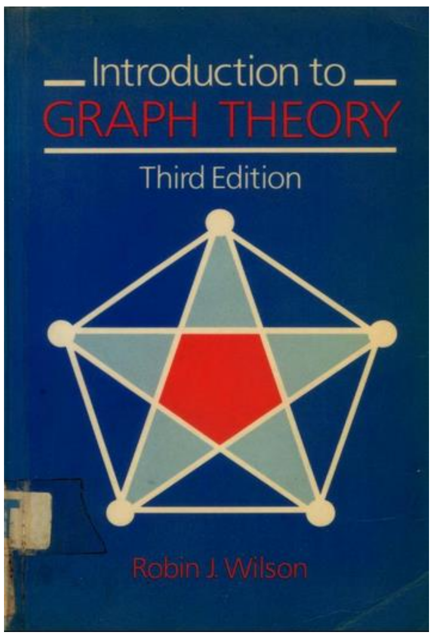
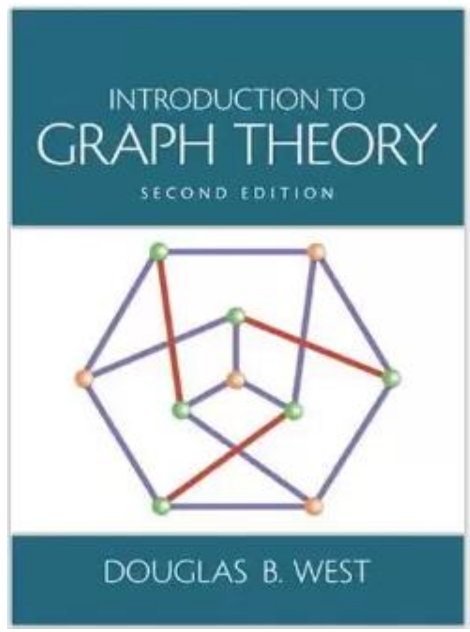
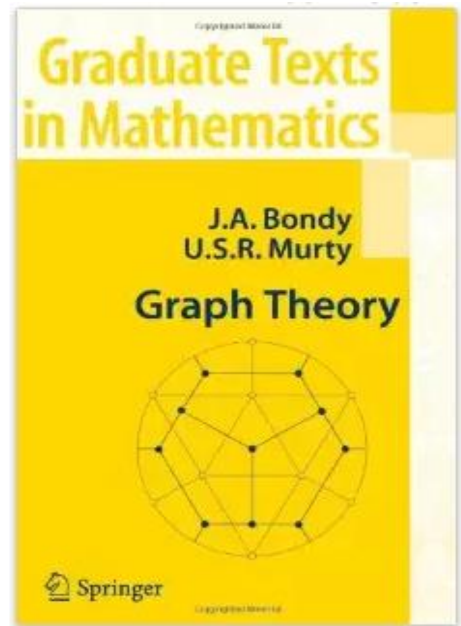
Graph Coloring

prepared and Instructed by

Shmuel Wimer
Eng. Faculty, Bar-Ilan University







Vertex Coloring

A k **-coloring** of a graph G is a labeling $f: V(G) \to \{1, \dots, k\}$.

A coloring is **proper** if no two vertices x and y connected with an edge have same color, i.e. $xy \in E(G) \Rightarrow f(x) \neq f(y)$.

G is k**-colorable** if it has proper k-coloring.

The **chromatic number** $\chi(G)$ is the smallest k such that G has proper k-coloring. G is called k**-chromatic**.

If $\chi(G) = k$, but $\chi(H) < k$ for every proper subgraph H, then G is k**-critical**.

The vertices having same color in a proper k-coloring must be independent. Therefore, $\chi(G)$ is the **minimum** number of **independent sets** covering G.

Hence, G is k-colorable iff G is k**-partite**.

Examples. Every bipartite graph is 2-colorable.
Every even cycle graph is 2-colorable (it is bipartite).
Every odd cycle graph is 3-colorable and 3-critical.

