# 5 Sectional Curvature.

Throughout this section, we let $(M, g)$ an $m$-dimensional (pseudo) Riemannian manifold, and $\nabla$ the covariant derivative defined in (4.12).

**Tensors.** A correspondence $S : M \ni \mathrm{P} \mapsto S_{\mathrm{P}}$ of a point P and a multi-linear map $S_{\mathrm{P}} \colon (T_{\mathrm{P}}M)^k \to \mathbb{R}$ is called a ($k$-th order covariant) *tensor field* or a *tensor* on $M$. For such a tensor field $S$ trivially induces a map

$$\hat{S} \colon \bigl(\mathfrak{X}(M)\bigr)^p \ni (X_1, \dots, X_p) \mapsto S(X_1, \dots, X_p) \in \mathcal{F}(M) \tag{5.1}$$

where $\mathfrak{X}(M)$ is the set ($C^\infty(M)$-module) of $C^\infty$-vector fields of $M$, and $\mathcal{F}(M)$ is the set of real-valued function on $M$. The tensor field $S$ is said to be *smooth* of *class* $C^\infty$ if $S(X_1, \dots, X_p)$ as in (5.1) is of class $C^\infty$ for an arbitrary $X_1, \dots, X_p$.

**Example 5.1.** The (pseudo) Riemannian metric $g$ is a smooth tensor field.

**Example 5.2.** A smooth 1-form on $M$ is a smooth, first-order covariant tensor field on $M$.

We denote by

$$\Gamma\,(\otimes^p T^*M) = \Gamma(T^*M \otimes \cdots \otimes T^*M) \tag{5.2}$$

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the set of $p$-th order covariant tensor fields on $M$. [7] The set $\Gamma\,(\otimes^p T^*M)$ is a vector space over $\mathbb{R}$. Moreover, for each $S \in \Gamma(\otimes^p T^*M)$ and $f \in C^\infty(M)$, $fS := (\mathrm{P} \mapsto f(\mathrm{P})S_{\mathrm{P}})$ is also an element of $\Gamma(\otimes^p T^*M)$. That is, $\Gamma(\otimes^p T^*M)$ is a $C^\infty(M)$-module.

**Lemma 5.3.** *Let $S$ be a $p$-th covariant tensor field on $M$ and $\hat{S} \colon \mathfrak{X}(M)^p \to C^\infty(M)$ the map induced by $S$ as in* (5.1) *Then for an arbitrary $f \in C^\infty(M)$, it holds that*

$$\hat{S}(X_1, \dots, fX_j, \dots, X_p) = f\hat{S}(X_1, \dots, X_j, \dots, X_p), \tag{5.3}$$

*where $X_1, \dots, X_p \in \mathfrak{X}(M)$.*

*Proof.* For each $\mathrm{P} \in M$,

$$\begin{aligned} S(X_1, \dots, &fX_j, \dots, X_p)(\mathrm{P}) \\ &= S_{\mathrm{P}}\bigl((X_1)_{\mathrm{P}}, \dots, f(\mathrm{P})(X_j)_{\mathrm{P}}, \dots, (X_p)_{\mathrm{P}}\bigr) \\ &= f(\mathrm{P})S_{\mathrm{P}}\bigl((X_1)_{\mathrm{P}}, \dots, (X_j)_{\mathrm{P}}, \dots, (X_p)_{\mathrm{P}}\bigr) \\ &= (fS)(X_1, \dots, X_p)(\mathrm{P}). \qquad \square \end{aligned}$$

**Proposition 5.4.** *A multi-linear map $\hat{S} \colon \bigl(\mathfrak{X}(M)\bigr)^p \to C^\infty(M)$ is induced from a certain $S \in \Gamma(\otimes^p T^*M)$ as in* (5.1) *if $\hat{S}$ is $C^\infty(M)$-multi-linear, that is,* (5.3) *holds for any $f \in C^\infty(M)$ and $X_1, \dots, X_p \in \mathfrak{X}(M)$.*

[7] The symbol "$\otimes$" in (5.2) means the *tensor product*. For example, $T^*M \otimes T^*M$ is a tensor product of the cotangent bundles, which is a certain vector bundle over $M$. The notion $\Gamma(*)$ means the set of sections of the vector bundle "$*$". Anyway, we do not give a precise meaning of these notations.

