1 Linear Ordinary Differential Equations

Preliminaries: Matrix Norms. Denote by $\mathrm{M}_n(\mathbb{R})$ the set of $n \times n$ matrix with real components, which can be identified the vector space $\mathbb{R}^{n^2}$. In particular, the Euclidean norm of $\mathbb{R}^{n^2}$ induces a norm

$$(1.1)\qquad |X|_{\mathrm{E}} = \sqrt{\operatorname{tr}({}^t XX)} = \sqrt{\sum_{i,j=1}^{n} x_{ij}^2}$$

on $\mathrm{M}_n(\mathbb{R})$. On the other hand, we let

$$(1.2)\qquad |X|_{\mathrm{M}} := \sup\left\{\frac{|X\boldsymbol{v}|}{|\boldsymbol{v}|}\,;\,\boldsymbol{v}\in\mathbb{R}^n\setminus\{\boldsymbol{0}\}\right\},$$

where $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^n$.

Lemma 1.1. (1) *The map $X \mapsto |X|_{\mathrm{M}}$ is a norm of $\mathrm{M}_n(\mathbb{R})$.*

(2) *For X, $Y \in \mathrm{M}_n(\mathbb{R})$, it holds that $|XY|_{\mathrm{M}} \leqq |X|_{\mathrm{M}}\,|Y|_{\mathrm{M}}$.*

(3) *Let $\lambda = \lambda(X)$ be the maximum eigenvalue of semi-positive definite symmetric matrix ${}^tXX$. Then $|X|_{\mathrm{M}} = \sqrt{\lambda}$ holds.*

(4) $(1/\sqrt{n})|X|_{\mathrm{E}} \leqq |X|_{\mathrm{M}} \leqq |X|_{\mathrm{E}}$.

(5) *The map $|\cdot|_{\mathrm{M}}\colon \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ is continuous with respect to the Euclidean norm.*

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Proof. Since $|X\boldsymbol{v}|/|\boldsymbol{v}|$ is invariant under scalar multiplications to $\boldsymbol{v}$, we have $|X|_{\mathrm{M}} = \sup\{|X\boldsymbol{v}|\,;\,\boldsymbol{v}\in S^{n-1}\}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Since $S^{n-1} \ni \boldsymbol{x} \mapsto |A\boldsymbol{x}| \in \mathbb{R}$ is a continuous function defined on a compact space, it takes the maximum. Thus, the right-hand side of (1.2) is well-defined. It is easy to verify that $|\cdot|_{\mathrm{M}}$ satisfies the axiom of the norm.

Since $A := {}^tXX$ is positive semi-definite, the eigenvalues λ_j $(j = 1, \dots, n)$ are non-negative real numbers. In particular, there exists an orthonormal basis $[\boldsymbol{a}_j]$ of $\mathbb{R}^n$ satisfying $A\boldsymbol{a}_j = \lambda_j\boldsymbol{a}_j$ $(j = 1, \dots, n)$. Let λ be the maximum eigenvalues of A, and write $\boldsymbol{v} = v_1\boldsymbol{a}_1 + \dots + v_n\boldsymbol{a}_n$. Then it holds that

$$\langle X\boldsymbol{v}, X\boldsymbol{v}\rangle = \lambda_1 v_1^2 + \dots + \lambda_n v_n^2 \leqq \lambda\,\langle \boldsymbol{v}, \boldsymbol{v}\rangle\,,$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product of $\mathbb{R}^n$. The equality of this inequality holds if and only if $\boldsymbol{v}$ is the λ-eigenvector, proving (3). Noticing the norm (1.1) is invariant under conjugations $X \mapsto {}^tPXP$ $(P \in \mathrm{O}(n))$, we obtain $|X|_{\mathrm{E}} = \sqrt{\lambda_1^2 + \dots + \lambda_n^2}$ by diagonalizing ${}^tXX$ by an orthogonal matrix P. Then we obtain (4). Hence two norms $|\cdot|_{\mathrm{E}}$ and $|\cdot|_{\mathrm{M}}$ induce the same topology as $\mathrm{M}_n(\mathbb{R})$. In particular, we have (5). □

Preliminaries: Matrix-valued Functions.

Lemma 1.2. *Let X and Y be C^∞-maps defined on a domain $U \subset \mathbb{R}^m$ into $\mathrm{M}_n(\mathbb{R})$. Then*

(1) $\dfrac{\partial}{\partial u_j}(XY) = \dfrac{\partial X}{\partial u_j}Y + X\dfrac{\partial Y}{\partial u_j}$,

(2) $\dfrac{\partial}{\partial u_j}\det X = \operatorname{tr}\left(\widetilde{X}\dfrac{\partial X}{\partial u_j}\right)$, *and*

(3) $\dfrac{\partial}{\partial u_j}X^{-1} = -X^{-1}\dfrac{\partial X}{\partial u_j}X^{-1}$,

where $\widetilde{X}$ is the cofactor matrix of X, and we assume in (3) *that X is a regular matrix.*

Proposition 1.3. *Assume two C^∞ matrix-valued functions $X(t)$ and $\Omega(t)$ satisfy*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0. \tag{1.3}$$

Then

$$\det X(t) = (\det X_0)\exp\int_{t_0}^{t} \operatorname{tr}\Omega(\tau)\,d\tau \tag{1.4}$$

holds. In particular, if $X_0 \in \mathrm{GL}(n,\mathbb{R})$,[1] *then $X(t) \in \mathrm{GL}(n,\mathbb{R})$ for all t.*

Proof. By (2) of Lemma 1.2, we have

$$\begin{aligned}\frac{d}{dt}\det X(t) &= \operatorname{tr}\left(\widetilde{X}(t)\frac{dX(t)}{dt}\right) = \operatorname{tr}\left(\widetilde{X}(t)X(t)\Omega(t)\right)\\ &= \operatorname{tr}\bigl(\det X(t)\Omega(t)\bigr) = \det X(t)\operatorname{tr}\Omega(t).\end{aligned}$$

[1] $\mathrm{GL}(n,\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\,\det A \neq 0\}$: the general linear group.

Here, we used the relation $\widetilde{X}X = X\widetilde{X} = (\det X)\,\mathrm{id}$[2]. Hence $\frac{d}{dt}\bigl(\rho(t)^{-1}\det X(t)\bigr) = 0$, where $\rho(t)$ is the right-hand side of (1.4). □

Proposition 1.4. *Assume $\Omega(t)$ in* (1.3) *is skew-symmetric for all t, that is, ${}^t\Omega + \Omega$ is identically O. If $X_0 \in \mathrm{O}(n)$ (resp. $X_0 \in \mathrm{SO}(n)$)*[3]*, $X(t) \in \mathrm{O}(n)$ (resp. $X(t) \in \mathrm{SO}(n)$) for all t.*

Proof. By (1) in Lemma 1.2,

$$\begin{aligned}\frac{d}{dt}(X\,{}^tX) &= \frac{dX}{dt}\,{}^tX + X\,{}^t\!\left(\frac{dX}{dt}\right)\\ &= X\Omega\,{}^tX + X\,{}^t\Omega\,{}^tX = X(\Omega + {}^t\Omega)\,{}^tX = O.\end{aligned}$$

Hence $X\,{}^tX$ is constant, that is, if $X_0 \in \mathrm{O}(n)$,

$$X(t)\,{}^tX(t) = X(t_0)\,{}^tX(t_0) = X_0\,{}^tX_0 = \mathrm{id}\,.$$

If $X_0 \in \mathrm{O}(n)$, this proves the first case of the proposition. Since $\det A = \pm 1$ when $A \in \mathrm{O}(n)$, the second case follows by continuity of $\det X(t)$. □

Preliminaries: Norms of Matrix-Valued functions. Let $I = [a,b]$ be a closed interval, and denote by $C^0(I,\mathrm{M}_n(\mathbb{R}))$ the set of continuous functions $X\colon I \to \mathrm{M}_n(\mathbb{R})$. For any fixed number k, we define

$$||X||_{I,k} := \sup\left\{e^{-kt}|X(t)|_{\mathrm{M}}\,;\, t \in I\right\} \tag{1.5}$$

[2] In this lecture, id denotes the identity matrix.

[3] $\mathrm{O}(n) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\,{}^tAA = A\,{}^tA = \mathrm{id}\}$: the orthogonal group; $\mathrm{SO}(n) = \{A \in \mathrm{O}(n)\,;\,\det A = 1\}$: the special orthogonal group.

for $X \in C^0(I, \mathrm{M}_n(\mathbb{R}))$. When $k = 0$, $||\cdot||_{I,0}$ is the *uniform norm* for continuous functions, which is complete. Similarly, one can prove the following in the same way:

Lemma 1.5. *The norm $||\cdot||_{I,k}$ on $C^0(I, \mathrm{M}_n(\mathbb{R}))$ is complete.*

Linear Ordinary Differential Equations. We prove the fundamental theorem for *linear* ordinary differential equations.

Proposition 1.6. *Let $\Omega(t)$ be a C^∞-function valued in $\mathrm{M}_n(\mathbb{R})$ defined on an interval I. Then for each $t_0 \in I$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0,\mathrm{id}}(t)$ such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = \mathrm{id}\,. \tag{1.6}$$

Proof. Uniqueness: Assume $X(t)$ and $Y(t)$ satisfy (1.6). Then

$$\begin{aligned} Y(t) - X(t) &= \int_{t_0}^{t} \bigl(Y'(\tau) - X'(\tau)\bigr)\,d\tau \\ &= \int_{t_0}^{t} \bigl(Y(\tau) - X(\tau)\bigr)\Omega(\tau)\,d\tau \qquad \left({}' = \frac{d}{dt}\right) \end{aligned}$$

holds. Hence for an arbitrary closed interval $J \subset I$,

$$\begin{aligned} |Y(t) - X(t)|_{\mathrm{M}} &\leqq \left| \int_{t_0}^{t} \left|\bigl(Y(\tau) - X(\tau)\bigr)\Omega(\tau)\right|_{\mathrm{M}} d\tau \right| \\ &\leqq \left| \int_{t_0}^{t} |Y(\tau) - X(\tau)|_{\mathrm{M}}\, |\Omega(\tau)|_{\mathrm{M}}\, d\tau \right| \end{aligned}$$

$$\begin{aligned} &= \left| \int_{t_0}^{t} e^{-k\tau}\, |Y(\tau) - X(\tau)|_{\mathrm{M}}\, e^{k\tau}\, |\Omega(\tau)|_{\mathrm{M}}\, d\tau \right| \\ &\leqq ||Y - X||_{J,k} \sup_J |\Omega|_{\mathrm{M}} \left| \int_{t_0}^{t} e^{k\tau}\, d\tau \right| \\ &= ||Y - X||_{J,k} \frac{\sup_J |\Omega|_{\mathrm{M}}}{|k|} e^{kt} \left| 1 - e^{-k(t-t_0)} \right| \\ &\leqq ||Y - X||_{J,k} \sup_J |\Omega|_{\mathrm{M}} \frac{e^{kt}}{|k|} \end{aligned}$$

holds for $t \in J$. Thus, for an appropriate choice of $k \in \mathbb{R}$, it holds that

$$||Y - X||_{J,k} \leqq \frac{1}{2} ||Y - X||_{J,k},$$

that is, $||Y - X||_{J,k} = 0$, proving $Y(t) = X(t)$ for $t \in J$. Since J is arbitrary, $Y = X$ holds on I.

Existence: Let $J := [t_0, a] \subset I$ be a closed interval, and define a sequence $\{X_j\}$ of matrix-valued functions defined on I satisfying $X_0(t) = \mathrm{id}$ and

$$X_{j+1}(t) = \mathrm{id} + \int_{t_0}^{t} X_j(\tau)\Omega(\tau)\,d\tau \quad (j = 0, 1, 2, \dots). \tag{1.7}$$

Let $k := 2\sup_J |\Omega|_{\mathrm{M}}$. Then

$$\begin{aligned} |X_{j+1}(t) - X_j(t)|_{\mathrm{M}} &\leqq \int_{t_0}^{t} |X_j(\tau) - X_{j-1}(\tau)|_{\mathrm{M}} |\Omega(\tau)|_{\mathrm{M}}\, d\tau \\ &\leqq \frac{e^{k(t-t_0)}}{|k|} \sup_J |\Omega|_{\mathrm{M}} ||X_j - X_{j-1}||_{J,k} \end{aligned}$$

for an appropriate choice of $k \in \mathbb{R}$, and hence $\|X_{j+1} - X_j\|_{J,k} \leqq \frac{1}{2}\|X_j - X_{j-1}\|_{J,k}$, that is, $\{X_j\}$ is a Cauchy sequence with respect to $\|\cdot\|_{J,k}$. Thus, by completeness (Lemma 1.5), it converges to some $X \in C^0(J, \mathrm{M}_n(\mathbb{R}))$. By (1.7), the limit X satisfies

$$X(t_0) = \mathrm{id}, \qquad X(t) = \mathrm{id} + \int_{t_0}^{t} X(\tau)\Omega(\tau)\,d\tau.$$

Applying the fundamental theorem of calculus, we can see that X satisfies $X'(t) = X(t)\Omega(t)$ ($' = d/dt$). Since J can be taken arbitrarily, existence of the solution on I is proven.

Finally, we shall prove that X is of class C^∞. Since $X'(t) = X(t)\Omega(t)$, the derivative X' of X is continuous. Hence X is of class C^1, and so is $X(t)\Omega(t)$. Thus we have that $X'(t)$ is of class C^1, and then X is of class C^2. Iterating this argument, we can prove that $X(t)$ is of class C^r for arbitrary r. □

Corollary 1.7. *Let $\Omega(t)$ be a matrix-valued C^∞-function defined on an interval I. Then for each $t_0 \in I$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X_{t_0,X_0}(t)$ defined on I such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \quad X(t_0) = X_0 \quad \bigl(X(t) := X_{t_0,X_0}(t)\bigr) \tag{1.8}$$

In particular, $X_{t_0,X_0}(t)$ is of class C^∞ in X_0 and t.

Proof. We rewrite $X(t)$ in Proposition 1.6 as $Y(t) = X_{t_0,\mathrm{id}}(t)$. Then the function

$$X(t) := X_0 Y(t) = X_0 X_{t_0,\mathrm{id}}(t), \tag{1.9}$$

is desired one. Conversely, assume $X(t)$ satisfies the conclusion. Noticing $Y(t)$ is a regular matrix for all t because of Proposition 1.3,

$$W(t) := X(t)Y(t)^{-1}$$

satisfies

$$\begin{aligned}\frac{dW}{dt} &= \frac{dX}{dt}Y^{-1} - XY^{-1}\frac{dY}{dt}Y^{-1}\\ &= X\Omega Y^{-1} - XY^{-1}Y\Omega Y^{-1} = O.\end{aligned}$$

Hence

$$W(t) = W(t_0) = X(t_0)Y(t_0)^{-1} = X_0.$$

Hence the uniqueness is obtained. The final part is obvious by the expression (1.9). □

Proposition 1.8. *Let $\Omega(t)$ and $B(t)$ be matrix-valued C^∞-functions defined on I. Then for each $t_0 \in I$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function defined on I satisfying*

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0. \tag{1.10}$$

Proof. Rewrite X in Proposition 1.6 as $Y := X_{t_0,\mathrm{id}}$. Then

$$X(t) = \left(X_0 + \int_{t_0}^{t} B(\tau)Y^{-1}(\tau)\,d\tau\right)Y(t) \tag{1.11}$$

satisfies (1.10). Conversely, if X satisfies (1.10), $W := XY^{-1}$ satisfies

$$X' = W'Y + WY' = W'Y + WY\Omega, \quad X\Omega + B = WY\Omega + B,$$

and then we have $W' = BY^{-1}$. Since $W(t_0) = X_0$,

$$W = X_0 + \int_{t_0}^{t} B(\tau)Y^{-1}(\tau)\,d\tau.$$

Thus we obtain (1.11). □

Theorem 1.9. *Let I and U be an interval and a domain in $\mathbb{R}^m$, respectively, and let $\Omega(t,\boldsymbol{\alpha})$ and $B(t,\boldsymbol{\alpha})$ be matrix-valued C^∞-functions defined on $I \times U$ ($\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_m)$). Then for each $t_0 \in I$, $\boldsymbol{\alpha} \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0,X_0,\boldsymbol{\alpha}}(t)$ defined on I such that*

$$\text{(1.12)} \qquad \frac{dX(t)}{dt} = X(t)\Omega(t,\boldsymbol{\alpha}) + B(t,\boldsymbol{\alpha}), \qquad X(t_0) = X_0.$$

Moreover,

$$I \times I \times \mathrm{M}_n(\mathbb{R}) \times U \ni (t, t_0, X_0, \boldsymbol{\alpha}) \mapsto X_{t_0,X_0,\boldsymbol{\alpha}}(t) \in \mathrm{M}_n(\mathbb{R})$$

is a C^∞-map.