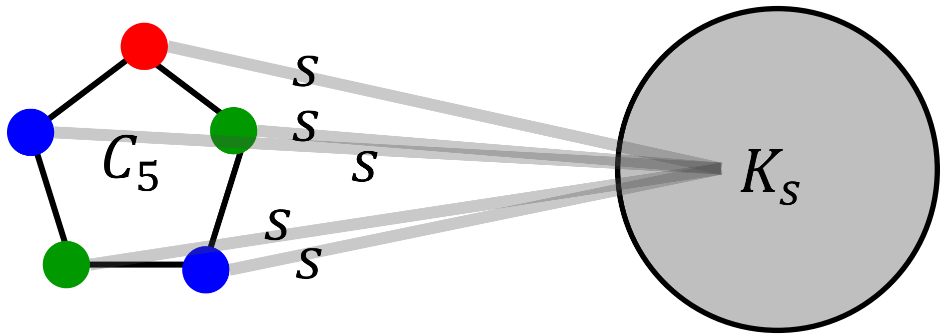
2-colorability can be tested with BFS. (how?)

We compute the distance from a vertex u. A connected graph is bipartite iff $G[X]$ and $G[Y]$ are independent sets, where X and Y are vertices of even and odd distance from u, respectively.

The largest clique size $\omega(G)$ satisfies $\chi(G) \geq \omega(G)$.

The largest independent set size $\alpha(G)$ satisfies $\chi(G) \geq n(G)/\alpha(G)$, since every color class is an independent set, therefore having at most $\alpha(G)$ vertices.

Is $\chi(G) > \omega(G)$ possible? Yes!



Proper coloring of K_s requites s colors. $\omega(G) = s + 2$, but $\chi(G) = s + 3$, hence $\chi(G) > \omega(G)$.

Could it be constructed with C_3 rather than C_5?

Example. **Minimizing exam period in school**. How to schedule exams in minimum parallel sessions, where no two concurrent exams have a common student?

Define $G(V,E)$, where $v \in V$ corresponds to course, and $e(u,v) \in E$ iff courses u and v have a common student.

An independent set of vertices implies a parallel exam session. $\chi(G)$ is the smallest number of parallel sessions.

Example. **Chemical storage**. Store n different chemicals. The interaction between some pairs is explosive.

What is the smallest required number of compartments in the storage? $\chi(G)$.

Upper Bounds of Chromatic Number

Easy bounds are $\chi(G) \leq n(G)$, $\chi(G) \geq \omega(G)$, and $\chi(G) \geq n(G)/\alpha(G)$, all hold with **equality** for **cliques**.

Better than $\chi(G) \leq n(G)$ upper bounds can be obtained by coloring algorithms.

A **greedy** algorithm w.r.t $V(G) = \{v_1, \dots, v_n\}$ assigns to v_i the smallest color index not incident so far to v_i.

Proposition. There is $\chi(G) \leq \Delta(G) + 1$. ($\Delta(G)$ is the largest vertex degree.)

Proof. By construction. A vertex has no more than $\Delta(G)$ neighbors. Upon v_i coloring there must be at least one of $1, \dots, \Delta(G) + 1$ colors unused. ■

Different orderings may yield **smaller upper bounds**. Finding the best ordering is hard. Is there an ordering yielding $\chi(G)$? It can be shown that such exists.

Example. **Register allocation and interval graphs**.

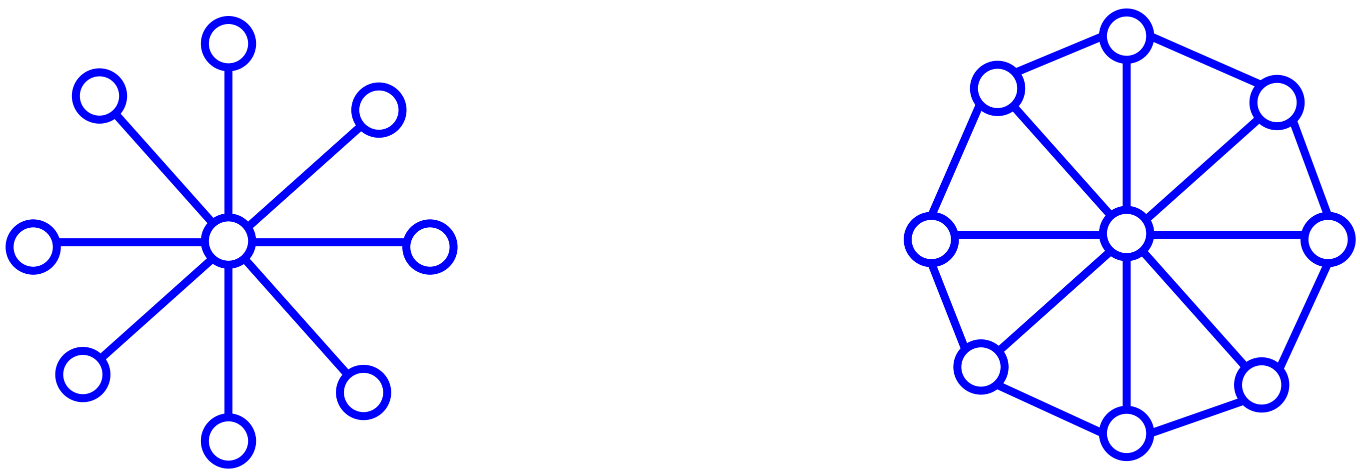
Consider the registers used by a compiler, each has start and end time. What is the smallest number of physical registers that can be used?