*Proof.* Assume that (5.1) holds. We fix $\mathrm{P} \in M$ and take a local coordinate system $(U; u^1, \dots, u^m)$ of $M$ around P. According to this coordinate system, we set $X_j = \sum_{l=1}^m \xi_j^l (\partial/\partial u^l)$ $(j = 1, \dots, p)$, where $\xi_j^l$'s are $C^\infty$-functions on $U$. Then by (5.1),

$$\hat{S}(X_1, \dots, X_p) = \sum \xi_1^{i_1} \dots \xi_p^{i_p} \hat{S}\left(\frac{\partial}{\partial u^{i_1}}, \dots, \frac{\partial}{\partial u^{i_p}}\right)$$

holds, where the sum in the right-hand side is taken over $i_j = 1, \dots, m$ $(j = 1, \dots, p)$. This means the value of the left-hand side at P is determined by $\xi_j^{i_l}(\mathrm{P})$, which depend only on $(X_j)_\mathrm{P}$. Hence, for each $\boldsymbol{v}_1, \dots, \boldsymbol{v}_p \in T_\mathrm{P} M$, we can define

$$S_\mathrm{P}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_p) := \hat{S}(X_1, \dots, X_m)(\mathrm{P}),$$

where $X_j$ is an arbitrary vector field on $M$ such that $X_j(\mathrm{P}) = \boldsymbol{v}_j$. Then $S\colon \mathrm{P} \mapsto S_\mathrm{P}$ is the desired one. □

If $\hat{S}$ is induced from a tensor field $S$, we say that $\hat{S}$ itself is a tensor field. From now on, we denote $\hat{S}$ in (5.1) by $S$ for a simplicity. Then, for each $S \in \Gamma(\otimes^p T^* M)$, the $C^\infty$-multi-linear map

$$S\colon \big(\mathfrak{X}(M)\big)^p \longrightarrow C^\infty(M) \tag{5.4}$$

is induced.

Taking a local coordinate system $(U; u^1, \dots, u^m)$ on $M$, we set

$$S_{i_1,\dots,i_p} := S\left(\frac{\partial}{\partial u^{i_1}}, \dots \frac{\partial}{\partial u^{i_p}}\right) \qquad (i_1, \dots, i_p = 1, \dots, m), \tag{5.5}$$



which are called the *components* of $S$ with respect to the local coordinate system $(u^j)$. Let $\{S_{a_1,\dots,a_p}\}$ be the components of $S$ with respect to another coordinate system $(x^a)$. Then it holds that

$$S_{a_1,\dots,a_p} = \sum_{i_1,\dots,i_p=1}^m \frac{\partial u^{i_1}}{\partial x^{a^1}} \dots \frac{\partial u^{i_p}}{\partial x^{a^p}} S_{i_1,\dots,i_p}. \tag{5.6}$$

**Proposition 5.5.** *Let* $S\colon \big(\mathfrak{X}(M)\big)^p \to C^\infty(M)$ *be a multi-linear map, and set* $S_{i_1,\dots,i_p}$ *by* (5.5)*. Then* $S$ *is a tensor field on* $M$ *if and only if it satisfy* (5.6) *for an arbitrary coordinate change* $(x^a) \mapsto (u^j)$*.*

*Proof.* Problem 5-1. □

**The Curvature Tensor.** Let $\nabla$ be the covariant derivative on $(M, g)$, as defined in (4.12), which is considered as

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \longmapsto \nabla_X Y \in \mathfrak{X}(M).$$

*Remark* 5.6. The tri-linear map

$$D\colon \big(\mathfrak{X}(M)\big)^3 \ni (X, Y, Z) \mapsto g(\nabla_X Y, Z) \in C^\infty(M)$$

is not a tensor field. In fact, (4.16) means that $D(X, fY, Z)$ and $fD(X, Y, Z)$ may not coincide. But for a fixed $Y \in \mathfrak{X}(M)$,

$$DY\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Z) \mapsto g(\nabla_X Y, Z) \in C^\infty(M)$$

is a tensor because of (4.15).