Proof. Let $\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}}) := \Omega(t+t_0,\boldsymbol{\alpha})$ and $\widetilde{B}(t,\tilde{\boldsymbol{\alpha}}) = B(t+t_0,\boldsymbol{\alpha})$, and let $\widetilde{X}(t) := X(t+t_0)$. Then (1.12) is equivalent to

$$\text{(1.13)} \qquad \frac{d\widetilde{X}(t)}{dt} = \widetilde{X}(t)\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}}) + \widetilde{B}(t,\tilde{\boldsymbol{\alpha}}), \quad \widetilde{X}(0) = X_0,$$

where $\tilde{\boldsymbol{\alpha}} := (t_0, \alpha_1, \dots, \alpha_m)$. There exists the unique solution $\widetilde{X}(t) = \widetilde{X}_{\mathrm{id},X_0,\tilde{\boldsymbol{\alpha}}}(t)$ of (1.13) for each $\tilde{\boldsymbol{\alpha}}$ because of Proposition 1.8. So it is sufficient to show differentiability with respect to the parameter $\tilde{\boldsymbol{\alpha}}$. We set $Z = Z(t)$ the unique solution of

$$\text{(1.14)} \qquad \frac{dZ}{dt} = Z\widetilde{\Omega} + \widetilde{X}\frac{\partial\widetilde{\Omega}}{\partial\alpha_j} + \frac{\partial\widetilde{B}}{\partial\alpha_j}, \qquad Z(0) = O.$$

Then it holds that $Z = \partial\widetilde{X}/\partial\alpha_j$ (Problem 1-1). In particular, by the proof of Proposition 1.8, it holds that

$$Z = \frac{\partial\widetilde{X}}{\partial\alpha_j} = \left(\int_0^t \left(\widetilde{X}(\tau)\frac{\partial\widetilde{\Omega}(\tau,\tilde{\boldsymbol{\alpha}})}{\partial\alpha_j} + \frac{\partial\widetilde{B}(\tau,\tilde{\boldsymbol{\alpha}})}{\partial\alpha_j}\right) Y^{-1}(\tau)d\tau\right) Y(t).$$

Here, $Y(t)$ is the unique matrix-valued C^∞-function satisfying $Y'(t) = Y(t)\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}})$, and $Y(0) = \mathrm{id}$. Hence $\widetilde{X}$ is a C^∞-function in $(t,\tilde{\boldsymbol{\alpha}})$. □

Fundamental Theorem for Space Curves. As an application, we prove the fundamental theorem for space curves. A C^∞-map $\gamma\colon I \to \mathbb{R}^3$ defined on an interval $I \subset \mathbb{R}$ into $\mathbb{R}^3$ is said to be a *regular curve* if $\dot\gamma \neq \mathbf{0}$ holds on I. For a regular curve $\gamma(t)$, there exists a parameter change $t = t(s)$ such that $\tilde\gamma(s) := \gamma(t(s))$ satisfies $|\tilde\gamma'(s)| = 1$. Such a parameter s is called the *arc-length parameter.*

Let $\gamma(s)$ be a regular curve in $\mathbb{R}^3$ parametrized by the arc-length satisfying $\gamma''(s) \neq \mathbf{0}$ for all s. Then

$$\boldsymbol{e}(s) := \gamma'(s), \qquad \boldsymbol{n}(s) := \frac{\gamma''(s)}{|\gamma''(s)|}, \qquad \boldsymbol{b}(s) := \boldsymbol{e}(s) \times \boldsymbol{n}(s)$$

forms a positively oriented orthonormal basis $\{\boldsymbol{e},\boldsymbol{n},\boldsymbol{b}\}$ of $\mathbb{R}^3$ for each s. Regarding each vector as column vector, we have the matrix-valued function

$$\mathcal{F}(s) := (\boldsymbol{e}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)) \in \mathrm{SO}(3). \tag{1.15}$$

in s, which is called the *Frenet frame* associated to the curve γ. Under the situation above, we set

$$\kappa(s) := |\gamma''(s)| > 0, \qquad \tau(s) := -\left\langle \boldsymbol{b}'(s), \boldsymbol{n}(s) \right\rangle,$$

which are called the *curvature* and *torsion*, respectively, of γ. Using these quantities, the Frenet frame satisfies

$$\frac{d\mathcal{F}}{ds} = \mathcal{F}\Omega, \qquad \Omega = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}. \tag{1.16}$$

Proposition 1.10. *The curvature and the torsion are invariant under the transformation* $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ *of* $\mathbb{R}^3$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$). Conversely, two curves $\gamma_1(s)$, $\gamma_2(s)$ parametrized by arc-length parameter have common curvature and torsion, there exist $A \in \mathrm{SO}(3)$ and $\boldsymbol{b} \in \mathbb{R}^3$ such that $\gamma_2 = A\gamma_1 + \boldsymbol{b}$.

Proof. Let κ, τ and $\mathcal{F}_1$ be the curvature, torsion and the Frenet frame of γ_1, respectively. Then the Frenet frame of $\gamma_2 = A\gamma_1 + \boldsymbol{b}$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$) is $\mathcal{F}_2 = A\mathcal{F}_1$. Hence both $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy (1.16), and then γ_1 and γ_2 have common curvature and torsion.

Conversely, assume γ_1 and γ_2 have common curvature and torsion. Then the frenet frame $\mathcal{F}_1$, $\mathcal{F}_2$ both satisfy (1.16). Let $\mathcal{F}$ be the unique solution of (1.16) with $\mathcal{F}(t_0) = \mathrm{id}$. Then by the proof of Corollary 1.7, we have $\mathcal{F}_j(t) = \mathcal{F}_j(t_0)\mathcal{F}(t)$ ($j = 1, 2$). In particular, since $\mathcal{F}_j \in \mathrm{SO}(3)$, $\mathcal{F}_2(t) = A\mathcal{F}_1(t)$ ($A := \mathcal{F}_2(t_0)\mathcal{F}_1(t_0)^{-1} \in \mathrm{SO}(3)$). Comparing the first column of these, $\gamma_2'(s) = A\gamma_1'(t)$ holds. Integrating this, the conclusion follows. □

Theorem 1.11 (The fundamental theorem for space curves)**.** *For given* C^∞*-functions* $\kappa(s)$ *and* $\tau(s)$ *defined on* I *such that* $\kappa(s) > 0$ *on* I*. Then there exists a space curve* $\gamma(s)$ *parametrized by arc-length whose curvature and torsion are* κ *and* τ*, respectively. Moreover, such a curve is unique up to transformation* $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$) *of* $\mathbb{R}^3$.

Proof. We have already shown the uniqueness in Proposition 1.10. We shall prove the existence: Let $\Omega(s)$ be as in (1.16), and $\mathcal{F}(s)$ the solution of (1.16) with $\mathcal{F}(s_0) = \mathrm{id}$. Since Ω is skew-symmetric, $\mathcal{F}(s) \in \mathrm{SO}(3)$ by Proposition 1.4. Denoting the column vectors of $\mathcal{F}$ by $\boldsymbol{e}$, $\boldsymbol{n}$, $\boldsymbol{b}$, and let

$$\gamma(s) := \int_{s_0}^{s} \boldsymbol{e}(\sigma)\, d\sigma.$$

Then $\mathcal{F}$ is the Frenet frame of γ, and κ, and τ are the curvature and torsion of γ, respectively (Problem 1-2). □

Exercises

1-1 Verify that Z in (1.14) coincides with $\partial\widetilde{X}/\partial\alpha_j$.

1-2 Complete the proof of Theorem 1.11.

1-3 Find an explicit expression of a space curve $\gamma(s)$ parametrized the arc-length s, whose curvature and torsion are $a/(1+s^2)$ and $b/(1+s^2)$, respectively, where a and b are constants.

2 Integrability Conditions

Let $U \subset \mathbb{R}^m$ be a domain of $(\mathbb{R}^m; u^1, \dots, u^m)$ and consider m-tuple of $n \times n$-matrix valued C^∞-maps

$$(2.1) \qquad \Omega_j \colon \mathbb{R}^m \supset U \longrightarrow \mathrm{M}_n(\mathbb{R}) \qquad (j = 1, \dots, m).$$

In this section, we consider an initial value problem of a system of linear partial differential equations

$$(2.2) \qquad \frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0,$$

where $\mathrm{P}_0 = (u_0^1, \dots, u_0^m) \in U$ is a fixed point, X is an $n \times n$-matrix valued unknown, and $X_0 \in \mathrm{M}_n(\mathbb{R})$. The chain rule yields the following:

Lemma 2.1. *Let $X \colon U \to \mathrm{M}_n(\mathbb{R})$ be a C^∞-map satisfying* (2.2)*. Then for each smooth path $\gamma\colon I \to U$ defined on an interval $I \subset \mathbb{R}$, $\hat{X} := X \circ \gamma : I \to \mathrm{M}_n(\mathbb{R})$ satisfies the ordinary differential equation*

$$(2.3) \quad \frac{d\hat{X}}{dt}(t) = \hat{X}(t)\Omega_\gamma(t) \qquad \left(\Omega_\gamma(t) := \sum_{j=1}^{n} \Omega_j \circ \gamma(t) \frac{du^j}{dt}(t)\right)$$

on I, where $\gamma(t) = \bigl(u^1(t), \dots, u^m(t)\bigr)$.

Proposition 2.2. *If a C^∞-map $X \colon U \to \mathrm{M}_n(\mathbb{R})$ defined on a domain $U \subset \mathbb{R}^m$ satisfies* (2.2) *with $X_0 \in \mathrm{GL}(n, \mathbb{R})$, then*

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$X(\mathrm{P}) \in \mathrm{GL}(n, \mathbb{R})$ for all $\mathrm{P} \in U$. In addition, if Ω_j $(j = 1, \dots, m)$ are skew-symmetric and $X_0 \in \mathrm{SO}(n)$, then $X(\mathrm{P}) \in \mathrm{SO}(n)$ holds for all $\mathrm{P} \in U$.

Proof. Since U is connected, there exists a continuous path $\gamma_0\colon [0,1] \to U$ such that $\gamma_0(0) = \mathrm{P}_0$ and $\gamma_0(1) = \mathrm{P}$. By Whitney's approximation theorem (cf. Theorem 10.16 in [2-3]), there exists a smooth path $\gamma\colon [0,1] \to U$ joining P_0 and P approximating γ_0. Since $\hat{X} := X \circ \gamma$ satisfies (2.3) with $\hat{X}(0) = X_0$, Proposition 1.3 yields that $\det \hat{X}(1) \neq 0$ whenever $\det X_0 \neq 0$. The latter half follows from Proposition 1.4. □

Proposition 2.3. *If a matrix-valued C^∞ function $X \colon U \to \mathrm{GL}(n, \mathbb{R})$ satisfies* (2.2)*, it holds that*

$$(2.4) \qquad \frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j \Omega_k - \Omega_k \Omega_j$$

for each (j, k) with $1 \leqq j < k \leqq m$.

Proof. Differentiating (2.2) by u^k, we have

$$\frac{\partial^2 X}{\partial u^k \partial u^j} = \frac{\partial X}{\partial u^k}\Omega_j + X\frac{\partial \Omega_j}{\partial u^k} = X\left(\frac{\partial \Omega_j}{\partial u^k} + \Omega_k \Omega_j\right).$$

On the other hand, switching the roles of j and k, we get

$$\frac{\partial^2 X}{\partial u^j \partial u^k} = X\left(\frac{\partial \Omega_k}{\partial u^j} + \Omega_j \Omega_k\right).$$

Since X is of class C^∞, the left-hand sides of these equalities coincide, and so are the right-hand sides. Since $X \in \mathrm{GL}(n, \mathbb{R})$, the conclusion follows. □

The equality (2.4) is called the *integrability condition* or *compatibility condition* of (2.2).

Lemma 2.4. *Let $\Omega_j\colon U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ be C^∞-maps defined on a domain $U \subset \mathbb{R}^m$ which satisfy (2.4). Then for each smooth map*

$$\sigma\colon D \ni (t, w) \longmapsto \sigma(t, w) = (u^1(t, w), \dots, u^m(t, w)) \in U$$

defined on a domain $D \subset \mathbb{R}^2$, it holds that

$$\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} - TW + WT = 0, \tag{2.5}$$

where

$$T := \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial t}, \quad W := \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial w} \quad (\widetilde{\Omega}_j := \Omega_j \circ \sigma). \tag{2.6}$$

Proof. By the chain rule, we have

$$\begin{aligned}
\frac{\partial T}{\partial w} &= \sum_{j,k=1}^m \frac{\partial \Omega_j}{\partial u^k} \frac{\partial u^k}{\partial w} \frac{\partial u^j}{\partial t} + \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial^2 u^j}{\partial w \partial t}, \\
\frac{\partial W}{\partial t} &= \sum_{j,k=1}^m \frac{\partial \Omega_j}{\partial u^k} \frac{\partial u^k}{\partial t} \frac{\partial u^j}{\partial w} + \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial^2 u^j}{\partial t \partial w} \\
&= \sum_{j,k=1}^m \frac{\partial \Omega_k}{\partial u^j} \frac{\partial u^j}{\partial t} \frac{\partial u^k}{\partial w} + \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial^2 u^j}{\partial t \partial w}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} &= \sum_{j,k=1}^m \left(\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} \right) \frac{\partial u^k}{\partial w} \frac{\partial u^j}{\partial t} \\
&= \sum_{j,k=1}^m \left(\widetilde{\Omega}_j \widetilde{\Omega}_k - \widetilde{\Omega}_k \widetilde{\Omega}_j \right) \frac{\partial u^k}{\partial w} \frac{\partial u^j}{\partial t} \\
&= \left(\sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial t} \right) \left(\sum_{k=1}^m \widetilde{\Omega}_k \frac{\partial u^k}{\partial w} \right) - \left(\sum_{k=1}^m \widetilde{\Omega}_k \frac{\partial u^k}{\partial w} \right) \left(\sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial t} \right) \\
&= TW - WT. \qquad \square
\end{aligned}$$

Integrability of linear systems. In this section, we shall prove the following

Theorem 2.5. *Let $\Omega_j\colon U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ be C^∞-functions defined on a simply connected domain $U \subset \mathbb{R}^m$ satisfying (2.4). Then for each $\mathrm{P}_0 \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique $n \times n$-matrix valued function $X\colon U \to \mathrm{M}_n(\mathbb{R})$ satisfying (2.2). Moreover,*

- *if $X_0 \in \mathrm{GL}(n, \mathbb{R})$, $X(\mathrm{P}) \in \mathrm{GL}(n, \mathbb{R})$ holds on U,*
- *if $X_0 \in \mathrm{SO}(n)$ and Ω_j $(j = 1, \dots, m)$ are skew-symmetric matrices, $X \in \mathrm{SO}(n)$ holds on U.*

Proof. The latter half is a direct conclusion of Proposition 2.2. We show the existence of X: Take a smooth path $\gamma\colon [0, 1] \to U$ joining P_0 and P. Then by Theorem 1.9, there exists a unique C^∞-map $\hat{X}\colon [0, 1] \to \mathrm{M}_n(\mathbb{R})$ satisfying (2.3) with initial condition $\hat{X}(0) = X_0$.