Assign the symbols a, b, c, … to the registers in the code, and draw their usage time intervals.

Proposition. If G is an interval graph then $\chi(G) = \omega(G)$.

Proof. By left-to-right traversal of the time intervals, pre sorted by their starting time. Initializing $k = 0$. Increasing to $k + 1$ at starting point and decreasing to $k - 1$ at ending point. ■

The bound $\chi(G) \leq \Delta(G) + 1$ may still be very poor.



For $(n + 1)$-vertex star $\Delta(G) = n$, whereas $\chi(G) = 2$.

For $(n + 1)$-vertex wheel $\Delta(G) = n$, whereas $\chi(G) \leq 4$.

The bound $\chi(G) \leq \Delta(G) + 1$ can be further improved by considering the vertices with high degree first.

Proposition. (Welsh-Powell 1967) If the vertices are ordered in non increasing degree, $d_1 \geq d_2 \geq \cdots \geq d_n$, then $\chi(G) \leq 1 + \max_i \min\{d_i, i-1\}$.

Proof. When vertex i is colored, its already colored neighbors have at most $\min\{d_i, i-1\}$ distinct colors.

Its (proper) color is therefore $1 + \min\{d_i, i-1\}$. Maximization over i yields the upper bound. ■

The **minimum degree** $\delta(G)$ in G can also be used to deduce upper bounds.

Lemma. If G is k-critical graph, then $\delta(G) \geq k - 1$.

Proof. Assume in contrary that $\delta(G) < k - 1$. Let $x \in G$ be a vertex for which $d_G(x) < k - 1$.

Since G is k-critical, $G - x$ is by definition $(k - 1)$-colorable. Use any $k - 1$ colors to color properly $G - x$.

Since $d_G(x) < k - 1$, $N(x)$ consume $k - 2$ colors at most. Let us color x by one not consumed by $N(x)$.

We colored properly G with $k - 1$ colors, which contradicts with G being k-critical graph ($\chi(G) = k$). ■

Corollary. (Szekeres-Wilf 1968)

$$\chi(G) \le 1 + \max_{H \subseteq G} \delta(H)$$

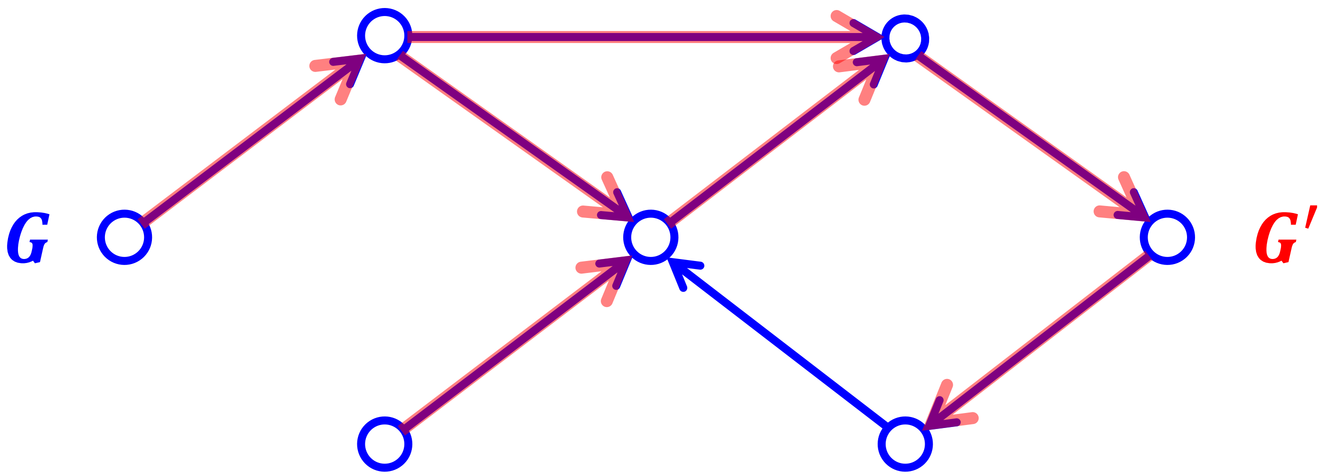
Proof. Let $k = \chi(G)$ and H' be a k-critical subgraph of G. By the above lemma $\delta(H') \ge k - 1 = \chi(G) - 1$.

