that

$$g(\nabla_X \boldsymbol{e}_j, \boldsymbol{e}_k) = Xg(\boldsymbol{e}_j, \boldsymbol{e}_k) - g(\boldsymbol{e}_j, \nabla_X \boldsymbol{e}_k) = -g(\boldsymbol{e}_j, \nabla_X \boldsymbol{e}_k)$$

holds for $j, k = 1, \dots, m$. Then we have

$$\begin{aligned}
\Omega_i^j(X,Y) &= d\omega_i^j(X,Y) + \sum_{k=1}^m \bigl(\omega_i^k(X)\omega_k^j(Y) - \omega_i^k(Y)\omega_k^j(Y)\bigr) \\
&= X\omega_i^j(Y) - Y\omega_i^j(X) - \omega_i^j([X,Y]) \\
&\quad \sum_{k=1}^m \bigl(\omega_i^k(X)\omega_k^j(Y) - \omega_i^k(Y)\omega_k^j(Y)\bigr) \\
&= Xg(\nabla_Y \boldsymbol{e}_i, \boldsymbol{e}_j) - Yg(\nabla_X \boldsymbol{e}_i, \boldsymbol{e}_j) - g(\nabla_{[X,Y]} \boldsymbol{e}_i, \boldsymbol{e}_j) \\
&\quad + \sum_{k=1}^m \Bigl(g(\nabla_X \boldsymbol{e}_i, \boldsymbol{e}_k) g(\nabla_Y \boldsymbol{e}_k, \boldsymbol{e}_j) - g(\nabla_Y \boldsymbol{e}_i, \boldsymbol{e}_k) g(\nabla_X \boldsymbol{e}_k, \boldsymbol{e}_j) \Bigr) \\
&= g(\nabla_X \nabla_Y \boldsymbol{e}_i, \boldsymbol{e}_j) + g(\nabla_Y \boldsymbol{e}_i, \nabla_X \boldsymbol{e}_j) \\
&\quad - g(\nabla_Y \nabla_X \boldsymbol{e}_i, \boldsymbol{e}_j) - g(\nabla_X \boldsymbol{e}_i, \nabla_Y \boldsymbol{e}_j) - g(\nabla_{[X,Y]} \boldsymbol{e}_i, \boldsymbol{e}_j) \\
&\quad - \sum_{k=1}^m \Bigl(g(\nabla_X \boldsymbol{e}_i, \boldsymbol{e}_k) g(\boldsymbol{e}_k, \nabla_Y \boldsymbol{e}_j) - g(\nabla_Y \boldsymbol{e}_i, \boldsymbol{e}_k) g(\boldsymbol{e}_k, \nabla_X \boldsymbol{e}_j) \Bigr) \\
&= R(X, Y, \boldsymbol{e}_i, \boldsymbol{e}_j),
\end{aligned}$$

where we used the relation

$$\sum_{k=1}^m g(\boldsymbol{v}, \boldsymbol{e}_k) g(\boldsymbol{w}, \boldsymbol{e}_k) = g(\boldsymbol{v}, \boldsymbol{w}). \qquad \Box$$

Space of Constant Sectional Curvature. The goal of this lecture is to prove the following

Theorem 6.7. *Let $U \subset \mathbb{R}^m$ be a simply connected domain and let g be a Riemannian metric on U with constant sectional curvature k. Then there exists a local diffeomorphism*

$$f : U \longrightarrow M^m(k)$$

such that the Riemannian metric g coincide with the metric on U induced from $M^m(k)$ by f, where

$$M^m(k) := \begin{cases} S^m(k) & (\text{when } k > 0, \text{ cf. Example 5.16}), \\ \mathbb{R}^m & (\text{when } k = 0, \text{ the Euclidean } m\text{-space}), \\ H^m(k) & (\text{when } k < 0, \text{ cf. Example 5.17}). \end{cases}$$

Remark 6.8. The theorem can be generalized for simply connected Riemannian manifolds (M, g) of constant sectional curvature. Moreover, one can show that f is injective. Hence, we can say that *a simply connected Riemannian m-manifold (M, g) can be identified as a subset of $M^m(k)$*. In particular, if (M, g) is complete, it coincides with $M^m(k)$.