As seen in the proof in Proposition 4.10, the 4-linear map $R\colon \bigl(\mathfrak{X}(M)\bigr)^4 \to C^\infty(M)$ defined by

$$R(X,Y,Z,W) := g\bigl(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W\bigr), \tag{5.7}$$

where $[X,Y]$ denotes the *Lie bracket* of the vector fields, is a tensor field, which we call the *curvature tensor*, or the *Riemann-Christoffel curvature tensor* of $(M,g)$.

**Proposition 5.7.** *The curvature tensor $R$ has the following symmetricity:*

(1) $R(Y,X,Z,W) = -R(X,Y,Z,W)$.

(2) $R(X,Y,W,Z) = -R(X,Y,Z,W)$.

(3) $R(X,Y,Z,W) = R(Z,W,X,Y)$.

(4) $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$,

*where $X$, $Y$, $Z$ and $W$ are vector fields.*

*Proof.* The equality (1) follows from the property of the Lie bracket $[Y,X] = -[X,Y]$. The equality (4) can be proved by the property (4.17) and the Jacobi identity

$$\bigl[[X,Y],Z\bigr] + \bigl[[Y,Z],X\bigr] + \bigl[[Z,X],Y\bigr] = 0$$

for the Lie bracket. The property (2) can be shown by applying (4.18) and (4.17) (Problem 5-2). The property (3) follows from (4), (1) and (2). In fact, summing up

$$\begin{aligned}
R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) &= 0,\\
R(Y,Z,W,X) + R(Z,W,Y,X) + R(W,Y,Z,X) &= 0,\\
R(Z,W,X,Y) + R(W,X,Z,Y) + R(X,Z,W,Y) &= 0,\\
R(W,X,Y,Z) + R(X,Y,W,Z) + R(Y,W,X,Z) &= 0,
\end{aligned}$$

(3) follows. □



**Proposition 5.8.** *Assume two tensors $R_1$ and $R_2 \in \Gamma(\otimes^4 T^*M)$ satisfy the symmetricity as in Proposition 5.7. If*

$$R_1(X,Y,Y,X) = R_2(X,Y,Y,X)$$

*holds for all $X$, $Y \in \mathfrak{X}(M)$, then $R_1 = R_2$.*

*Proof.* Expanding

$$\begin{aligned}
&R_1(X+sZ, Y+tW, Y+tW, X+sZ)\\
&\qquad = R_2(X+sZ, Y+tW, Y+tW, X+sZ),
\end{aligned}$$

we have the conclusion from the coefficients of $st$. □

**Sectional Curvature.**

**Lemma 5.9.** *Let $R$ be the curvature tensor of a (pseudo) Riemannian manifold $(M,g)$. Then, for each $\mathrm{P} \in M$,*

$$\frac{R(\boldsymbol{v},\boldsymbol{w},\boldsymbol{w},\boldsymbol{v})}{g(\boldsymbol{v},\boldsymbol{v})g(\boldsymbol{w},\boldsymbol{w}) - g(\boldsymbol{v},\boldsymbol{w})^2} \qquad (\boldsymbol{v},\boldsymbol{w} \in T_{\mathrm{P}}M)$$

*depends only on the* 2*-dimensional subspace of* $T_PM$ *spanned by* $\{\boldsymbol{v}, \boldsymbol{w}\}$ *whenever*

$$g(\boldsymbol{v},\boldsymbol{v})g(\boldsymbol{w},\boldsymbol{w}) - g(\boldsymbol{v},\boldsymbol{w})^2 \neq 0. \tag{5.8}$$