There is also $\delta(H') \le \max_{H \subseteq G} \delta(H)$, yielding the desired bound. ■

Show a graph where $\chi(G) \le 1 + \max_{H \subseteq G} \delta(H) < 1 + \Delta(G)$. (homework)

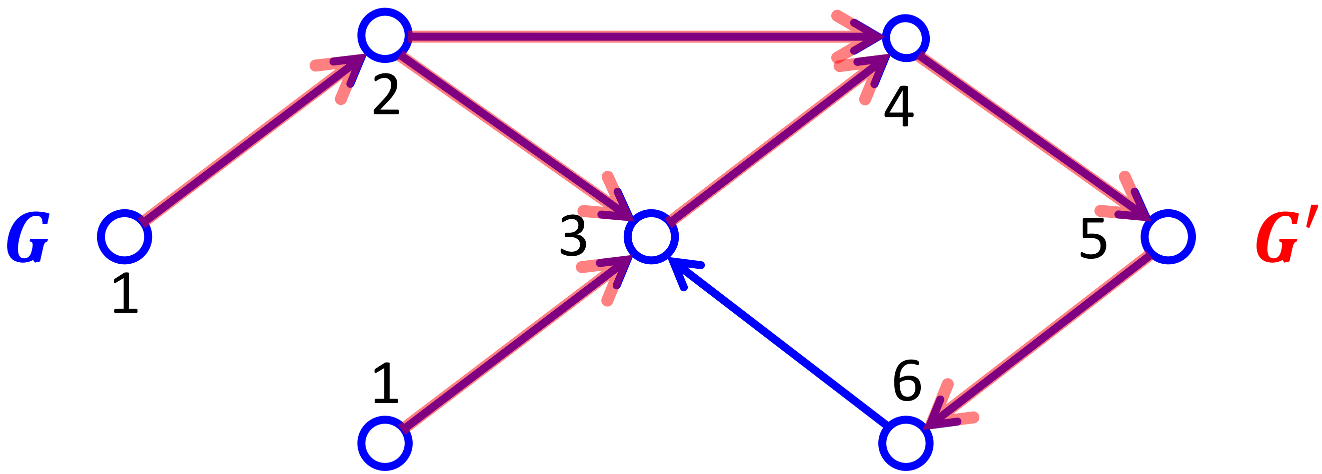
Coloring of Directed Graphs

Theorem. Let a graph G be directed with **longest path** $l(G)$, then $\chi(G) \leq 1 + l(G)$. Furthermore, there are orientations of G's edges such that equality holds.



Proof. Let G' be a maximal acyclic sub digraph of G (not necessarily a tree).

G' must have some vertices with outgoing arcs only.



Define $f(v)$ to be a coloring function assigning color $1 + l(v)$ to vertex v (longest path from an outgoing vertex).

f strictly increases along a path in G' using the colors $1 + l(G')$ on $V(G) = V(G')$.

For each edge $uv \in E(G)$ there exists a path in G' between u and v, since either there was $uv \in E(G')$ or uv is closing a cycle of G.