Proof of Theorem 6.7 (for the case $k = 0$): This is an alternative proof of Theorem 3.13 in Section 3. Take an orthonormal frame field $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ on U, and let $\boldsymbol{\omega}$ be the connection form with respect to the basis. Fix a base point $\mathrm{P}_0 \in U$, and consider the system of differential equations

$$(6.7) \qquad \frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\boldsymbol{\omega}_j, \qquad \mathcal{F}(\mathrm{P}_0) = \mathrm{id} \qquad (j = 1, \dots, m),$$

where

$$(6.8) \qquad \boldsymbol{\omega}_j := \boldsymbol{\omega}\left(\frac{\partial}{\partial u^j}\right).$$

By Lemma 6.6, the assumption $k=0$ implies

$$\begin{aligned} O &= \left(R\left(\frac{\partial}{\partial u^i},\frac{\partial}{\partial u^j},\boldsymbol{e}_k,\boldsymbol{e}_l\right)\right)_{k,l=1,\dots,m} \\ &= (d\boldsymbol{\omega}+\boldsymbol{\omega}\wedge\boldsymbol{\omega})\left(\frac{\partial}{\partial u^i},\frac{\partial}{\partial u^j}\right) = \frac{\partial\boldsymbol{\omega}_j}{\partial u^i}-\frac{\partial\boldsymbol{\omega}_i}{\partial u^j}+\boldsymbol{\omega}_i\boldsymbol{\omega}_j-\boldsymbol{\omega}_j\boldsymbol{\omega}_i. \end{aligned}$$

Hence by Theorem 2.5, there exists a unique solution $\mathcal{F}$ of (6.7). Moreover, since $\boldsymbol{\omega}_j$ is skew-symmetric because of (6.3), the solution gives a smooth map $\mathcal{F}\colon U\to \mathrm{SO}(m)$. Decompose $\mathcal{F}$ into the column vectors as $\mathcal{F}=(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)$. Since $\mathcal{F}$ is an orthogonal matrix, $\{\boldsymbol{x}_j(\mathrm{P})\}$ is an orthonormal basis at each P.

Define an $\mathbb{R}^m$-valued 1-form

$$\varphi := \sum_{i=1}^m\left(\sum_{k=1}^m g_i^k\boldsymbol{x}_k\right)du^i,\qquad g_i^k = g\left(\frac{\partial}{\partial u^i},\boldsymbol{e}_k\right),$$

where $\{\boldsymbol{e}_j\}$ is the orthonormal frame on U we took in the beginning of the proof. Then φ is a closed on U. In fact, by (4.18) and (4.17), we have

$$\begin{aligned} \frac{\partial}{\partial u^j}\left(\sum_{k=1}^m g_i^k\boldsymbol{x}_k\right) &= \sum_{k=1}^m\left(\frac{\partial}{\partial u^j}g\left(\frac{\partial}{\partial u^i},\boldsymbol{e}_k\right)+g_i^k\boldsymbol{x}_k\right) \\ &= \sum_{k=1}^m\left[\left(g\left(\nabla_{\frac{\partial}{\partial u^j}}\frac{\partial}{\partial u^i},\boldsymbol{e}_k\right)+g\left(\frac{\partial}{\partial u^i},\nabla_{\frac{\partial}{\partial u^j}}\boldsymbol{e}_k\right)\right)\boldsymbol{x}_k+g_i^k\frac{\partial\boldsymbol{x}_k}{\partial u^j}\right] \\ &= \sum_{k=1}^m g\left(\nabla_{\frac{\partial}{\partial u^j}}\frac{\partial}{\partial u^i},\boldsymbol{e}_k\right)\boldsymbol{x}_k+\sum_{k=1}^m g\left(\frac{\partial}{\partial u^i},\sum_{l=1}^m\omega_k^l\left(\frac{\partial}{\partial u^j}\right)\boldsymbol{e}_l\right)\boldsymbol{x}_k \end{aligned}$$