We shall show that the value $\hat{X}(1)$ does not depend on choice of paths joining P_0 and P. To show this, choose another smooth path $\tilde{\gamma}$ joining P_0 and P. Since U is simply connected, there exists a homotopy between γ and $\tilde{\gamma}$, that is, there exists a continuous map $\sigma_0\colon [0,1]\times[0,1] \ni (t,w) \mapsto \sigma(t,w) \in U$ satisfying

$$\begin{aligned} \sigma_0(t,0) &= \gamma(t), & \sigma_0(t,1) &= \tilde{\gamma}(t), \\ \sigma_0(0,w) &= \mathrm{P}_0, & \sigma_0(1,w) &= \mathrm{P}. \end{aligned} \tag{2.7}$$

Then, by Whitney's approximation theorem (Theorem 10.16 in [2-3]) again, there exists a smooth map $\sigma\colon [0,1]\times[0,1] \to U$ satisfying the same boundary conditions as (2.7). We set T and W as in (2.6). For each fixed $w \in [0,1]$, there exists $X_w\colon [0,1] \to \mathrm{M}_n(\mathbb{R})$ such that

$$\frac{dX_w}{dt}(t) = X_w(t)T(t,w), \qquad X_w(0) = X_0.$$

Since $T(t,w)$ is smooth in t and w, the map

$$\check{X}\colon [0,1]\times[0,1] \ni (t,w) \mapsto X_w(t) \in \mathrm{M}_n(\mathbb{R})$$

is a smooth map. To show that $\hat{X}(1) = \check{X}(1,0)$ does not depend on choice of paths, it is sufficient to show that

$$\frac{\partial \check{X}}{\partial w} = \check{X}W \tag{2.8}$$

holds on $[0,1]\times[0,1]$. In fact, by (2.7), $W(1,w) = 0$ for all $w \in [0,1]$, and then (2.8) implies that $\check{X}(1,w)$ is constant.

We prove (2.8): By definition, it holds that

$$\frac{\partial \check{X}}{\partial t} = \check{X}T, \qquad \check{X}(0,w) = X_0 \tag{2.9}$$

for each $w \in [0,1]$. Hence by (2.5),

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial \check{X}}{\partial w} &= \frac{\partial^2 \check{X}}{\partial t \partial w} = \frac{\partial^2 \check{X}}{\partial w \partial t} = \frac{\partial}{\partial w}\check{X}T \\ &= \frac{\partial \check{X}}{\partial w}T + \check{X}\frac{\partial T}{\partial w} = \frac{\partial \check{X}}{\partial w}T + \check{X}\left(\frac{\partial W}{\partial t} + TW - WT\right) \\ &= \frac{\partial \check{X}}{\partial w}T + \check{X}\frac{\partial W}{\partial t} + \frac{\partial \check{X}}{\partial t}W - \check{X}WT \\ &= \frac{\partial}{\partial t}(\check{X}W) + \left(\frac{\partial \check{X}}{\partial w} - \check{X}W\right)T. \end{aligned}$$

So, the function $Y_w(t) := \partial\check{X}/\partial w - \check{X}W$ satisfies the ordinary differential equation

$$\frac{dY_w}{dt}(t) = Y_w(t)T(t,w), \quad Y_w(0) = O$$

holds for each $w \in [0,1]$. Thus, by the uniqueness of the solution, $Y_w(t) = O$ holds on $[0,1]\times[0,1]$. Hence we have (2.8).

Thus, $\hat{X}(1)$ depends only the end point P of the path. Hence we can set $X(\mathrm{P}) := \hat{X}(1)$ for each $\mathrm{P} \in U$, and obtain a map $X\colon U \to \mathrm{M}_m(\mathbb{R})$. Finally we show that X is the desired solution. The initial condition $X(\mathrm{P}_0) = X_0$ is obviously satisfied. On the other hand, if we set

$$Z(\delta) := X(u^1,\dots,u^j+\delta,\dots,u^m) - X(u^1,\dots,u^m),$$

$Z(\delta)$ satisfies the equation (2.3) for the path $\gamma(\delta) := (u^1, \dots, u^j + \delta, \dots, u^m)$ with $Z(0) = X(\mathrm{P})$. Since $\Omega_\gamma = \Omega_j$,

$$\frac{\partial X}{\partial u^j} = \frac{dZ}{d\delta} = Z\Omega_j = X\Omega_j,$$

which completes the proof. □

Application: Poincaré's lemma.

Theorem 2.6 (Poincaré's lemma)**.** *If a differential* 1*-form*

$$\omega = \sum_{j=1}^{m} \alpha_j(u^1, \dots, u^m)\, du^j$$

defined on a simply connected domain $U \subset \mathbb{R}^m$ *is closed, that is,* $d\omega = 0$ *holds, then there exists a* C^∞*-function* f *on* U *such that* $df = \omega$*. Such a function* f *is unique up to additive constants.*

Proof. The assumption is equivalent to

$$\frac{\partial \alpha_j}{\partial u_i} - \frac{\partial \alpha_i}{\partial u_j} = 0 \qquad (1 \leqq i < j \leqq m). \tag{2.10}$$

Consider a system of linear partial differential equations with unknown ξ, a 1×1-matrix valued function (i.e. a real-valued function), as

$$\frac{\partial \xi}{\partial u^j} = \xi \alpha_j \quad (j = 1, \dots, m), \qquad \xi(u_0^1, \dots, u_0^m) = 1. \tag{2.11}$$

Then it satisfies (2.4) because of (2.10). Hence by Theorem 2.5, there exists a smooth function $\xi(u^1, \dots, u^m)$ satisfying (2.11).

In particular, Proposition 1.3 yields $\xi = \det \xi$ never vanishes. Here, $\xi(u_0^1, \dots, u_0^m) = 1 > 0$ means that $\xi > 0$ holds on U. Letting $f := \log \xi$, we have the function f satisfying $df = \omega$.

Next, we show the uniqueness: if two functions f and g satisfy $df = dg = \omega$, it holds that $d(f - g) = 0$. Hence by connectivity of U, $f - g$ must be constant. □

Application: Conjugation of Harmonic functions. In this paragraph, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. It is well-known that a function

$$f\colon U \ni u + \mathrm{i}\, v \longmapsto \xi(u, v) + \mathrm{i}\, \eta(u, v) \in \mathbb{C} \qquad (\mathrm{i} = \sqrt{-1}) \tag{2.12}$$

defined on a domain $U \subset \mathbb{C}$ is *holomorphic* if and only if it satisfies the following relation, called the *Cauchy-Riemann equations*:

$$\frac{\partial \xi}{\partial u} = \frac{\partial \eta}{\partial v}, \qquad \frac{\partial \xi}{\partial v} = -\frac{\partial \eta}{\partial u}. \tag{2.13}$$

Definition 2.7. A function $f\colon U \to \mathbb{R}$ defined on a domain $U \subset \mathbb{R}^2$ is said to be *harmonic* if it satisfies

$$\Delta f = f_{uu} + f_{vv} = 0.$$

The operator Δ is called the *Laplacian.*

Proposition 2.8. *If function* f *in* (2.12) *is holomorphic,* $\xi(u, v)$ *and* $\eta(u, v)$ *are harmonic functions.*

Proof. By (2.13), we have

$$\xi_{uu} = (\xi_u)_u = (\eta_v)_u = \eta_{vu} = \eta_{uv} = (\eta_u)_v = (-\xi_v)_v = -\xi_{vv}.$$

Hence $\Delta\xi = 0$. Similarly,

$$\eta_{uu} = (-\xi_v)_u = -\xi_{vu} = -\xi_{uv} = -(\xi_u)_v = -(\eta_v)_v = -\eta_{vv}.$$

Thus $\Delta\eta = 0$. □

Theorem 2.9. *Let $U \subset \mathbb{C} = \mathbb{R}^2$ be a simply connected domain and $\xi(u,v)$ a C^∞-function harmonic on U*[4]*. Then there exists a C^∞ harmonic function η on U such that $\xi(u,v) + \mathrm{i}\,\eta(u,v)$ is holomorphic on U.*

Proof. Let $\alpha := -\xi_v\,du + \xi_u\,dv$. Then by the assumption,

$$d\alpha = (\xi_{vv} + \xi_{uu})\,du \wedge dv = 0$$

holds, that is, α is a closed 1-form. Hence by simple connectivity of U and the Poincaré's lemma (Theorem 2.6), there exists a function η such that $d\eta = \eta_u\,du + \eta_v\,dv = \alpha$. Such a function η satisfies (2.13) for given ξ. Hence $\xi + \mathrm{i}\,\eta$ is holomorphic in $u + \mathrm{i}\,v$. □

Definition 2.10. The harmonic function η in Theorem 2.9 is called the *conjugate* harmonic function of ξ.

The fundamental theorem for Surfaces. Let $p\colon U \to \mathbb{R}^3$ be a parametrization of a *regular surface* defined on a domain $U \subset \mathbb{R}^2$. That is, $p = p(u,v)$ is a C^∞-map such that p_u and p_v are linearly independent at each point on U. Then $\nu := (p_u \times p_v)/|p_u \times p_v|$ is the *unit normal vector field* to the surface.

[4]The theorem holds under the assumption of C^2-differentiablity.

The matrix-valued function $\mathcal{F} := (p_u, p_v, \nu)\colon U \to \mathrm{M}_3(\mathbb{R})$ is called the *Gauss frame* of p. We set

$$(2.14)\qquad \begin{aligned} ds^2 &:= E\,du^2 + 2F\,du\,dv + G\,dv^2,\\ I\!I &:= L\,du^2 + 2M\,du\,dv + N\,dv^2, \end{aligned}$$

where

$$\begin{aligned} &E = p_u \cdot p_u && F = p_u \cdot p_v && G = p_v \cdot p_v\\ &L = p_{uu} \cdot \nu && M = p_{uv} \cdot \nu && N = p_{vv} \cdot \nu. \end{aligned}$$

We call ds^2 (resp. $I\!I$) the *first* (resp. *second*) *fundamental form.* Note that linear independence of p_u and p_v implies

$$(2.15)\qquad E > 0, \qquad G > 0 \qquad \text{and} \qquad EG - F^2 > 0.$$

Set

$$(2.16)\qquad \begin{aligned} &\Gamma^1_{11} := \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, && \Gamma^2_{11} := \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)},\\ &\Gamma^1_{12} = \Gamma^1_{21} := \frac{GE_v - FG_u}{2(EG - F^2)}, && \Gamma^2_{12} = \Gamma^2_{21} := \frac{EG_u - FE_v}{2(EG - F^2)},\\ &\Gamma^1_{22} := \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}, && \Gamma^2_{22} := \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}. \end{aligned}$$

and

$$(2.17)\qquad A = \begin{pmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{pmatrix} := \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

The functions Γ^k_{ij} and the matrix A are called the *Christoffel symbols* and the *Weingarten matrix*. We state the following the *fundamental theorem for surfaces*, and give a proof (for a special case) in the following section.

Theorem 2.11 (The Fundamental Theorem for Surfaces)**.** *Let $p\colon U \ni (u,v) \mapsto p(u,v) \in \mathbb{R}^3$ be a parametrization of a regular surface defined on a domain $U \subset \mathbb{R}^2$. Then the Gauss frame $\mathcal{F} := \{p_u, p_v, \nu\}$ satisfies the equations*

$$\frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\varOmega, \qquad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\varLambda, \tag{2.18}$$

$$\varOmega := \begin{pmatrix} \varGamma_{11}^1 & \varGamma_{12}^1 & -A_1^1 \\ \varGamma_{11}^2 & \varGamma_{12}^2 & -A_1^2 \\ L & M & 0 \end{pmatrix}, \quad \varLambda := \begin{pmatrix} \varGamma_{21}^1 & \varGamma_{22}^1 & -A_2^1 \\ \varGamma_{21}^2 & \varGamma_{22}^2 & -A_2^2 \\ M & N & 0 \end{pmatrix},$$

where $\varGamma_{jk}^i$ $(i,j,k=1,2)$, A_l^k $(k,l=1,2)$ and L, M, N are the Christoffel symbols, the entries of the Weingarten matrix and the entries of the second fundamental form, respectively.

Theorem 2.12. *Let $U \subset \mathbb{R}^2$ be a simply connected domain, E, F, G, L, M, N C^∞-functions satisfying (2.15), and $\varGamma_{ij}^k$, A_i^j the functions defined by (2.16) and (2.17), respectively. If $\varOmega$ and $\varLambda$ satisfies*

$$\varOmega_v - \varLambda_u = \varOmega\varLambda - \varLambda\varOmega, \tag{2.19}$$

there exists a parameterization $p\colon U \to \mathbb{R}^3$ of regular surface whose fundamental forms are given by (2.14). Moreover, such a surface is unique up to orientation preserving isometries of $\mathbb{R}^3$.

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Exercises

2-1 Let $\xi(u,v) = \log\sqrt{u^2+v^2}$ be a function defined on $U = \mathbb{R}^2 \setminus \{(0,0)\}$

(1) Show that ξ is harmonic on U.

(2) Find the conjugate harmonic function η of ξ on

$$V = \mathbb{R}^2 \setminus \{(u,0) \mid u \leqq 0\} \subset U.$$

(3) Show that there exists no conjugate harmonic function of ξ defined on U.

2-2 Let $\theta = \theta(u,v)$ be a smooth function on a domain $U \subset \mathbb{R}^2$ such that $0 < \theta < \pi$, and

$$ds^2 := du^2 + 2\cos\theta\,du\,dv + dv^2, \quad I\!I := 2\sin\theta\,du\,dv.$$

Show that the condition (2.19) is equivalent to

$$\theta_{uv} = \sin\theta.$$

3 Flatness.

Riemannian manifolds. A *Riemannian manifold* (resp. *pseudo Riemannian manifold*) is a pair (M, g) of a C^∞-manifold M and a *Riemannian metric* (resp. *pseudo Riemannian metric*) on M, that is, g is a collection $\{g_{\mathrm{P}}\,;\,\mathrm{P}\in M\}$ of positive definite (resp. non-degenerate) inner products on $T_{\mathrm{P}}M$ such that, for each pair of C^∞-vector fields (X, Y) on M, the map

$$M \ni \mathrm{P} \longmapsto g_{\mathrm{P}}(X_{\mathrm{P}}, Y_{\mathrm{P}}) \in \mathbb{R}$$

is a C^∞-function, where X_{P} and Y_{P} are values of X and Y at P, respectively.

Example 3.1 ((Pseudo) Euclidean spaces.)**.** Let $\mathbb{R}^n_s$ be a pseudo Euclidean vector space with inner product $\langle\ ,\ \rangle$ of signature $(n-s, s)$. Identifying the tangent space $T_{\mathrm{P}}\mathbb{R}^n_s$ of $\mathbb{R}^n_s$ itself by translations, $\langle\ ,\ \rangle$ gives a pseudo Riemannian metric of the manifold $\mathbb{R}^n_s$. Such a pseudo Riemannian manifold is called the *pseudo Euclidean space* of signature $(n-s, s)$. In particular, $\mathbb{R}^n := \mathbb{R}^n_0$, which is a Riemannian manifold, is called the Euclidean space.

Example 3.2. Let M be a submanifold of the Euclidean space $\mathbb{R}^n$, that is, M is a subset of $\mathbb{R}^n$ and has a structure of C^∞-manifold such that the inclusion map $\iota\colon M \to \mathbb{R}^n$ is an immersion. Then $T_{\mathrm{P}}M$ is considered as a linear subspace of $\mathbb{R}^n (= T_{\mathrm{P}}\mathbb{R}^n)$, and then the restriction of the inner product $\langle\ ,\ \rangle$ of $\mathbb{R}^n$ gives a Riemannian metric on M. Such a Riemannian metric is called the *induced metric*.

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Example 3.3 (Spheres)**.** For positive real number k, a subset $S^n(k) := \{\boldsymbol{x} \in \mathbb{R}^{n+1}\,;\, \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 1/k\}$ is an n-dimensional C^∞-submanifold (hypersurface)[5] of the Euclidean space $\mathbb{R}^{n+1}$, called the n-dimensional *sphere* of curvature[6] k.