*Proof.* Set

$$(\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{v},\boldsymbol{w})A,$$

where $A \in \mathrm{GL}(2,\mathbb{R})$. Then

$$R(\boldsymbol{x},\boldsymbol{y},\boldsymbol{y},\boldsymbol{x}) = (\det A)^2 R(\boldsymbol{v},\boldsymbol{w},\boldsymbol{w},\boldsymbol{v}),$$
$$g(\boldsymbol{x},\boldsymbol{x})g(\boldsymbol{y},\boldsymbol{y}) - g(\boldsymbol{x},\boldsymbol{y})^2 = (\det A)^2\bigl(g(\boldsymbol{v},\boldsymbol{v})g(\boldsymbol{w},\boldsymbol{w}) - g(\boldsymbol{v},\boldsymbol{w})^2\bigr).$$

Hence the conclusion follows. □

*Remark* 5.10. When $g$ is positive definite (i.e., $(M,g)$ is a Riemannian manifold), (5.8) holds if and only if $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent. On the other hand, when $g$ is indefinite, the left-hand side of (5.8) may vanish even if $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent. In this case, (5.8) holds if and only if $\mathrm{Span}\{\boldsymbol{v},\boldsymbol{w}\}$ is a non-degenerate subspace of $T_PM$.

**Definition 5.11.** For a 2-dimensional non-degenerate subspace $\Pi_P \subset T_PM$, we set

$$K(\Pi_P) := \frac{R(\boldsymbol{v},\boldsymbol{w},\boldsymbol{w},\boldsymbol{v})}{g(\boldsymbol{v},\boldsymbol{v})g(\boldsymbol{w},\boldsymbol{w}) - g(\boldsymbol{v},\boldsymbol{w})^2)},$$

where $\{\boldsymbol{v},\boldsymbol{w}\}$ is a basis of $\Pi_P$. We call it the *sectional curvature* at $\Pi_P$.



*Remark* 5.12. The set of 2-dimensional subspaces on an $n$-dimensional vector space $V$ can be endowed with the structure of a compact $(2n-4)$-dimensional manifold, denoted by $\mathrm{Gr}_2(V)$, which is called the 2-*Grassmanian manifold* over $V$. So, when $(M,g)$ is a Riemannian, the sectional curvature can be considered as a smooth map

$$K\colon \mathrm{Gr}_2(TM) := \bigcup_{P\in M} \mathrm{Gr}_2(T_PM) \to \mathbb{R}.$$

**Example 5.13.** Let $(M,g)$ be a 2-dimensional Riemannian manifold. Since $\mathrm{Gr}_2(T_PM)$ consists of one point, the sectional curvature $K$ can be regarded as a function defined on $M$ itself. In this case, the sectional curvature is written as

$$\begin{aligned} K = &\frac{E(E_vG_v - 2F_uG_v + G_u^2)}{4(EG-F^2)^2} \\ &+ \frac{F(E_uG_v - E_vG_u - 2E_vF_v - 2F_uG_u + 4F_uF_v)}{4(EG-F^2)^2} \\ &+ \frac{G(E_uG_u - 2E_uF_v + E_v^2)}{4(EG-F^2)^2} - \frac{E_{vv} - 2F_{uv} + G_{uu}}{2(EG-F^2)}, \end{aligned}$$

where $(u^1,u^2) = (u,v)$ is a local coordinate system and

$$E = g_{11}, \qquad F = g_{12} = g_{21}, \qquad G = g_{22}.$$

**Constant Sectional Curvature.** A Riemannian manifold $(M,g)$ is said to be a *space of constant sectional curvature* if $K$ is constant everywhere.