$$\begin{aligned} &\quad+\sum_{k,l=1}^m g_i^k\omega_k^l\left(\frac{\partial}{\partial u^j}\right)\boldsymbol{x}_l \\ &= \sum_{k=1}^m\left[g\left(\nabla_{\frac{\partial}{\partial u^j}}\frac{\partial}{\partial u^i},\boldsymbol{e}_k\right)+\sum_{l=1}^m(\omega_k^l+\omega_l^k)\left(\frac{\partial}{\partial u^j}\right)g_i^l+\right]\boldsymbol{x}_k \\ &= \sum_{k=1}^m\left[g\left(\nabla_{\frac{\partial}{\partial u^j}}\frac{\partial}{\partial u^i},\boldsymbol{e}_k\right)\right]\boldsymbol{x}_k, \end{aligned}$$

$$\frac{\partial}{\partial u^i}\left(\sum_{k=1}^m g_j^k\boldsymbol{x}_k\right) = \sum_{k=1}^m\left[g\left(\nabla_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^j},\boldsymbol{e}_k\right)\right]\boldsymbol{x}_k.$$

Hence by (4.17), we have

$$\frac{\partial}{\partial u^j}\left(\sum_{k=1}^m g_i^k\boldsymbol{x}_k\right) = \frac{\partial}{\partial u^i}\left(\sum_{k=1}^m g_j^k\boldsymbol{x}_k\right),$$

that is, $d\varphi=0$. Hence by Poincaré's lemma, there exists $f\colon U\to\mathbb{R}^m$ satisfying $df=\varphi$. This f is desired one. To show this, it is sufficient to show

$$df(\boldsymbol{e}_j)=\boldsymbol{x}_j,\qquad (j=1,\dots,m). \tag{6.9}$$

In fact, if (6.9) holds,

$$g(\boldsymbol{e}_i,\boldsymbol{e}_j)=\delta_{ij}=\langle\boldsymbol{x}_i,\boldsymbol{x}_j,=\rangle\,\langle df(\boldsymbol{e}_i),df(\boldsymbol{e}_j),,\rangle$$

and then the induced metric coincides with g. We show (6.9):

$$df(\boldsymbol{e}_j)=\varphi(\boldsymbol{e}_j)=\sum_{i=1}^m\left(\sum_{k=1}^m g_i^k\boldsymbol{x}_k\right)du^i(\boldsymbol{e}_j)$$

$$= \sum_{i,k=1}^{m} g\left(\frac{\partial}{\partial u^i}, \boldsymbol{e}_k\right) du^i(\boldsymbol{e}_j)\boldsymbol{x}_k$$
$$= \sum_{k=1}^{m} g\left(\sum_{i=1}^{m} du^i(\boldsymbol{e}_j)\frac{\partial}{\partial u^i}, \boldsymbol{e}_k\right)\boldsymbol{x}_k = \sum_{k=1}^{m} g(\boldsymbol{e}_j, \boldsymbol{e}_k)\boldsymbol{x}_k = \boldsymbol{x}_j.$$

Here, we used the formula

$$\sum_{i=1}^{m} du^i(\boldsymbol{v})\frac{\partial}{\partial u^i} = \boldsymbol{v}. \qquad \square$$