Example 3.4. A linear subspace L of $\mathbb{R}^n_s$ is said to be *non-degenerate* if the restriction of the inner product $\langle\ ,\ \rangle$ of $\mathbb{R}^n_s$ to L is non-degenerate. A submanifold $M \subset \mathbb{R}^n_s$ is said to be *non-degenerate* if $T_{\mathrm{P}}M$ is a non-degenerate subspace of $\mathbb{R}^n_s$ for each $\mathrm{P} \in M$. In this case, the restriction of $\langle\ ,\ \rangle$ on $T_{\mathrm{P}}M$ is a (non-degenerate) inner product of $T_{\mathrm{P}}M$.

Example 3.5 (Hyperbolic spaces.)**.** For positive real number k, a subset

$$H^n(-k) := \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1\,;\, \langle \boldsymbol{x}, \boldsymbol{x}\rangle = -1/k, x^0 > 0\}$$

is a connected C^∞-hypersurface of the Lorentz-Minkowski space $\mathbb{R}^{n+1}_1$, where $\langle\ ,\ \rangle$ is the inner product of signature $(-, +, \dots, +)$, and $\boldsymbol{x} = {}^t(x^0, x^1, \dots, x^n)$. The tangent space $T_{\boldsymbol{x}}H^n(-k) = \{\boldsymbol{v} \in \mathbb{R}^{n+1}_1\,;\, \langle \boldsymbol{v}, \boldsymbol{x}\rangle = 0\} = \boldsymbol{x}^\perp$ is non-degenerate subspace in $\mathbb{R}^{n+1}_1$ and the restriction of $\langle\ ,\ \rangle$ to $T_{\boldsymbol{x}}H^n(-k)$ is positive definite. Thus, we obtain a Riemannian manifold $H^n(-k)$, which is called the *hyperbolic space* of curvature $-k$.

Geodesics. Let $M \subset \mathbb{R}^{n+1}_s$ be a non-degenerate submanifold of dimension m. By non-degeneracy, the orthogonal decompo-

[5] An n-dimensional submanifold of $(n+1)$-dimensional manifold (i.e., a submanifold of codimension one) is called a *hypersurface*.

[6] The word *curvature* is undefined at the moment.

sition

$$\mathbb{R}^{n+1}_s = T_{\mathrm{P}}\mathbb{R}^{n+1}_s = T_{\mathrm{P}}M \oplus N_{\mathrm{P}}, \big(N_{\mathrm{P}} := (T_{\mathrm{P}}M)^{\perp}\big) \tag{3.1}$$

holds for each $\mathrm{P} \in M$. Take a curve γ on M, that is, γ is a C^∞-map

$$\gamma\colon J \ni t \longmapsto \gamma(t) \in M \subset \mathbb{R}^{n+1}_s$$

where $J \subset \mathbb{R}$ is an interval. From now on, by a word *smooth*, we mean "of class C^∞".

Definition 3.6. Let $\gamma\colon J \to M \subset \mathbb{R}^{n+1}_s$ be a smooth curve on M. A *smooth vector field on M along γ* is a map

$$X\colon J \ni t \longmapsto X(t) \in T_{\gamma(t)}M \subset \mathbb{R}^{n+1}_s$$

which is of class C^∞ as a map from J to $\mathbb{R}^{n+1}_s$.

Example 3.7. Let $\gamma\colon J \to M \subset \mathbb{R}^{n+1}_s$ be a smooth curve. Then

$$\dot\gamma\colon J \ni t \longmapsto \dot\gamma(t) = \frac{d\gamma}{dt}(t) \in T_{\gamma(t)}M$$

is a smooth vector field along γ, called the *velocity* vector field of the curve γ.

Definition 3.8. Let X be a smooth vector field along a smooth curve γ on M. Then the vector field

$$\frac{\nabla}{dt}X(t) := \nabla_{\dot\gamma(t)}X(t) := \Big[\dot X(t)\Big]^{\mathrm{T}} \in T_{\gamma(t)}M$$

of M along γ is called the *covariant derivative* of X along γ, where $[*]^{\mathrm{T}}$ denotes the tangential component as in (3.1).

Definition 3.9. The covariant derivative

$$\frac{\nabla}{dt}\dot\gamma(t) = \nabla_{\dot\gamma(t)}\dot\gamma(t) := [\ddot\gamma(t)]^{\mathrm{T}} \in T_{\gamma(t)}M \tag{3.2}$$

of $\dot\gamma$ along γ is called the *acceleration* of the curve γ.

Definition 3.10. A curve γ on a non-degenerate submanifold $M \subset \mathbb{R}^{n+1}_s$ is called a *geodesic* if $\nabla_{\dot\gamma}\dot\gamma$ vanishes identically.

Local and intrinsic expressions. Let $M \subset \mathbb{R}^{n+1}_s$ be a non-degenerate submanifold and take a local coordinate neighborhood $(U; u^1, \dots, u^m)$ of M, where $m = \dim M$. Then the inclusion map $\iota\colon M \to \mathbb{R}^{n+1}_s$ induces an immersion

$$f\colon U \ni (u^1, \dots, u^m) \longmapsto f(u^1, \dots, u^m) \in M \subset \mathbb{R}^{n+1}_s, \tag{3.3}$$

here we identify the coordinate neighborhood $U \subset M$ with a region of $\mathbb{R}^m$. We call such an f a (local) *parametrization* of M. Under this parametrization, the canonical basis $\{(\partial/\partial u^j)_{\mathrm{P}}\}$ of $T_{\mathrm{P}}M$ (in the abstract way) is identified with

$$\left\{\frac{\partial f}{\partial u^1}(\mathrm{P}), \dots, \frac{\partial f}{\partial u^m}(\mathrm{P})\right\} \subset T_{\mathrm{P}}M \subset \mathbb{R}^{n+1}_s.$$

We set, for $i, j = 1, \dots, m$,

$$g_{ij} := g\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = \left\langle \frac{\partial f}{\partial u^i}, \frac{\partial f}{\partial u^j}\right\rangle (= g_{ji}) \tag{3.4}$$

which is a component of the induced metric $g := \langle\ ,\ \rangle|_{T_{\mathrm{P}}M}$ with respect to the canonical basis $\{\partial/\partial u^j\}$. Since the induced metric is non-degenerate, the $m \times m$-matrix (g_{ij}) is a regular matrix at each point $\mathrm{P} \in M$. In particular, when the induced metric is positive definite, (g_{ij}) is positive definite. We denote by (g^{ij}) the inverse matrix of (g_{ij}):

$$\sum_{k=1}^{m} g_{ik} g^{kj} = \delta_i^j = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{3.5}$$

Then, as we have seen in Section 5 of "Advanced Topics in Geometry A1, 2019" (the previous quarter), we have

Lemma 3.11. *Let γ is a curve in $U \subset M$ and express*

$$\gamma(t) = f\bigl(u^1(t), \dots, u^m(t)\bigr),$$

where $f\colon U \to M$ is a local parametrization of M as in (3.3)*.*

$$\dot\gamma = \sum_{j=1}^{m} \frac{du^j}{dt}\frac{\partial f}{\partial u^j} \tag{3.6}$$

$$\nabla_{\dot\gamma}\dot\gamma = \sum_{j=1}^{m}\left(\frac{d^2u^j}{dt^2} + \sum_{k,l=1}^{m} \Gamma^j_{kl}\frac{du^k}{dt}\frac{du^l}{dt}\right)\frac{\partial f}{\partial u^j} \tag{3.7}$$

hold, where

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{m} g^{kl}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{lj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right). \tag{3.8}$$

The functions Γ^k_{ij} of (3.8) are called the *Christoffel symbols* with respect to the local coordinate system $(u^1, \dots, u^m)$.

Christoffel symbols. By definition (3.8), the Christoffel symbols Γ^k_{ij} are functions defined on the coordinate neighborhood U which are determined only by the coefficients (g_{ij}) of the (pseudo) Riemannian metric. That is, the definition of Γ^k_{ij} does not require the knowledge of γ.

Proposition 3.12. *Let $M \subset \mathbb{R}^{n+1}_s$ be a non-degenerate submanifold with induced metric $\langle\ ,\ \rangle$, and take a local coordinate system $(U; u^1, \dots, u^m)$ of M. We write parametrization of M with respect to (u^j) as* (3.3)*. Then the Christoffel symbols Γ^k_{ij} with respect to (u^j) satisfy*

$$\Gamma^k_{ij} = \Gamma^k_{ji} \tag{3.9}$$

$$\left\langle \frac{\partial^2 f}{\partial u^i \partial u^j}, \frac{\partial f}{\partial u^l} \right\rangle = \sum_{k=1}^{m} g_{lk}\Gamma^k_{ij}, \tag{3.10}$$

$$\frac{\partial g_{ij}}{\partial u^l} = \sum_{k=1}^{m}\left(g_{kj}\Gamma^k_{il} + g_{ik}\Gamma^k_{jl}\right), \tag{3.11}$$

where g_{ij}'s are the components of the induced metric defined in (3.4)*.*

Proof. The first equality (3.9) is obvious from the definition

(3.8) and the symmetricity of (g_{ij}). Next, we prove (3.10). Since

$$
\begin{aligned}
\left\langle \frac{\partial^2 f}{\partial u^i \partial u^j}, \frac{\partial f}{\partial u^l} \right\rangle &= \frac{\partial}{\partial u^i} \left\langle \frac{\partial f}{\partial u^j}, \frac{\partial f}{\partial u^l} \right\rangle - \left\langle \frac{\partial f}{\partial u^j}, \frac{\partial^2 f}{\partial u^i \partial u^l} \right\rangle \\
&= \frac{\partial g_{jl}}{\partial u^i} - \left\langle \frac{\partial f}{\partial u^j}, \frac{\partial^2 f}{\partial u^l \partial u^i} \right\rangle \\
&= \frac{\partial g_{lj}}{\partial u^i} - \frac{\partial}{\partial u^l} \left\langle \frac{\partial f}{\partial u^j}, \frac{\partial f}{\partial u^i} \right\rangle + \left\langle \frac{\partial^2 f}{\partial u^l \partial u^j}, \frac{\partial f}{\partial u^i} \right\rangle \\
&= \frac{\partial g_{lj}}{\partial u^i} - \frac{\partial g_{ji}}{\partial u^l} + \left\langle \frac{\partial^2 f}{\partial u^j \partial u^l}, \frac{\partial f}{\partial u^i} \right\rangle \\
&= \frac{\partial g_{lj}}{\partial u^i} - \frac{\partial g_{ji}}{\partial u^l} + \frac{\partial}{\partial u^j} \left\langle \frac{\partial f}{\partial u^l}, \frac{\partial f}{\partial u^i} \right\rangle - \left\langle \frac{\partial f}{\partial u^l}, \frac{\partial^2 f}{\partial u^j \partial u^i} \right\rangle \\
&= \frac{\partial g_{lj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} + \frac{\partial g_{li}}{\partial u^j} - \left\langle \frac{\partial^2 f}{\partial u^i \partial u^j}, \frac{\partial f}{\partial u^l} \right\rangle,
\end{aligned}
$$

we have

$$
\begin{aligned}
\left\langle \frac{\partial^2 f}{\partial u^i \partial u^j}, \frac{\partial f}{\partial u^l} \right\rangle &= \frac{1}{2} \left(\frac{\partial g_{lj}}{\partial u^i} + \frac{\partial g_{li}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^l} \right) \\
&= \frac{1}{2} \sum_{p=1}^{m} \delta_l^p \left(\frac{\partial g_{pj}}{\partial u^i} + \frac{\partial g_{pi}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^p} \right) \\
&= \frac{1}{2} \sum_{k,p=1}^{m} g_{lp} g^{pk} \left(\frac{\partial g_{pj}}{\partial u^i} + \frac{\partial g_{pi}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^p} \right) = \sum_{k=1}^{m} g_{kl} \mathit{\Gamma}_{ij}^k.
\end{aligned}
$$

Using this, we have

$$
\begin{aligned}
\frac{\partial g_{ij}}{\partial u^l} &= \frac{\partial}{\partial u^l} \left\langle \frac{\partial f}{\partial u^i}, \frac{\partial f}{\partial u^j} \right\rangle = \left\langle \frac{\partial^2 f}{\partial u^l \partial u^i}, \frac{\partial f}{\partial u^j} \right\rangle + \left\langle \frac{\partial f}{\partial u^i}, \frac{\partial^2 f}{\partial u^l \partial u^j} \right\rangle \\
&= \sum_{k=1}^{m} g_{kj} \mathit{\Gamma}_{li}^k + \sum_{k=1}^{m} g_{ik} \mathit{\Gamma}_{lj}^k,
\end{aligned}
$$

proving (3.11). □

Flatness. We shall prove the following:

Theorem 3.13. *Let (M, g) be a Riemannian manifold of dimension m (resp. pseudo Riemannian manifold of signature $(m-s, s)$), and $(U; u^1, \dots, u^m)$ a local coordinate system. Assume there exists an immersion $f\colon U \to \mathbb{R}^m$ (resp. $\mathbb{R}^m_s$) into the Euclidean space (resp. the pseudo Euclidean space) with the same dimension m as M whose induced metric is g. Then the Christoffel symbols $\mathit{\Gamma}_{ij}^k$ with respect to the coordinate system (u^j) satisfy*

$$
\frac{\partial \mathit{\Gamma}_{ij}^l}{\partial u^k} - \frac{\partial \mathit{\Gamma}_{ik}^l}{\partial u^j} + \sum_{p=1}^{m} \left(\mathit{\Gamma}_{ij}^p \mathit{\Gamma}_{pk}^l - \mathit{\Gamma}_{ik}^p \mathit{\Gamma}_{pj}^l \right) = 0 \tag{3.12}
$$

holds for $i, j, k, l = 1, \dots, m$. Conversely, when U is simply connected and (3.12) holds, there exists an immersion $f\colon U \to \mathbb{R}^m$ (resp. $\mathbb{R}^m_s$) such that the induced metric by f coincides with the metric g.

Lemma 3.14. *Let $f\colon U \to \mathbb{R}^m$ (resp. $\mathbb{R}^m_s$) be an immersion of a domain $(U; u^1, \dots, u^m) \subset \mathbb{R}^m$, and set*

$$\mathcal{F} := \left(\frac{\partial f}{\partial u^1}, \dots, \frac{\partial f}{\partial u^m}\right) : U \to \mathrm{M}_m(\mathbb{R}).$$

Then $\mathcal{F}$ satisfies

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j, \quad \Omega_j := \begin{pmatrix} \Gamma^1_{1j} & \dots & \Gamma^1_{mj} \\ \vdots & \ddots & \vdots \\ \Gamma^m_{1j} & \dots & \Gamma^m_{mj} \end{pmatrix} \tag{3.13}$$

for $j = 1, \dots, m$, where Γ^k_{ij}'s are the Christoffel symbols of the induced metric with respect to the coordinate system (u^j). Moreover, the Christoffel symbols satisfy (3.12).

Proof. Problem 3-1. □

Lemma 3.15. *Let $A \in \mathrm{M}_m(\mathbb{R})$ be a symmetric matrix such that the quadratic form $\boldsymbol{x} \mapsto {}^t\boldsymbol{x}A\boldsymbol{x}$ has a signature $(s, m-s)$. Then there exists a regular matrix P such that*

$${}^tPJ_{s,m-s}P = A, \qquad J_{s,m-s} = \begin{pmatrix} -\mathrm{id}_s & O \\ O & \mathrm{id}_{m-s} \end{pmatrix},$$

where id_k is the $k \times k$ identity matrix and O's are zero matrices.