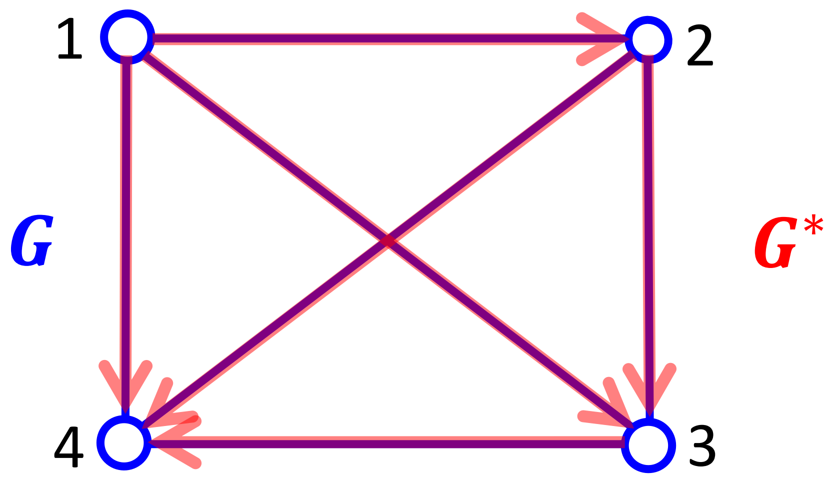
That implies $f(u) \neq f(v)$ since f increases along paths of G'. Consequently, f is a proper coloring and $\chi(G) \leq 1 + l(G)$.

To prove the existence of an orientations of G's edges satisfying $\chi(G) = 1 + l(G)$, an orientation satisfying $\chi(G) \geq 1 + l(G)$ is shown.

Let f be an optimal coloring satisfying $f(G) = \chi(G)$. We derive a digraph G^* as follows.

Each edge $uv \in E(G^*)$ is oriented $u \rightarrow v$ iff $f(u) < f(v)$. Since f is a proper coloring, this defines an orientation.

Since the color labels along paths in G^* strictly increase, and there are only $\chi(G)$ labels, there is $l(G^*) \leq \chi(G^*) - 1$, hence $\chi(G^*) = 1 + l(G^*)$. ■



Brooks' Theorem

The bound $\chi(G) \leq \Delta(G) + 1$ holds for any graph.

Brook showed that **cliques** and **odd** cycles are essentially the only graphs where $\chi(G) = \Delta(G) + 1$ holds.

Theorem. (Brooks 1941) If G is connected and other than a clique or an odd cycle, then $\chi(G) \leq \Delta(G)$.

Proof. Let G have n nodes and be connected, neither a clique, nor an odd cycle.

Let $k = \Delta(G)$ and assume $k \geq 3$, as otherwise for $k = 1$ it is single vertex, and cycle for $k = \Delta(G) = 2$.

Consider first the case where G is **not *k*-regular**.

Choose a vertex v_n for which $d(v_n) < k$ and grow a spanning tree rooted at v_n (by any search, e.g. BSF).

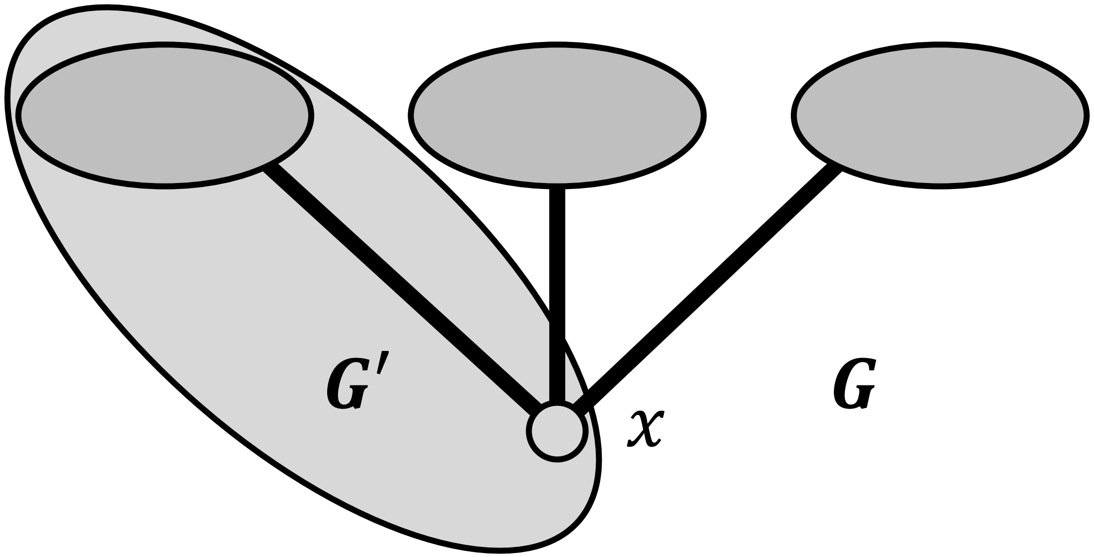
Index the vertices in **decreasing** order as they are being reached by the search, yielding the order $v_1, v_2, \ldots, v_n$.