Proof of Theorem 6.7 (for the case $k > 0$): Since $k > 0$, there exists a real number c such that $k = c^2$. Taking the orthonormal frame field $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ on U, we set

$$\hat{\boldsymbol{\omega}}_j := \begin{pmatrix} 0 & -c\,{}^t\boldsymbol{g}_j \\ c\boldsymbol{g}_j & \boldsymbol{\omega}_j \end{pmatrix}, \tag{6.10}$$

for each $j = 1, \dots, m$, which is an $(n+1)\times(n+1)$-skew symmetric matrix-valued function, here

$$\boldsymbol{g}_j := \begin{pmatrix} g_j^1 \\ \vdots \\ g_j^m \end{pmatrix} = \begin{pmatrix} g(\partial/\partial u^j, \boldsymbol{e}_1) \\ \vdots \\ g(\partial/\partial u^j, \boldsymbol{e}_m) \end{pmatrix} \tag{6.11}$$

and $\boldsymbol{\omega}_j$ is as in (6.8). By the assumption, (5.9) holds. Hence one can show easily that

$$\frac{\partial \hat{\boldsymbol{\omega}}_j}{\partial u^i} - \frac{\partial \hat{\boldsymbol{\omega}}_i}{\partial u^j} + \hat{\boldsymbol{\omega}}_i\hat{\boldsymbol{\omega}}_j - \hat{\boldsymbol{\omega}}_j\hat{\boldsymbol{\omega}}_i = O$$

for each $i, j = 1, \dots, m$. Hence there exists a smooth map

$$\mathcal{F} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_m) \colon U \to \mathrm{SO}(m+1)$$

satisfying

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\hat{\boldsymbol{\omega}}_j \qquad (j = 1, \dots, m)$$

with $\mathcal{F}(\mathrm{P}_0) = \mathrm{id}$. Then

$$f := \frac{1}{c}\boldsymbol{x}_0$$

is the desired map. In fact,

$$df(\boldsymbol{e}_j) = \boldsymbol{x}_j \qquad (j = 1, \dots, m)$$

holds. $\square$

Proof of Theorem 6.7 (for the case $k < 0$): Since $k < 0$, there exists a real number c such that $k = -c^2$. Taking the orthonormal frame field $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ on U, we set

$$\hat{\boldsymbol{\omega}}_j := \begin{pmatrix} 0 & c\,{}^t\boldsymbol{g}_j \\ c\boldsymbol{g}_j & \boldsymbol{\omega}_j \end{pmatrix}, \tag{6.12}$$

for each $j = 1, \dots, m$, which is an $(n+1)\times(n+1)$-matrix-valued function, here

$$\boldsymbol{g}_j := \begin{pmatrix} g_j^1 \\ \vdots \\ g_j^m \end{pmatrix} = \begin{pmatrix} g(\partial/\partial u^j, \boldsymbol{e}_1) \\ \vdots \\ g(\partial/\partial u^j, \boldsymbol{e}_m) \end{pmatrix} \tag{6.13}$$

and $\boldsymbol{\omega}_j$ is as in (6.8). Since

$$Y\boldsymbol{\omega}_j - {}^t\omega_j Y = 0 \qquad (j = 1, \dots, m)$$

holds, where $Y := \operatorname{diag}(-1, 1, \dots, 1)$. This implies that there exists

$$\mathcal{F}\colon U \to \mathrm{SO}(m+1, 1),$$

where

$$\mathrm{SO}(m+1,1) = \\ \left\{a = (a_{ij})_{i,j=0,\dots,m} \in \mathrm{M}_n(\mathbb{R})\,;\, {}^t aYa = Y, \det a = 1, a_{00} > 0.\right\}$$

Then there exists $\mathcal{F}\colon U \to \mathrm{SO}(m+1)$ satisfying

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\hat{\boldsymbol{\omega}}_j \qquad (j = 1, \dots, m)$$

with $\mathcal{F}(\mathrm{P}_0) = \mathrm{id}$. Then

$$f := \frac{1}{c}\boldsymbol{x}_0$$

is the desired map. □

Exercises

6-1 Prove Lemma 6.5.

6-2 Prove Theorem 6.7