Proof. By the assumptions, A has $(m - s)$ positive eigenvalues and s negative eigenvalues, and A can be diagonalized by an orthogonal matrix Q:

$$A = {}^tQ \begin{pmatrix} \Lambda_- & O \\ O & \Lambda_+ \end{pmatrix} Q,$$

where $\Lambda_- := \mathrm{diag}(-a_1^2, \dots, -a_s^2)$, $\Lambda_+ := \mathrm{diag}(a_{s+1}^2, \dots, a_m^2)$ and a_j's ($j = 1, \dots, m$) are non-zero real numbers. Let $D := \mathrm{diag}(a_1, \dots, a_m)$ and $P := DQ$, we have the conclusion. □

Proof of Theorem 3.13. The first assertion has been proved in Lemma 3.14. We assume (3.12) holds for each $i, j, k, l = 1, \dots, m$, and fix $\mathrm{P}_0 \in U$. We let $A = (g_{ij}(\mathrm{P}_0))$, which is a symmetric matrix such that the corresponding quadratic form is of signature $(m - s, s)$. Then there exists a regular matrix P as in Lemma 3.15. Then by Theorem 2.5, there exists $\mathcal{F}\colon U \to \mathrm{M}_m(\mathbb{R})$ satisfying (3.13) with initial condition $\mathcal{F}(\mathrm{P}_0) = P$. We set

$$\omega := \sum_{j=1}^m \boldsymbol{f}_j \, du^j, \qquad \text{where} \quad \mathcal{F} = (\boldsymbol{f}_1, \dots, \boldsymbol{f}_m).$$

Then by (3.9), we know that ω is a vector-valued closed one form. Hence by Poincaré's Lemma (Theorem 2.6), there exists a C^∞-function $f\colon M \to \mathbb{R}^m_s$ such that $df = \omega$, that is,

$$\frac{\partial f}{\partial u^j} = \boldsymbol{f}_j \qquad (j = 1, \dots, m).$$

We shall prove that this f is the desired immersion, that is, our goal is to prove

$$g_{ij} = \langle \boldsymbol{f}_i, \boldsymbol{f}_j \rangle \qquad (i, j = 1, \dots, m).$$

To do it, we set

$$\kappa_{ij} := g_{ij} - \langle \boldsymbol{f}_i, \boldsymbol{f}_j \rangle \qquad (i, j = 1, \dots, m).$$

So, by a choice of the initial condition, we have

$$\kappa_{ij}(\mathrm{P}_0) = 0 \qquad (i, j = 1, \dots, m). \tag{3.14}$$

Then, by (3.13) and (3.11), it holds that

$$\frac{\partial \kappa_{ij}}{\partial u^l} = \sum_{k=1}^{m} \left(\kappa_{ik} \mathit{\Gamma}_{jl}^k + \kappa_{kj} \mathit{\Gamma}_{il}^k \right) \qquad (i, j, l = 1, \dots, m). \tag{3.15}$$

Let $\mathrm{P} \in U$ and take a path $\gamma(t)$ $(0 \leqq t \leqq 1)$ in U satisfying $\gamma(0) = \mathrm{P}_0$ and $\gamma(1) = \mathrm{P}$. Then the functions $\tilde{\kappa}_{ij}(t)$ satisfy a system of ordinary differential equations

$$\frac{d\tilde{\kappa}_{ij}}{dt} = \sum_{l=1}^{m} \sum_{k=1}^{m} \left(\tilde{\kappa}_{ik} \mathit{\Gamma}_{jl}^k \circ \gamma + \tilde{\kappa}_{kj} \mathit{\Gamma}_{il}^k \circ \gamma \right) \frac{du^l}{dt} \quad (i, j = 1, \dots, m),$$

where $\gamma(t) = (u^1(t), \dots, u^m(t))$. Since $\tilde{\kappa}_{ij}(t) = 0$ $(i, j = 1, \dots, m)$ satisfy the equation with initial condition (3.14), uniqueness theorem implies that $\tilde{\kappa}_{ij}(1) = \kappa_{ij}(\mathrm{P}) = 0$, proving the theorem.

Remark 3.16. As we see in the following section, the condition (3.12) does not depend on choice of local coordinate systems. We say a (pseudo) Riemmanian manifold (M, g) to be *flat* if (3.12) holds on M.

Exercises

3-1 Show Lemma 3.14.

3-2 Let $M := \mathbb{R}_+ \times \mathbb{R} = \{(u^1, u^2)\,;\, u^1 > 0\}$, and consider a Riemannian metric g on M whose components are

$$g_{11} = 1, \qquad g_{12} = 0, \qquad g_{22} = \left\{\varphi(u^1)\right\}^2,$$

where $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function.

- Find a function φ satisfying (3.12) and $\lim_{t \to 0+} \varphi(t) = 0$.
- Under the situation above, find $f\colon U \to \mathbb{R}^2$ on an appropriate domain U on M such that the induced metric coincides with g.

4 The Curvature Tensor.

Change of Coordinate Systems. Throughout this section, we let (M,g) be a (pseudo) Riemannian m-manifold and take a local coordinate system $(U;u^1,\dots,u^m)$ on a neighborhood of $\mathrm{P}\in U$. Choose another coordinate system $(V;x^1,\dots,x^m)$ on a neighborhood V of P. Then the coordinate change

$$(4.1)\qquad \boldsymbol{x}=(x^1,\dots,x^m)\mapsto \boldsymbol{u}(\boldsymbol{x})=(u^1(x^1,\dots,x^m),\dots,u^m(x^1,\dots,x^m))$$

is defined as a C^∞-map between certain domains in $\mathbb{R}^m$. Since the transformation (4.1) is a diffeomorphism, the inverse

$$(4.2)\qquad \boldsymbol{u}=(u^1,\dots,u^m)\mapsto \boldsymbol{x}(\boldsymbol{u})=(x^1(u^1,\dots,u^m),\dots,x^m(u^1,\dots,u^m))$$

is also C^∞. Thus, the Jacobian matrix

$$J:=\begin{pmatrix}\dfrac{\partial u^1}{\partial x^1} & \dots & \dfrac{\partial u^1}{\partial x^m}\\ \vdots & \ddots & \vdots\\ \dfrac{\partial u^m}{\partial x^1} & \dots & \dfrac{\partial u^m}{\partial x^m}\end{pmatrix}=\left(\frac{\partial u^i}{\partial x^a}\right)_{i,a=1,\dots,m}$$

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is invertible on each point of the domain of $\boldsymbol{u}(\boldsymbol{x})$, and the Jacobian matrix of the inverse map $\boldsymbol{x}=\boldsymbol{x}(\boldsymbol{u})$ is obtained as

$$\begin{pmatrix}\dfrac{\partial x^1}{\partial u^1} & \dots & \dfrac{\partial x^1}{\partial u^m}\\ \vdots & \ddots & \vdots\\ \dfrac{\partial x^m}{\partial u^1} & \dots & \dfrac{\partial x^m}{\partial u^m}\end{pmatrix}=\left(\frac{\partial x^a}{\partial u^i}\right)_{a,i=1,\dots,m}=J^{-1},$$

where the inverse matrix of the right-hand side is evaluated at $\boldsymbol{u}(\boldsymbol{x})$, that is,

$$(4.3)\qquad \sum_{i=1}^m\frac{\partial x^a}{\partial u^i}\frac{\partial u^i}{\partial x^b}=\delta^a_b,\qquad\text{and}\qquad \sum_{a=1}^m\frac{\partial u^i}{\partial x^a}\frac{\partial x^a}{\partial u^j}=\delta^i_j$$

hold, where δ denotes Kronecker's delta.

Components of Vector fields and Differential forms. Let X be a vector field on M. Then it can be expressed on coordinate neighborhoods $(U;\boldsymbol{u})$ and $(V;\boldsymbol{x})$ as

$$X=\sum_{i=1}^m X^i\frac{\partial}{\partial u^i}=\sum_{a=1}^m\widetilde{X}^a\frac{\partial}{\partial x^a}.$$

Since

$$(4.4)\qquad \frac{\partial}{\partial u^i}=\sum_{a=1}^m\frac{\partial x^a}{\partial u^i}\frac{\partial}{\partial x^a}\quad\text{and}\quad\frac{\partial}{\partial x^a}=\sum_{i=1}^m\frac{\partial u^i}{\partial x^a}\frac{\partial}{\partial u^i},$$

we have the following transformation formula for the components of X:

$$\widetilde{X}^a = \sum_{i=1}^m \frac{\partial x^a}{\partial u^i} X^i, \qquad X^i = \sum_{a=1}^m \frac{\partial u^i}{\partial x^a} \widetilde{X}^a. \tag{4.5}$$

Thus,

Lemma 4.1. *The components (g_{ij}) and $(\tilde{g}_{ab})$ of the (pseudo) Riemannian metric g with respect to the coordinates $(u^1, \dots, u^m)$ and $(x^1, \dots, x^m)$, respectively, are related as*

$$\tilde{g}_{ab} = \sum_{i,j=1}^m \frac{\partial u^i}{\partial x^a} \frac{\partial u^j}{\partial x^b} g_{ij}. \tag{4.6}$$

Moreover, the inverse matrices (g^{ij}) and $(\tilde{g}^{ab})$ of (g_{ij}) and $(\tilde{g}_{ab})$, respectively, satisfy

$$g^{ij} = \sum_{a,b=1}^m \frac{\partial u^i}{\partial x^a} \frac{\partial u^j}{\partial x^b} \tilde{g}^{ab}.$$

Proof. By (4.4), we have

$$\begin{aligned} \tilde{g}_{ab} := g\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) &= \sum_{i,j=1}^m \frac{\partial u^i}{\partial x^a} \frac{\partial u^j}{\partial x^b} g\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) \\ &= \sum_{i,j=1}^m \frac{\partial u^i}{\partial x^a} \frac{\partial u^j}{\partial x^b} g_{ij} \end{aligned}$$

proving the first assertion. The second assertion follows from (4.3). □

Corollary 4.2. *The Christoffel symbols Γ_{ij}^k as in (3.8) with respect to the coordinate system $(u^1, \dots, u^m)$ and the Christoffel symbols $\widetilde{\Gamma}_{ab}^c$ with respect to $(x^1, \dots, x^m)$ are related as*

$$\widetilde{\Gamma}_{ab}^c = \sum_{k=1}^m \frac{\partial x^c}{\partial u^k} \left[\frac{\partial^2 u^k}{\partial x^a \partial x^b} + \sum_{i,j=1}^m \frac{\partial u^i}{\partial x^a} \frac{\partial u^j}{\partial x^b} \Gamma_{ij}^k \right].$$

Proof. The definition (3.8) and Lemma 4.1 yields the conclusion through a direct computation. □

The following corollary is essentially a rephrasing of Theorem 3.13. Namely the proof of the corollary gives an alternative proof of Theorem 3.13.

Corollary 4.3. *Let (M, g) be an m-dimensional (pseudo) Riemannian manifold. Then, for each* P*, there exists a coordinate neighborhood $(U; u^1, \dots, u^m)$ of* P *such that the components (g_{ij}) of the metric g satisfy $g_{ij} = \pm\delta_{ij}$ if and only if the Christoffel symbols $\widetilde{\Gamma}_{ab}^c$ of any coordinate system $(x^1, \dots, x^m)$ satisfy* (3.12).

Proof. Let $(V; x^1, \dots, x^m)$ be a coordinate system at P, and denote the Christoffel symbol with respect to (x^a) by $\widetilde{\Gamma}_{ab}^c$. Consider a system of partial differential equations

$$\frac{\partial \mathcal{F}}{\partial x^a} = \mathcal{F} \Omega_a, \tag{4.7}$$

where Ω_a's are matrices defined by (3.13) for $\{\widetilde{\Gamma}_{ab}^c\}$ and $\mathcal{F} = (\boldsymbol{v}_1, \dots, \boldsymbol{v}_m)$. Then the integrability condition of (4.7) is equivalent to (3.12) for the Christoffel symbols $\{\widetilde{\Gamma}_{ab}^c\}$ satisfies (3.12).

That is, if $\{\widetilde{\Gamma}_{ab}^c\}$ satisfies (3.12), there exists a solution $\mathcal{F}$ (4.7) with the initial value $\mathcal{F}(\mathrm{P}) = F_0$. In addition, if F_0 is a regular matrix, $\mathcal{F}$ is valued in $\mathrm{GL}(m, \mathbb{R})$.

Moreover, noticing $\widetilde{\Gamma}_{ab}^c = \widetilde{\Gamma}_{ba}^c$, there exists a vector-valued function $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ such that

$$d\boldsymbol{u} = \sum_{a=1}^m \boldsymbol{v}_a dx^a$$

because the right-hand side is a closed one form. Since $\mathcal{F}$ is the Jacobian matrix of $\boldsymbol{x} \mapsto \boldsymbol{u}$, which is valued in $\mathrm{GL}(m, \mathbb{R})$, $\boldsymbol{u} = (u^1, \dots, u^m)$ is a new coordinate system around P.

By Corollary 4.2, the Christoffel symbols with respect to $(u^1, \dots, u^m)$ vanishes identically. This means that g_{ij}'s are constants because of (3.11). Since (g_{ij}) is a constant matrix, a linear transformation of the coordinate system yields the conclusion. □

The Curvature Tensor. Set

(4.8)
$$R_{ijkl} := \sum_{q=1}^m g_{ql} \left(\frac{\partial \Gamma_{ki}^q}{\partial u^j} - \frac{\partial \Gamma_{kj}^q}{\partial u^i} + \sum_{p=1}^m \left(\Gamma_{ik}^p \Gamma_{pj}^q - \Gamma_{kj}^p \Gamma_{pi}^q \right) \right)$$

for $i, j, k, l = 1, \dots, m$, where Γ_{ij}^k's are the Christoffel symbols. Obviously, it holds that

Lemma 4.4. *The (pseudo) Riemannian manifold (M, g) is flat if and only if, for each point $\mathrm{P} \in M$, there exists a coordinate system (u^j) around* P *such that R_{ijkl} $(i, j, k, l = 1, \dots, m)$ vanish identically.*

Here, the condition "there exists a coordinate system" in Lemma 4.4 can be replaced by "for any coordinate systems", because of the following lemma:

Lemma 4.5. *Let $(x^1, \dots, x^m)$ be another coordinate system, and define $\widetilde{R}_{abcd}$ by (4.8) replacing Γ with $\widetilde{\Gamma}$, u with x. Then*

$$\widetilde{R}_{abcd} = \sum_{i,j,k,l=1}^m \frac{\partial u^i}{\partial x^a} \frac{\partial u^j}{\partial x^b} \frac{\partial u^k}{\partial x^c} \frac{\partial u^l}{\partial x^d} R_{ijkl} \tag{4.9}$$

holds for each a, b, c, $d = 1, \dots, m$.

Proof. By tedious but simple computation, the conclusion follows. □

The relation (4.9) looks similar to (4.6), where the metric g is a notion which is independent of choice of coordinates. In fact, by (4.5) and (4.3), we have

Corollary 4.6. *Let $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ and $\boldsymbol{w} \in T_{\mathrm{P}}M$ and write them by*

$$\boldsymbol{x} = \sum_{i=1}^m x^i \left(\frac{\partial}{\partial u^i} \right)_{\mathrm{P}}, \qquad \boldsymbol{y} = \sum_{j=1}^m y^j \left(\frac{\partial}{\partial u^j} \right)_{\mathrm{P}}$$
$$\boldsymbol{z} = \sum_{k=1}^m z^k \left(\frac{\partial}{\partial u^k} \right)_{\mathrm{P}}, \qquad \boldsymbol{w} = \sum_{l=1}^m w^l \left(\frac{\partial}{\partial u^l} \right)_{\mathrm{P}}.$$

Then

$$R(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w}) := \sum_{i,j,k,l=1}^{m} x^i y^j z^k w^l R_{ijkl} \tag{4.10}$$

does not depend on choice of coordinates.

Thus, we can define a 4-linear map

$$R\colon T_{\mathrm{P}}M \times T_{\mathrm{P}}M \times T_{\mathrm{P}}M \times T_{\mathrm{P}}M \longrightarrow \mathbb{R},$$

and

$$R\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M),$$

where $C^\infty(M)$ is the commutative ring consists of C^∞-functions on M, and $\mathfrak{X}(M)$ is the $C^\infty(M)$-module consists of smooth vector fields on M. In fact, for X, Y, Z, W, we define

$$R(X,Y,Z,W)\colon M \ni \mathrm{P} \mapsto R(X_{\mathrm{P}}, Y_{\mathrm{P}}, Z_{\mathrm{P}}, W_{\mathrm{P}}) \in \mathbb{R}.$$

Then R is $C^\infty(M)$-linear in each entry, namely, for X, Y, Z ,$W \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$,

$$\begin{aligned} R(fX,Y,Z,W) &= R(X,fY,Z,W) = R(X,Y,fZ,W) \\ &= R(X,Y,Z,fW) = fR(X,Y,Z,W) \end{aligned} \tag{4.11}$$

holds. We call this R the *curvature tensor* of (M,g). Theorem 3.13 can be restated as following "coordinate free" form.