Each vertex other than v_n has a **higher-indexed neighbor** along its path to root, hence it has **at most** $k-1$ lower-indexed neighbors.

Using the greedy coloring with the vertex decreasing order obtains **proper $\boldsymbol{k}$-coloring**.

In the remaining cases G is $\boldsymbol{k}$**-regular**. 3 cases possible.

1st case: G is **1-connected**. Let x be a cut-vertex.



Let G' be a component of $G - x$ together with x.

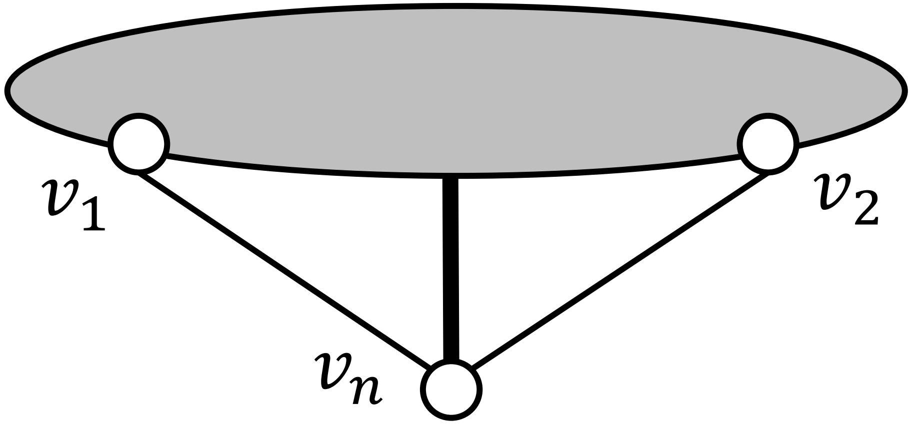
The degree of x in G' is **less than** $\boldsymbol{k}$ and a proper k-coloring is possible.

That can repeat for every components of $G - x$, yielding k-proper coloring for each (x included).

By permuting colors of the subgraphs, we can make the colorings agree on x, yielding k-proper coloring of G.

2nd case: *G* is **not 2-connected**.

Find a vertex v_n with two non adjacent neighbors v_1 and v_2 (why such exist?) whose deletion leaves a **connected** subgraph (otherwise *G* was 2-conncted).



$G - \{v_1, v_2\}$ is connected and a spanning tree rooted at v_n can be constructed (e.g. BFS).

The labels $n, \ldots, 3$ are assigned to the vertices in decreasing order as they are reached.

Starting coloring from v_1 and v_2, they use same color.

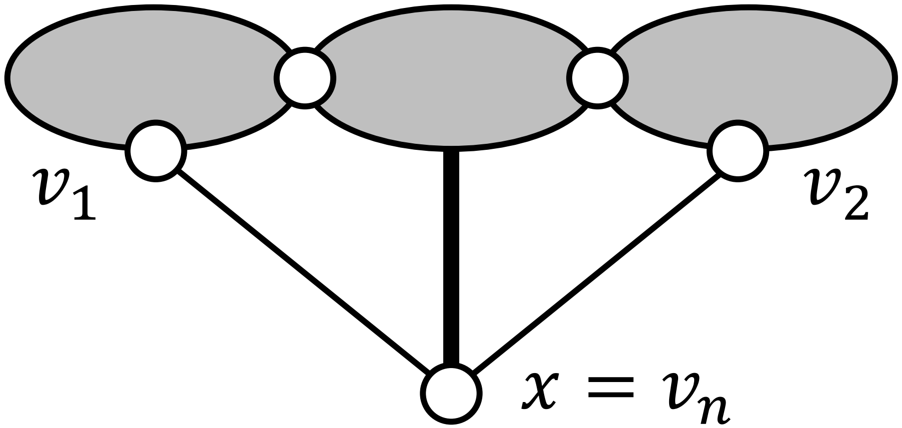
Each vertex other than v_n has at most $k - 1$ lower-indexed neighbors so k colors can be used for those.

v_n has k neighbors, of which v_1 and v_2 already used the same color. The rest neighbors used at most other $k - 2$ colors, and v_n can therefore be properly colored.