**Proposition 5.14.** *A (pseudo) Riemannian manifold $(M, g)$ has constant sectional curvature $k$ if and only if its curvature tensor $R$ satisfies*

$$R(X, Y, Z, W) = k(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)). \tag{5.9}$$

*Proof.* If $R$ satisfies (5.9), $K = k$ is constant obviously. Conversely, assume $K = k$ is constant. Then (5.9) holds for $Z = Y$, $W = X$. Since the right-hand side has the symmetric property as in Proposition 5.7, Proposition 5.8 yields (5.9). □

**Example 5.15.** The curvature tensor of the Euclidean space $\mathbb{R}^n$ vanishes identically, because $R$ vanishes identically. The covariant derivative of $\mathbb{R}^n$ is identified with the directional derivative $D$. This means that

$$D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z = O$$

holds for vector fields $X$, $Y$ and $Z$.

**Example 5.16.** Let $k > 0$ and

$$S^n(k) := \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1} \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = \frac{1}{k} \right\}.$$

As seen in Example 3.3, this is an $n$-dimensional submanifold of $\mathbb{R}^{n+1}$, and then is a Riemannian manifold with the induced metric from $\mathbb{R}^{n+1}$.

We compute the sectional curvature $S^n(k)$: The unit normal vector of $S^n(k)$ at $\boldsymbol{x}$ is $\boldsymbol{n} := \boldsymbol{x}/\sqrt{k}$. Then, regarding vector field



on $S^n(k)$ as a vector field of $\mathbb{R}^{n+1}$ along $S^n(k)$, we have by Proposition 4.8

$$\begin{aligned}
\nabla_Y Z = [D_Y Z]^{\mathrm{T}} &= D_Y Z - \frac{1}{\sqrt{k}} \langle D_Y Z, \boldsymbol{x} \rangle \boldsymbol{x} \\
&= D_Y Z - \frac{1}{\sqrt{k}} Y \langle Z, \boldsymbol{x} \rangle \boldsymbol{x} + \frac{1}{\sqrt{k}} \langle Z, D_Y \boldsymbol{x} \rangle \boldsymbol{x} \\
&= D_Y Z - \frac{1}{\sqrt{k}} Y \langle Z, \boldsymbol{x} \rangle \boldsymbol{x} + \frac{1}{\sqrt{k}} \langle Z, Y \rangle \boldsymbol{x} \\
&= D_Y Z + \frac{1}{\sqrt{k}} \langle Z, Y \rangle \boldsymbol{x},
\end{aligned}$$

where we used the relation $D_Y \boldsymbol{x} = Y$. Using this relation, we can show that the curvature tensor $R$ satisfy

$$\begin{aligned}
&R(X, Y, Z, W) \\
&\qquad = \big\langle D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, W \big\rangle \\
&\qquad\qquad + k\big(g(X, W)g(Y, Z) - g(X, Z)g(Y, T)\big).
\end{aligned}$$

Hence by Proposition 5.14, we obtain that the sectional curvature of $S^n(k)$ is $k$.

**Example 5.17.** Let $k$ is a positive constant and

$$H^n(-k) := \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1}_1 \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -\frac{1}{k}, x^0 > 0 \right\},$$

where $\boldsymbol{x} = (x^0, \dots, x^n)$. Then $H^n(-k)$ is a space-like hypersurface in the Lorentz-Minkowski space $\mathbb{R}^{n+1}_1$, as seen in Example 3.5, called the *hyperbolic space*. Since $H^n(-k)$ is a space-like hypersurface, the induced metric gives a Riemannian metric, and then $H^n(-k)$ is a Riemannian manifold.

By the completely same method as in the previous example, one can show that $H^n(-k)$ has constant sectional curvature $-k$.



***Exercises***

**5-1** Prove Proposition 5.5.

**5-2** Show (2) in Proposition 5.7.

**5-3** Compute the sectional curvature of a Riemannian 2-manifold $(M, g)$ with

$$g_{11} = g_{22} = \frac{4}{(1 + k(u^2 + v^2))^2}, \qquad g_{12} = g_{21} = 0,$$

where $(u, v) = (u^1, u^2)$ is a local coordinate system.