Corollary 4.7. *The Riemannian manifold is flat if and only if its curvature tensor vanishes identically.*

Covariant Derivatives. To define the curvature tensor in the coordinate-free form, we introduce the notion of *covariant derivatives* of vector fields.

For a vector field Y and tangent vector $\boldsymbol{v} \in T_{\mathrm{P}}M$, we define

$$\nabla_{\boldsymbol{v}} Y := \sum_{j=1}^{m} \left[\sum_{k=1}^{m} v^k \left(\frac{\partial Y^j}{\partial u^k} + \sum_{l=1}^{m} \Gamma^j_{lk} Y^l \right) \right] \left(\frac{\partial}{\partial u^j} \right)_{\mathrm{P}}, \tag{4.12}$$

where $Y = \sum_{i=1}^m Y^i(\partial/\partial u^i)$ and $\boldsymbol{v} = \sum_{i=1}^m v^i(\partial/\partial u^i)_{\mathrm{P}}$, and Γ^k_{ij}'s are the Christoffel symbols defined in (3.8).

Proposition 4.8. *Assume M is a (non-degenerate) submanifold of the (pseudo) Euclidean space $\mathbb{R}^{n+1}_s$, and take a vector field X on M defined on a neighborhood of $\mathrm{P} \in M$. Then*

$$\nabla_{\boldsymbol{v}} X = [D_{\boldsymbol{v}} X]^{\mathrm{T}}$$

holds, where $D_{\boldsymbol{v}} X$ is a directional derivative of $\mathbb{R}^{n+1}_s$-valued function with respect to $\boldsymbol{v}$, and $[*]^{\mathrm{T}}$ denotes the tangential component of it, as in (3.1).

Proof. Let $f = f(u^1,\dots,u^m)$ be a parametrization of M with respect to the local coordinate system (u^j) and we let

$$\left[\frac{\partial^2 f}{\partial u^i \partial u^j} \right]^{\mathrm{T}} = \sum_{k=1}^{m} G^k_{ij} \frac{\partial f}{\partial u^k}.$$

Then by (3.10) and (3.4), we have

$$\sum_{k=1}^{m} g_{kl} G^k_{ij} = \sum_{k=1}^{m} g_{kl} \Gamma^k_{ij}.$$

Since (g_{ij}) is a regular matrix, we have $G_{ij}^k = \mathit{\Gamma}_{ij}^k$. In other words,

$$\left[\frac{\partial^2 f}{\partial u^i \partial u^j}\right]^{\mathrm{T}} = \sum_{k=1}^m \mathit{\Gamma}_{ij}^k \frac{\partial f}{\partial u^k} \tag{4.13}$$

holds. Thus, identifying $\partial/\partial u^j$ with $\partial f/\partial u^j$, we have

$$\left[D_{\partial/\partial u^i} \frac{\partial f}{\partial u^j}\right]^{\mathrm{T}} = \left[\frac{\partial^2 f}{\partial u^i \partial u^k}\right]^{\mathrm{T}} = \sum_{k=1}^m \mathit{\Gamma}_{ij}^k \frac{\partial f}{\partial u^k} = \nabla_{\partial/\partial u^i} \frac{\partial f}{\partial u^j}.$$

Applying this, the conclusion follows. □

Using covariant derivative, we obtain the bilinear

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \mapsto \nabla_X Y \in \mathfrak{X}(M), \tag{4.14}$$

which is also called the *covariant derivative*, alternatively, the *Riemannian connection* or the *Levi-Civita connection*.

Proposition 4.9. *For each X, $Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$,*

$$\nabla_{fX} Y = f \nabla_X Y, \tag{4.15}$$

$$\nabla_X fY = (Xf)Y + f\nabla_X Y, \tag{4.16}$$

$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{4.17}$$

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{4.18}$$

where $[\ ,\ ]$ denotes the Lie-bracket for vector fields.

Proof. The first two assertions are direct conclusion of the definition of ∇. The third assertion follows because $\mathit{\Gamma}_{ij}^k = \mathit{\Gamma}_{ji}^k$ (cf. (3.9)) The last assertion can be proved by

$$\frac{\partial g_{ij}}{\partial u^l} = \sum_{k=1}^m \left(g_{kj} \mathit{\Gamma}_{il}^k + g_{ik} \mathit{\Gamma}_{jl}^k\right),$$

as seen in (3.11). □

Proposition 4.10. *For X, Y, Z and $W \in \mathfrak{X}(M)$, it holds that*

$$R(X, Y, Z, W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W), \tag{4.19}$$

where R is the curvature tensor as in (4.10).

Proof. Denote the right-hand side of (4.19) by $S(X, Y, Z, W)$. Then by Proposition 4.9, it holds that

$$\begin{aligned} S(fX, Y, Z, W) = S(X, fY, Z, W) &= S(X, Y, fZ, W) \\ S(X, Y, Z, fW) &= fS(X, Y, Z, W). \end{aligned} \tag{4.19a}$$

Then by (4.19a) it is sufficient to show the conclusion for

$$X = \frac{\partial}{\partial u^i}, \quad Y = \frac{\partial}{\partial u^j}, \quad Z = \frac{\partial}{\partial u^k}, \quad W = \frac{\partial}{\partial u^l}. \tag{4.20}$$

In fact,

$$\nabla_{\partial/\partial u^j}\frac{\partial}{\partial u^i} = \sum_{l=1}^{m}\Gamma_{ij}^l\frac{\partial}{\partial u^k},$$

$$\begin{aligned}\nabla_{\partial/\partial u^k}\nabla_{\partial/\partial u^j}\frac{\partial}{\partial u^i} &= \sum_{l=1}^{m}\left[\frac{\partial\Gamma_{ij}^l}{\partial u^k}\frac{\partial}{\partial u^l} + \Gamma_{ij}^l\nabla_{\partial/\partial u^k}\frac{\partial}{\partial u^l}\right]\\ &= \left[\frac{\partial\Gamma_{ij}^l}{\partial u^k} + \Gamma_{ij}^p\Gamma_{pk}^l\right]\frac{\partial}{\partial u^l},\end{aligned}$$

and

$$\left[\frac{\partial}{\partial u^i},\frac{\partial}{\partial u^j}\right] = 0$$

yield the conclusion. □

Exercises

4-1 Prove Corollary 4.2.

4-2 We consider a Riemannian metric g on a domain $U \subset \mathbb{R}^2$ with

$$g_{11} = g_{22} = e^{2\sigma}, \qquad g_{12} = g_{21} = 0,$$

with respect to the canonical coordinate system (u^1, u^2), where σ is a smooth function on U.

(1) Show that (U, g) is flat if and only if σ is a harmonic function, that is, it satisfies

$$\frac{\partial^2\sigma}{(\partial u^1)^2} + \frac{\partial^2\sigma}{(\partial u^2)^2} = 0.$$

(2) Compute R_{ijkl} for

$$e^{2\sigma} = \frac{4}{(1 + k(u^2 + v^2))^2},$$

where k is a constant and $(u, v) = (u^1, u^2)$ is the canonical coordinate system on $\mathbb{R}^2$.

5 Sectional Curvature.

Throughout this section, we let (M, g) an m-dimensional (pseudo) Riemannian manifold, and ∇ the covariant derivative defined in (4.12).

Tensors. A correspondence $S : M \ni \mathrm{P} \mapsto S_{\mathrm{P}}$ of a point P and a multi-linear map $S_{\mathrm{P}} : (T_{\mathrm{P}}M)^k \to \mathbb{R}$ is called a (k-th order covariant) *tensor field* or a *tensor* on M. For such a tensor field S trivially induces a map

$$\hat{S} : \bigl(\mathfrak{X}(M)\bigr)^p \ni (X_1, \dots, X_p) \mapsto S(X_1, \dots, X_p) \in \mathcal{F}(M) \tag{5.1}$$

where $\mathfrak{X}(M)$ is the set ($C^\infty(M)$-module) of C^∞-vector fields of M, and $\mathcal{F}(M)$ is the set of real-valued function on M. The tensor field S is said to be *smooth* of *class* C^∞ if $S(X_1, \dots, X_p)$ as in (5.1) is of class C^∞ for an arbitrary $X_1, \dots, X_p$.

Example 5.1. The (pseudo) Riemannian metric g is a smooth tensor field.

Example 5.2. A smooth 1-form on M is a smooth, first-order covariant tensor field on M.

We denote by

$$\Gamma\,(\otimes^p T^*M) = \Gamma(T^*M \otimes \dots \otimes T^*M) \tag{5.2}$$

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the set of p-th order covariant tensor fields on M. [7] The set $\Gamma\,(\otimes^p T^*M)$ is a vector space over $\mathbb{R}$. Moreover, for each $S \in \Gamma(\otimes^p T^*M)$ and $f \in C^\infty(M)$, $fS := (\mathrm{P} \mapsto f(\mathrm{P})S_{\mathrm{P}})$ is also an element of $\Gamma(\otimes^p T^*M)$. That is, $\Gamma(\otimes^p T^*M)$ is a $C^\infty(M)$-module.

Lemma 5.3. *Let S be a p-th covariant tensor field on M and $\hat{S} : \mathfrak{X}(M)^p \to C^\infty(M)$ the map induced by S as in* (5.1) *Then for an arbitrary $f \in C^\infty(M)$, it holds that*

$$\hat{S}(X_1, \dots, fX_j, \dots, X_p) = f\hat{S}(X_1, \dots, X_j, \dots, X_p), \tag{5.3}$$

where $X_1, \dots, X_p \in \mathfrak{X}(M)$.

Proof. For each $\mathrm{P} \in M$,

$$\begin{aligned} S(X_1, \dots, &fX_j, \dots, X_p)(\mathrm{P}) \\ &= S_{\mathrm{P}}\bigl((X_1)_{\mathrm{P}}, \dots, f(\mathrm{P})(X_j)_{\mathrm{P}}, \dots, (X_p)_{\mathrm{P}}\bigr) \\ &= f(\mathrm{P})S_{\mathrm{P}}\bigl((X_1)_{\mathrm{P}}, \dots, (X_j)_{\mathrm{P}}, \dots, (X_p)_{\mathrm{P}}\bigr) \\ &= (fS)(X_1, \dots, X_p)(\mathrm{P}). \qquad \Box \end{aligned}$$

Proposition 5.4. *A multi-linear map $\hat{S} : \bigl(\mathfrak{X}(M)\bigr)^p \to C^\infty(M)$ is induced from a certain $S \in \Gamma(\otimes^p T^*M)$ as in* (5.1) *if $\hat{S}$ is $C^\infty(M)$-multi-linear, that is,* (5.3) *holds for any $f \in C^\infty(M)$ and $X_1, \dots, X_p \in \mathfrak{X}(M)$.*

[7] The symbol "$\otimes$" in (5.2) means the *tensor product*. For example, $T^*M \otimes T^*M$ is a tensor product of the cotangent bundles, which is a certain vector bundle over M. The notion $\Gamma(*)$ means the set of sections of the vector bundle "$*$". Anyway, we do not give a precise meaning of these notations.

Proof. Assume that (5.1) holds. We fix P $\in M$ and take a local coordinate system $(U; u^1, \dots, u^m)$ of M around P. According to this coordinate system, we set $X_j = \sum_{l=1}^m \xi_j^l (\partial/\partial u^l)$ $(j = 1, \dots, p)$, where ξ_j^l's are C^∞-functions on U. Then by (5.1),

$$\hat{S}(X_1, \dots, X_p) = \sum \xi_1^{i_1} \dots \xi_p^{i_p} \hat{S}\left(\frac{\partial}{\partial u^{i_1}}, \dots, \frac{\partial}{\partial u^{i_p}}\right)$$

holds, where the sum in the right-hand side is taken over $i_j = 1, \dots, m$ $(j = 1, \dots, p)$. This means the value of the left-hand side at P is determined by $\xi_j^{i_l}(\mathrm{P})$, which depend only on $(X_j)_{\mathrm{P}}$. Hence, for each $\boldsymbol{v}_1, \dots, \boldsymbol{v}_p \in T_{\mathrm{P}}M$, we can define

$$S_{\mathrm{P}}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_p) := \hat{S}(X_1, \dots, X_m)(\mathrm{P}),$$

where X_j is an arbitrary vector field on M such that $X_j(\mathrm{P}) = \boldsymbol{v}_j$. Then $S\colon \mathrm{P} \mapsto S_{\mathrm{P}}$ is the desired one. □

If $\hat{S}$ is induced from a tensor field S, we say that $\hat{S}$ itself is a tensor field. From now on, we denote $\hat{S}$ in (5.1) by S for a simplicity. Then, for each $S \in \Gamma(\otimes^p T^*M)$, the C^∞-multi-linear map

$$S\colon \big(\mathfrak{X}(M)\big)^p \longrightarrow C^\infty(M) \tag{5.4}$$

is induced.

Taking a local coordinate system $(U; u^1, \dots, u^m)$ on M, we set

$$S_{i_1,\dots,i_p} := S\left(\frac{\partial}{\partial u^{i_1}}, \dots \frac{\partial}{\partial u^{i_p}}\right) \qquad (i_1, \dots, i_p = 1, \dots, m), \tag{5.5}$$

which are called the *components* of S with respect to the local coordinate system (u^j). Let $\{S_{a_1,\dots,a_p}\}$ be the components of S with respect to another coordinate system (x^a). Then it holds that

$$S_{a_1,\dots,a_p} = \sum_{i_1,\dots,i_p=1}^m \frac{\partial u^{i_1}}{\partial x^{a^1}} \dots \frac{\partial u^{i_p}}{\partial x^{a^p}} S_{i_1,\dots,i_p}. \tag{5.6}$$

Proposition 5.5. *Let $S\colon \big(\mathfrak{X}(M)\big)^p \to C^\infty(M)$ be a multi-linear map, and set $S_{i_1,\dots,i_p}$ by* (5.5). *Then S is a tensor field on M if and only if it satisfy* (5.6) *for an arbitrary coordinate change $(x^a) \mapsto (u^j)$.*

Proof. Problem 5-1. □

The Curvature Tensor. Let ∇ be the covariant derivative on (M, g), as defined in (4.12), which is considered as

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \longmapsto \nabla_X Y \in \mathfrak{X}(M).$$

Remark 5.6. The tri-linear map

$$D\colon \big(\mathfrak{X}(M)\big)^3 \ni (X, Y, Z) \mapsto g(\nabla_X Y, Z) \in C^\infty(M)$$

is not a tensor field. In fact, (4.16) means that $D(X, fY, Z)$ and $fD(X, Y, Z)$ may not coincide. But for a fixed $Y \in \mathfrak{X}(M)$,

$$DY : \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Z) \mapsto g(\nabla_X Y, Z) \in C^\infty(M)$$

is a tensor because of (4.15).

As seen in the proof in Proposition 4.10, the 4-linear map $R\colon \bigl(\mathfrak{X}(M)\bigr)^4 \to C^\infty(M)$ defined by

$$(5.7)\quad R(X,Y,Z,W) := g\bigl(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W\bigr),$$

where $[X,Y]$ denotes the *Lie bracket* of the vector fields, is a tensor field, which we call the *curvature tensor*, or the *Riemann-Christoffel curvature tensor* of (M,g).

Proposition 5.7. *The curvature tensor R has the following symmetricity:*

(1) $R(Y,X,Z,W) = -R(X,Y,Z,W)$.

(2) $R(X,Y,W,Z) = -R(X,Y,Z,W)$.

(3) $R(X,Y,Z,W) = R(Z,W,X,Y)$.

(4) $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$,

where X, Y, Z and W are vector fields.

Proof. The equality (1) follows from the property of the Lie bracket $[Y,X] = -[X,Y]$. The equality (4) can be proved by the property (4.17) and the Jacobi identity

$$\bigl[[X,Y],Z\bigr] + \bigl[[Y,Z],X\bigr] + \bigl[[Z,X],Y\bigr] = 0$$

for the Lie bracket. The property (2) can be shown by applying (4.18) and (4.17) (Problem 5-2). The property (3) follows from

(4), (1) and (2). In fact, summing up

$$\begin{aligned}
R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) &= 0,\\
R(Y,Z,W,X) + R(Z,W,Y,X) + R(W,Y,Z,X) &= 0,\\
R(Z,W,X,Y) + R(W,X,Z,Y) + R(X,Z,W,Y) &= 0,\\
R(W,X,Y,Z) + R(X,Y,W,Z) + R(Y,W,X,Z) &= 0,
\end{aligned}$$

(3) follows. □

Proposition 5.8. *Assume two tensors R_1 and $R_2 \in \Gamma(\otimes^4 T^*M)$ satisfy the symmetricity as in Proposition 5.7. If*

$$R_1(X,Y,Y,X) = R_2(X,Y,Y,X)$$

holds for all X, $Y \in \mathfrak{X}(M)$, then $R_1 = R_2$.

Proof. Expanding

$$\begin{aligned}
&R_1(X+sZ, Y+tW, Y+tW, X+sZ)\\
&\qquad = R_2(X+sZ, Y+tW, Y+tW, X+sZ),
\end{aligned}$$

we have the conclusion from the coefficients of st. □

Sectional Curvature.

Lemma 5.9. *Let R be the curvature tensor of a (pseudo) Riemannian manifold (M,g). Then, for each $\mathrm{P} \in M$,*

$$\frac{R(\boldsymbol{v},\boldsymbol{w},\boldsymbol{w},\boldsymbol{v})}{g(\boldsymbol{v},\boldsymbol{v})g(\boldsymbol{w},\boldsymbol{w}) - g(\boldsymbol{v},\boldsymbol{w})^2} \qquad (\boldsymbol{v},\boldsymbol{w} \in T_{\mathrm{P}}M)$$

depends only on the 2*-dimensional subspace of* $T_{\mathrm{P}}M$ *spanned by* $\{\boldsymbol{v},\boldsymbol{w}\}$ *whenever*

$$g(\boldsymbol{v},\boldsymbol{v})g(\boldsymbol{w},\boldsymbol{w})-g(\boldsymbol{v},\boldsymbol{w})^2\neq 0. \tag{5.8}$$

Proof. Set

$$(\boldsymbol{x},\boldsymbol{y})=(\boldsymbol{v},\boldsymbol{w})A,$$

where $A\in \mathrm{GL}(2,\mathbb{R})$. Then

$$R(\boldsymbol{x},\boldsymbol{y},\boldsymbol{y},\boldsymbol{x})=(\det A)^2R(\boldsymbol{v},\boldsymbol{w},\boldsymbol{w},\boldsymbol{v}),$$
$$g(\boldsymbol{x},\boldsymbol{x})g(\boldsymbol{y},\boldsymbol{y})-g(\boldsymbol{x},\boldsymbol{y})^2=(\det A)^2\bigl(g(\boldsymbol{v},\boldsymbol{v})g(\boldsymbol{w},\boldsymbol{w})-g(\boldsymbol{v},\boldsymbol{w})^2\bigr).$$

Hence the conclusion follows. □

Remark 5.10. When g is positive definite (i.e., (M,g) is a Riemannian manifold), (5.8) holds if and only if $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent. On the other hand, when g is indefinite, the left-hand side of (5.8) may vanish even if $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent. In this case, (5.8) holds if and only if $\mathrm{Span}\{\boldsymbol{v},\boldsymbol{w}\}$ is a non-degenerate subspace of $T_{\mathrm{P}}M$.

Definition 5.11. For a 2-dimensional non-degenerate subspace $\Pi_{\mathrm{P}}\subset T_{\mathrm{P}}M$, we set

$$K(\Pi_{\mathrm{P}}):=\frac{R(\boldsymbol{v},\boldsymbol{w},\boldsymbol{w},\boldsymbol{v})}{g(\boldsymbol{v},\boldsymbol{v})g(\boldsymbol{w},\boldsymbol{w})-g(\boldsymbol{v},\boldsymbol{w})^2)},$$

where $\{\boldsymbol{v},\boldsymbol{w}\}$ is a basis of Π_{P}. We call it the *sectional curvature* at Π_{P}.

Remark 5.12. The set of 2-dimensional subspaces on an n-dimensional vector space V can be endowed with the structure of a compact $(2n-4)$-dimensional manifold, denoted by $\mathrm{Gr}_2(V)$, which is called the 2*-Grassmanian manifold* over V. So, when (M,g) is a Riemannian, the sectional curvature can be considered as a smooth map

$$K\colon \mathrm{Gr}_2(TM):=\bigcup_{\mathrm{P}\in M}\mathrm{Gr}_2(T_{\mathrm{P}}M)\to\mathbb{R}.$$

Example 5.13. Let (M,g) be a 2-dimensional Riemannian manifold. Since $\mathrm{Gr}_2(T_{\mathrm{P}}M)$ consists of one point, the sectional curvature K can be regarded as a function defined on M itself. In this case, the sectional curvature is written as

$$\begin{aligned}K=&\frac{E(E_vG_v-2F_uG_v+G_u^2)}{4(EG-F^2)^2}\\&+\frac{F(E_uG_v-E_vG_u-2E_vF_v-2F_uG_u+4F_uF_v)}{4(EG-F^2)^2}\\&+\frac{G(E_uG_u-2E_uF_v+E_v^2)}{4(EG-F^2)^2}-\frac{E_{vv}-2F_{uv}+G_{uu}}{2(EG-F^2)},\end{aligned}$$

where $(u^1,u^2)=(u,v)$ is a local coordinate system and

$$E=g_{11},\qquad F=g_{12}=g_{21},\qquad G=g_{22}.$$

Constant Sectional Curvature. A Riemannian manifold (M,g) is said to be a *space of constant sectional curvature* if K is constant everywhere.

Proposition 5.14. *A (pseudo) Riemannian manifold (M, g) has constant sectional curvature k if and only if its curvature tensor R satisfies*

$$R(X,Y,Z,W) = k(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)). \tag{5.9}$$

Proof. If R satisfies (5.9), $K = k$ is constant obviously. Conversely, assume $K = k$ is constant. Then (5.9) holds for $Z = Y$, $W = X$. Since the right-hand side has the symmetric property as in Proposition 5.7, Proposition 5.8 yields (5.9). □

Example 5.15. The curvature tensor of the Euclidean space $\mathbb{R}^n$ vanishes identically, because R vanishes identically. The covariant derivative of $\mathbb{R}^n$ is identified with the directional derivative D. This means that

$$D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z = O$$

holds for vector fields X, Y and Z.

Example 5.16. Let $k > 0$ and

$$S^n(k) := \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1}\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = \frac{1}{k} \right\}.$$

As seen in Example 3.3, this is an n-dimensional submanifold of $\mathbb{R}^{n+1}$, and then is a Riemannian manifold with the induced metric from $\mathbb{R}^{n+1}$.

We compute the sectional curvature $S^n(k)$: The unit normal vector of $S^n(k)$ at $\boldsymbol{x}$ is $\boldsymbol{n} := \boldsymbol{x}/\sqrt{k}$. Then, regarding vector field on $S^n(k)$ as a vector field of $\mathbb{R}^{n+1}$ along $S^n(k)$, we have by Proposition 4.8

$$\begin{aligned}
\nabla_Y Z = [D_Y Z]^{\mathrm{T}} &= D_Y Z - \frac{1}{\sqrt{k}} \langle D_Y Z, \boldsymbol{x} \rangle \boldsymbol{x} \\
&= D_Y Z - \frac{1}{\sqrt{k}} Y \langle Z, \boldsymbol{x} \rangle \boldsymbol{x} + \frac{1}{\sqrt{k}} \langle Z, D_Y \boldsymbol{x} \rangle \boldsymbol{x} \\
&= D_Y Z - \frac{1}{\sqrt{k}} Y \langle Z, \boldsymbol{x} \rangle \boldsymbol{x} + \frac{1}{\sqrt{k}} \langle Z, Y \rangle \boldsymbol{x} \\
&= D_Y Z + \frac{1}{\sqrt{k}} \langle Z, Y \rangle \boldsymbol{x},
\end{aligned}$$

where we used the relation $D_Y \boldsymbol{x} = Y$. Using this relation, we can show that the curvature tensor R satisfy

$$\begin{aligned}
&R(X,Y,Z,W) \\
&\qquad = \left\langle D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, W \right\rangle \\
&\qquad\qquad + k\bigl(g(X,W)g(Y,Z) - g(X,Z)g(Y,T)\bigr).
\end{aligned}$$

Hence by Proposition 5.14, we obtain that the sectional curvature of $S^n(k)$ is k.

Example 5.17. Let k is a positive constant and

$$H^n(-k) := \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1}_1\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -\frac{1}{k}, x^0 > 0 \right\},$$

where $\boldsymbol{x} = (x^0, \dots, x^n)$. Then $H^n(-k)$ is a space-like hypersurface in the Lorentz-Minkowski space $\mathbb{R}^{n+1}_1$, as seen in Example 3.5, called the *hyperbolic space*. Since $H^n(-k)$ is a space-like hypersurface, the induced metric gives a Riemannian metric, and then $H^n(-k)$ is a Riemannian manifold.

By the completely same method as in the previous example, one can show that $H^n(-k)$ has constant sectional curvature $-k$.

Exercises

5-1 Prove Proposition 5.5.

5-2 Show (2) in Proposition 5.7.

5-3 Compute the sectional curvature of a Riemannian 2-manifold (M, g) with

$$g_{11} = g_{22} = \frac{4}{(1 + k(u^2 + v^2))^2}, \qquad g_{12} = g_{21} = 0,$$

where $(u, v) = (u^1, u^2)$ is a local coordinate system.

6 Spaces of Constant Sectional Curvature.

Orthonormal Frame and Connection Forms. Let (M, g) be an orientable Riemannian m-manifold, and $(U; u^1, \dots, u^m)$ a local coordinate neighborhood.

Lemma 6.1. *There exists a m-tuple of vector fields $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ on M which forms a positively-oriented orthonormal basis of $T_{\mathrm{P}}M$ for each $\mathrm{P} \in U$.*

Proof. The procedure of the Gram-Schmidt orthogonalization works for the m-tuple of vector fields $\{\partial/\partial u^j\}_{j=1}^m$ on U. □

We call such a m-tuple $\{\boldsymbol{e}_j\}_{j=1}^m$ a positively-oriented *orthonormal frame field*, or a *frame field* for short, on U.

Lemma 6.2. *Let $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ be an orthonormal frame field on $U \subset M$. Then there exist C^∞-differential 1-forms ω_i^j $(i, j = 1, \dots, m)$ satisfying*

$$\nabla_X \boldsymbol{e}_i = \sum_{j=1}^m \omega_i^j(X)\boldsymbol{e}_i \qquad (i = 1, \dots, m), \tag{6.1}$$

$$\omega_i^j = -\omega_j^i \qquad (i, j = 1, \dots, m) \tag{6.2}$$

for an arbitrary vector field X on U, where ∇ denotes the covariant derivative (4.12).

Proof. We set

$$\omega_i^j(X) := g(\nabla_X \boldsymbol{e}_i, \boldsymbol{e}_j)$$

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for $X \in \mathfrak{X}(U)$. So by (4.15), $\omega_i^j(fX) = f\omega_i^j(X)$ holds for $f \in C^\infty(U)$. Hence $\omega_i^j(X)(\mathrm{P})$ depends only on X_{P} because of Lemma 5.3. Then each ω_i^j defines a 1-form on U. Smoothness of ω_i^j is obvious. Since $\{\boldsymbol{e}_j\}$ is an orthonormal basis, (6.1) follows.

Moreover, since $g(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta_{ij}$ is constant for each i and j, (4.18) implies

$$\begin{aligned} 0 = Xg(\boldsymbol{e}_i, \boldsymbol{e}_j) &= g(\nabla_X \boldsymbol{e}_i, \boldsymbol{e}_j) + g(\boldsymbol{e}_i, \nabla_X \boldsymbol{e}_j) \\ &= g\left(\sum_{k=1}^m \omega_i^k(X)\boldsymbol{e}_k, \boldsymbol{e}_j\right) + g\left(\boldsymbol{e}_i, \sum_{k=1}^m \omega_j^k(X)\boldsymbol{e}_k\right) \\ &= \sum_{k=1}^m \left(\omega_i^k(X)\delta_{kj} + \omega_j^k(X)\delta_{ik}\right) = \omega_i^j(X) + \omega_j^i(X). \end{aligned}$$

Hence (6.2) follows. □

We call $\{\omega_i^j\}$ in Lemma 6.2 the *connection forms* with respect to the frame $\{\boldsymbol{e}_j\}$.

By (6.2),

$$\boldsymbol{\omega} := \begin{pmatrix} \omega_1^1 & \dots & \omega_m^1 \\ \vdots & \ddots & \vdots \\ \omega_1^m & \dots & \omega_m^m \end{pmatrix} \quad \text{satisfies} \quad \boldsymbol{\omega} + {}^t\boldsymbol{\omega} = O, \tag{6.3}$$

in other words, $\boldsymbol{\omega}$ is a skew-symmetric matrix-valued 1-form.

Gauge transformations and the Curvature Form. Let $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ and $\{\boldsymbol{f}_1, \dots, \boldsymbol{f}_m\}$ be two positively-oriented orthonormal frames on $U \subset M$. Then there exists a smooth map $G = (G_{ij})\colon U \to \mathrm{SO}(m)$ such that[8]

$$(6.4)\quad (\boldsymbol{e}_1, \dots, \boldsymbol{e}_m) = (\boldsymbol{f}_1, \dots, \boldsymbol{f}_m)G = \left(\sum_{a=1}^m G_{1a}\boldsymbol{f}_a, \dots, \sum_{a=1}^m G_{ma}\boldsymbol{f}_a\right).$$

Let $\boldsymbol{\omega} = (\omega_i^j)$ (resp. $\widetilde{\boldsymbol{\omega}} = (\tilde{\omega}_a^b)$) be the connection forms with respect to the orthonormal frame $\{\boldsymbol{e}_j\}$ (resp. $\{\boldsymbol{f}_a\}$). The

Lemma 6.3. *Under the situation above, it holds that*

$$(6.5)\qquad \widetilde{\boldsymbol{\omega}} = G^{-1}dG + G^{-1}\boldsymbol{\omega}G.$$

Proof. By definition,

$$\nabla(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m) = (\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)\boldsymbol{\omega}, \qquad \text{and}$$
$$\nabla(\boldsymbol{f}_1, \dots, \boldsymbol{f}_m) = (\boldsymbol{f}_1, \dots, \boldsymbol{f}_m)\widetilde{\boldsymbol{\omega}}$$

hold. Hence, by (4.16), it holds that

$$\begin{aligned}(\boldsymbol{f}_1, \dots, \boldsymbol{f}_m)\widetilde{\boldsymbol{\omega}} &= (\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)G\widetilde{\boldsymbol{\omega}}\\ (\boldsymbol{f}_1, \dots, \boldsymbol{f}_m)\widetilde{\boldsymbol{\omega}} &= \nabla\bigl((\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)G\bigr)\\ &= (\nabla(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m))\,G + (\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)dG\end{aligned}$$

[8] As defined in Section 1, $\mathrm{SO}(m) = \{A \in \mathrm{M}_m(\mathbb{R})\,;\,{}^tAA = A^tA = \mathrm{id}, \det A = 1\}$ denotes the special orthogonal group. A map $G\colon U \to \mathrm{SO}(m)$ is said to be smooth (of class C^∞) if it is of class C^∞ as a map into $\mathrm{M}_m(\mathbb{R})$, the set of $m \times m$-real matrices, which is identified with $\mathbb{R}^{n^2}$.

$$\begin{aligned}&= (\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)\boldsymbol{\omega}G + (\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)dG\\ &= (\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)\,(\boldsymbol{\omega}G + dG)\,,\end{aligned}$$

where $\nabla\boldsymbol{v}$ means a 1-form $X \mapsto \nabla_X\boldsymbol{v}$. Since $\boldsymbol{e}_1, \dots, \boldsymbol{e}_m$ are linearly independent, the conclusion follows. □

The formula (6.5) is called the *Gauge transformation* of the connection forms.

Definition 6.4. The *curvature form* with respect to the frame field $\{\boldsymbol{e}_j\}$ is a skew-symmetric matrix-valued 2-form

$$(6.6)\qquad \boldsymbol{\Omega} := d\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega} = \left(d\omega_i^j + \sum_{k=1}^m \omega_i^k\omega_k^j\right)_{i,j=1,\dots,m}.$$

Lemma 6.5. *Under the transformation as in* (6.4)*, the curvature form $\boldsymbol{\Omega}$ and $\widetilde{\boldsymbol{\Omega}}$ with respect to the frame field $\{\boldsymbol{e}_j\}$ and $\{\boldsymbol{f}_a\}$, respectively, satisfy*

$$\widetilde{\boldsymbol{\Omega}} = G^{-1}\boldsymbol{\Omega}G.$$

Proof. Problem 6-1.

Lemma 6.6. *The curvature form $\boldsymbol{\Omega} = (\Omega_i^j)$ with respect to the frame field $\{\boldsymbol{e}_j\}$ satisfies*

$$\Omega_i^j(X, Y) = R(X, Y, \boldsymbol{e}_i, \boldsymbol{e}_j),$$

where R is the Riemann-Christoffel curvature tensor, and X, Y are vector fields.

Proof. Since $\{\boldsymbol{e}_j\}$ is an orthonormal basis, it holds that

$$g(\nabla_X \boldsymbol{e}_j, \boldsymbol{e}_k) = Xg(\boldsymbol{e}_j, \boldsymbol{e}_k) - g(\boldsymbol{e}_j, \nabla_X \boldsymbol{e}_k) = -g(\boldsymbol{e}_j, \nabla_X \boldsymbol{e}_k)$$

holds for $j, k = 1, \dots, m$. Then we have

$$\begin{aligned}
\Omega_i^j(X,Y) &= d\omega_i^j(X,Y) + \sum_{k=1}^m \bigl(\omega_i^k(X)\omega_k^j(Y) - \omega_i^k(Y)\omega_k^j(Y)\bigr)\\
&= X\omega_i^j(Y) - Y\omega_i^j(X) - \omega_i^j([X,Y])\\
&\quad \sum_{k=1}^m \bigl(\omega_i^k(X)\omega_k^j(Y) - \omega_i^k(Y)\omega_k^j(Y)\bigr)\\
&= Xg(\nabla_Y \boldsymbol{e}_i, \boldsymbol{e}_j) - Yg(\nabla_X \boldsymbol{e}_i, \boldsymbol{e}_j) - g(\nabla_{[X,Y]} \boldsymbol{e}_i, \boldsymbol{e}_j)\\
&\quad + \sum_{k=1}^m \Bigl(g(\nabla_X \boldsymbol{e}_i, \boldsymbol{e}_k)g(\nabla_Y \boldsymbol{e}_k, \boldsymbol{e}_j) - g(\nabla_Y \boldsymbol{e}_i, \boldsymbol{e}_k)g(\nabla_X \boldsymbol{e}_k, \boldsymbol{e}_j)\Bigr)\\
&= g(\nabla_X \nabla_Y \boldsymbol{e}_i, \boldsymbol{e}_j) + g(\nabla_Y \boldsymbol{e}_i, \nabla_X \boldsymbol{e}_j)\\
&\quad - g(\nabla_Y \nabla_X \boldsymbol{e}_i, \boldsymbol{e}_j) - g(\nabla_X \boldsymbol{e}_i, \nabla_Y \boldsymbol{e}_j) - g(\nabla_{[X,Y]} \boldsymbol{e}_i, \boldsymbol{e}_j)\\
&\quad - \sum_{k=1}^m \Bigl(g(\nabla_X \boldsymbol{e}_i, \boldsymbol{e}_k)g(\boldsymbol{e}_k, \nabla_Y \boldsymbol{e}_j) - g(\nabla_Y \boldsymbol{e}_i, \boldsymbol{e}_k)g(\boldsymbol{e}_k, \nabla_X \boldsymbol{e}_j)\Bigr)\\
&= R(X, Y, \boldsymbol{e}_i, \boldsymbol{e}_j),
\end{aligned}$$

where we used the relation

$$\sum_{k=1}^m g(\boldsymbol{v}, \boldsymbol{e}_k) g(\boldsymbol{w}, \boldsymbol{e}_k) = g(\boldsymbol{v}, \boldsymbol{w}). \qquad \square$$

Space of Constant Sectional Curvature. The goal of this lecture is to prove the following

Theorem 6.7. *Let $U \subset \mathbb{R}^m$ be a simply connected domain and let g be a Riemannian metric on U with constant sectional curvature k. Then there exists a local diffeomorphism*

$$f\colon U \longrightarrow M^m(k)$$

such that the Riemannian metric g coincide with the metric on U induced from $M^m(k)$ by f, where

$$M^m(k) := \begin{cases} S^m(k) & (\text{when } k > 0, \text{ cf. Example 5.16}),\\ \mathbb{R}^m & (\text{when } k = 0, \text{ the Euclidean } m\text{-space}),\\ H^m(k) & (\text{when } k < 0, \text{ cf. Example 5.17}). \end{cases}$$

Remark 6.8. The theorem can be generalized for simply connected Riemannian manifolds (M, g) of constant sectional curvature. Moreover, one can show that f is injective. Hence, we can say that *a simply connected Riemannian m-manifold (M, g) can be identified as a subset of $M^m(k)$*. In particular, if (M, g) is complete, it coincides with $M^m(k)$.

Proof of Theorem 6.7 (for the case $k = 0$): This is an alternative proof of Theorem 3.13 in Section 3. Take an orthonormal frame field $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ on U, and let $\boldsymbol{\omega}$ be the connection form with respect to the basis. Fix a base point $\mathrm{P}_0 \in U$, and consider the system of differential equations

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\boldsymbol{\omega}_j, \qquad \mathcal{F}(\mathrm{P}_0) = \mathrm{id} \qquad (j = 1, \dots, m), \tag{6.7}$$

where

$$\boldsymbol{\omega}_j := \boldsymbol{\omega}\left(\frac{\partial}{\partial u^j}\right). \tag{6.8}$$

By Lemma 6.6, the assumption $k=0$ implies

$$
\begin{aligned}
O &= \left(R\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}, \boldsymbol{e}_k, \boldsymbol{e}_l \right) \right)_{k,l=1,\dots,m} \\
&= (d\boldsymbol{\omega} + \boldsymbol{\omega}\wedge\boldsymbol{\omega}) \left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right) = \frac{\partial \boldsymbol{\omega}_j}{\partial u^i} - \frac{\partial \boldsymbol{\omega}_i}{\partial u^j} + \boldsymbol{\omega}_i\boldsymbol{\omega}_j - \boldsymbol{\omega}_j\boldsymbol{\omega}_i.
\end{aligned}
$$

Hence by Theorem 2.5, there exists a unique solution $\mathcal{F}$ of (6.7). Moreover, since $\boldsymbol{\omega}_j$ is skew-symmetric because of (6.3), the solution gives a smooth map $\mathcal{F}\colon U \to \mathrm{SO}(m)$. Decompose $\mathcal{F}$ into the column vectors as $\mathcal{F} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$. Since $\mathcal{F}$ is an orthogonal matrix, $\{\boldsymbol{x}_j(\mathrm{P})\}$ is an orthonormal basis at each P.

Define an $\mathbb{R}^m$-valued 1-form

$$
\varphi := \sum_{i=1}^m \left(\sum_{k=1}^m g_i^k \boldsymbol{x}_k \right) du^i, \qquad g_i^k = g\left(\frac{\partial}{\partial u^i}, \boldsymbol{e}_k \right),
$$

where $\{\boldsymbol{e}_j\}$ is the orthonormal frame on U we took in the beginning of the proof. Then φ is a closed on U. In fact, by (4.18) and (4.17), we have

$$
\begin{aligned}
\frac{\partial}{\partial u^j} \left(\sum_{k=1}^m g_i^k \boldsymbol{x}_k \right) &= \sum_{k=1}^m \left(\frac{\partial}{\partial u^j} g\left(\frac{\partial}{\partial u^i}, \boldsymbol{e}_k \right) + g_i^k \boldsymbol{x}_k \right) \\
&= \sum_{k=1}^m \left[\left(g\left(\nabla_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i}, \boldsymbol{e}_k \right) + g\left(\frac{\partial}{\partial u^i}, \nabla_{\frac{\partial}{\partial u^j}} \boldsymbol{e}_k \right) \right) \boldsymbol{x}_k + g_i^k \frac{\partial \boldsymbol{x}_k}{\partial u^j} \right] \\
&= \sum_{k=1}^m g\left(\nabla_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i}, \boldsymbol{e}_k \right) \boldsymbol{x}_k + \sum_{k=1}^m g\left(\frac{\partial}{\partial u^i}, \sum_{l=1}^m \omega_k^l \left(\frac{\partial}{\partial u^j} \right) \boldsymbol{e}_l \right) \boldsymbol{x}_k
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{k,l=1}^m g_i^k \omega_k^l \left(\frac{\partial}{\partial u^j} \right) \boldsymbol{x}_l \\
&= \sum_{k=1}^m \left[g\left(\nabla_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i}, \boldsymbol{e}_k \right) + \sum_{l=1}^m (\omega_k^l + \omega_l^k) \left(\frac{\partial}{\partial u^j} \right) g_i^l + \right] \boldsymbol{x}_k \\
&= \sum_{k=1}^m \left[g\left(\nabla_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i}, \boldsymbol{e}_k \right) \right] \boldsymbol{x}_k, \\
&\frac{\partial}{\partial u^i} \left(\sum_{k=1}^m g_j^k \boldsymbol{x}_k \right) = \sum_{k=1}^m \left[g\left(\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j}, \boldsymbol{e}_k \right) \right] \boldsymbol{x}_k.
\end{aligned}
$$

Hence by (4.17), we have

$$
\frac{\partial}{\partial u^j} \left(\sum_{k=1}^m g_i^k \boldsymbol{x}_k \right) = \frac{\partial}{\partial u^i} \left(\sum_{k=1}^m g_j^k \boldsymbol{x}_k \right),
$$

that is, $d\varphi = 0$. Hence by Poincaré's lemma, there exists $f\colon U \to \mathbb{R}^m$ satisfying $df = \varphi$. This f is desired one. To show this, it is sufficient to show

$$
\text{(6.9)} \qquad df(\boldsymbol{e}_j) = \boldsymbol{x}_j, \qquad (j = 1, \dots, m).
$$

In fact, if (6.9) holds,

$$
g(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta_{ij} = \langle \boldsymbol{x}_i, \boldsymbol{x}_j, = \rangle \, \langle df(\boldsymbol{e}_i), df(\boldsymbol{e}_j), , \rangle
$$

and then the induced metric coincides with g. We show (6.9):

$$
df(\boldsymbol{e}_j) = \varphi(\boldsymbol{e}_j) = \sum_{i=1}^m \left(\sum_{k=1}^m g_i^k \boldsymbol{x}_k \right) du^i(\boldsymbol{e}_j)
$$

$$= \sum_{i,k=1}^{m} g\left(\frac{\partial}{\partial u^i}, \boldsymbol{e}_k\right) du^i(\boldsymbol{e}_j)\boldsymbol{x}_k$$
$$= \sum_{k=1}^{m} g\left(\sum_{i=1}^{m} du^i(\boldsymbol{e}_j)\frac{\partial}{\partial u^i}, \boldsymbol{e}_k\right)\boldsymbol{x}_k = \sum_{k=1}^{m} g(\boldsymbol{e}_j, \boldsymbol{e}_k)\boldsymbol{x}_k = \boldsymbol{x}_j.$$

Here, we used the formula

$$\sum_{i=1}^{m} du^i(\boldsymbol{v})\frac{\partial}{\partial u^i} = \boldsymbol{v}. \qquad \square$$

Proof of Theorem 6.7 (for the case $k > 0$): Since $k > 0$, there exists a real number c such that $k = c^2$. Taking the orthonormal frame field $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ on U, we set

$$\hat{\boldsymbol{\omega}}_j := \begin{pmatrix} 0 & -c\,{}^t\boldsymbol{g}_j \\ c\boldsymbol{g}_j & \boldsymbol{\omega}_j \end{pmatrix}, \tag{6.10}$$

for each $j = 1, \dots, m$, which is an $(n+1)\times(n+1)$-skew symmetric matrix-valued function, here

$$\boldsymbol{g}_j := \begin{pmatrix} g_j^1 \\ \vdots \\ g_j^m \end{pmatrix} = \begin{pmatrix} g(\partial/\partial u^j, \boldsymbol{e}_1) \\ \vdots \\ g(\partial/\partial u^j, \boldsymbol{e}_m) \end{pmatrix} \tag{6.11}$$

and $\boldsymbol{\omega}_j$ is as in (6.8). By the assumption, (5.9) holds. Hence one can show easily that

$$\frac{\partial \hat{\boldsymbol{\omega}}_j}{\partial u^i} - \frac{\partial \hat{\boldsymbol{\omega}}_i}{\partial u^j} + \hat{\boldsymbol{\omega}}_i\hat{\boldsymbol{\omega}}_j - \hat{\boldsymbol{\omega}}_j\hat{\boldsymbol{\omega}}_i = O$$

for each $i, j = 1, \dots, m$. Hence there exists a smooth map

$$\mathcal{F} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_m)\colon U \to \mathrm{SO}(m+1)$$

satisfying

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\hat{\boldsymbol{\omega}}_j \qquad (j = 1, \dots, m)$$

with $\mathcal{F}(\mathrm{P}_0) = \mathrm{id}$. Then

$$f := \frac{1}{c}\boldsymbol{x}_0$$

is the desired map. In fact,

$$df(\boldsymbol{e}_j) = \boldsymbol{x}_j \qquad (j = 1, \dots, m)$$

holds. $\square$

Proof of Theorem 6.7 (for the case $k < 0$): Since $k < 0$, there exists a real number c such that $k = -c^2$. Taking the orthonormal frame field $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ on U, we set

$$\hat{\boldsymbol{\omega}}_j := \begin{pmatrix} 0 & c\,{}^t\boldsymbol{g}_j \\ c\boldsymbol{g}_j & \boldsymbol{\omega}_j \end{pmatrix}, \tag{6.12}$$

for each $j = 1, \dots, m$, which is an $(n+1)\times(n+1)$-matrix-valued function, here

$$\boldsymbol{g}_j := \begin{pmatrix} g_j^1 \\ \vdots \\ g_j^m \end{pmatrix} = \begin{pmatrix} g(\partial/\partial u^j, \boldsymbol{e}_1) \\ \vdots \\ g(\partial/\partial u^j, \boldsymbol{e}_m) \end{pmatrix} \tag{6.13}$$

and $\boldsymbol{\omega}_j$ is as in (6.8). Since

$$Y\boldsymbol{\omega}_j - {}^t\omega_j Y = 0 \qquad (j = 1, \dots, m)$$

holds, where $Y := \operatorname{diag}(-1, 1, \dots, 1)$. This implies that there exists

$$\mathcal{F}\colon U \to \mathrm{SO}(m+1, 1),$$

where

$$\mathrm{SO}(m+1,1) = \\ \left\{a = (a_{ij})_{i,j=0,\dots,m} \in \mathrm{M}_n(\mathbb{R})\,;\, {}^t aYa = Y, \det a = 1, a_{00} > 0.\right\}$$

Then there exists $\mathcal{F}\colon U \to \mathrm{SO}(m+1)$ satisfying

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\hat{\boldsymbol{\omega}}_j \qquad (j = 1, \dots, m)$$

with $\mathcal{F}(\mathrm{P}_0) = \mathrm{id}$. Then

$$f := \frac{1}{c}\boldsymbol{x}_0$$

is the desired map. □

Exercises

6-1 Prove Lemma 6.5.

6-2 Prove Theorem 6.7