All in all, k proper coloring of G has been obtained.

3rd case: G is **2-connected**. Choose a vertex x such that vertex connectivity $\kappa(G-x)=1$.

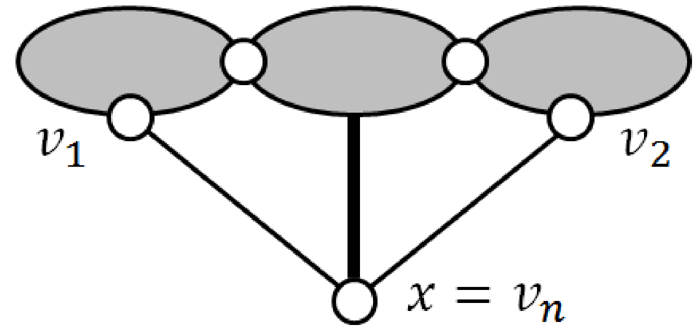
That is possible by choosing x to be one of the two disconnecting vertices.



x has a neighbor in every block of $G-x$ obtained by deleting the 2nd vertex in a cut-set, otherwise G was 1-connected rather than 2-connected.

There is no edge connecting v_1 and v_2 since they reside in different blocks.

$G-\{x, v_1, v_2\}$ is connected since blocks have no cut-vertices and v_1, v_2 are not such.



$k \geq 3$ implies $G-\{v_1, v_2\}$ is also connected.

All in all this is the same situation as the case of G not 2-connected. ■

Brooks' Theorem implies that the cliques and the odd cycles are the only $(k-1)$-regular k-critical graphs. (homework)

Example. Prove that for any graph G, there is a partition $V(G) = V_1 \cup V_2$, $(V_1 \cap V_2 = \emptyset)$ $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, such that $\chi(G[V_1]) + \chi(G[V_2]) = \chi(G)$.

Proof. Consider any coloring of G with $\chi(G)$ colors. Pick $k < \chi(G)$ color classes and denote by $V_1 \subset V$ the vertices of these color classes. Let $V_2 = V \backslash V_1$.

The above construction yields a proper coloring of V_1 by k colors and proper coloring of V_2 by $\chi(G) - k$ colors.

Consequently,

$$\chi(G[V_1]) \leq k \quad \text{and} \quad \chi(G[V_2]) \leq \chi(G) - k \ .$$

On the other hand, $G[V_1]$ cannot be colored with less than k colors.

Otherwise, together with the coloration of V_2 by $\chi(G) - k$ colors, G could be colored with less than $\chi(G)$ colors, which is impossible. Thus

$$\chi(G[V_1]) = k.$$

Similarly and symmetrically

$$\chi(G[V_2]) = \chi(G) - k.$$

In conclusion

$$\chi(G[V_1]) + \chi(G[V_2]) = \chi(G). \blacksquare$$

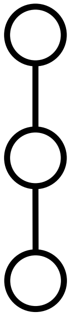
Chromatic Polynomials

We shall associate with any graph a function telling whether or not it is 4-colorable.

This study was motivated by the hope to prove the Four-Color Theorem, which by that time was a conjecture.

Let $P_G(k)$ denote the number of proper colorings of a graph G with k colors. $P_G(k)$ is called the **chromatic function** of G.

Example. $P_G(k) = k(k-1)^2$. The first vertex can be colored in k ways, while each of the other two in $k-1$ ways.



For a tree T of n vertices there is $P_T(k) = k(k-1)^{n-1}$.

For $G = K_3$ there is $P_G(k) = k(k-1)(k-2)$ and for $G = K_n$ there is $P_G(k) = k(k-1)(k-2) \dots (k-n+1)$.

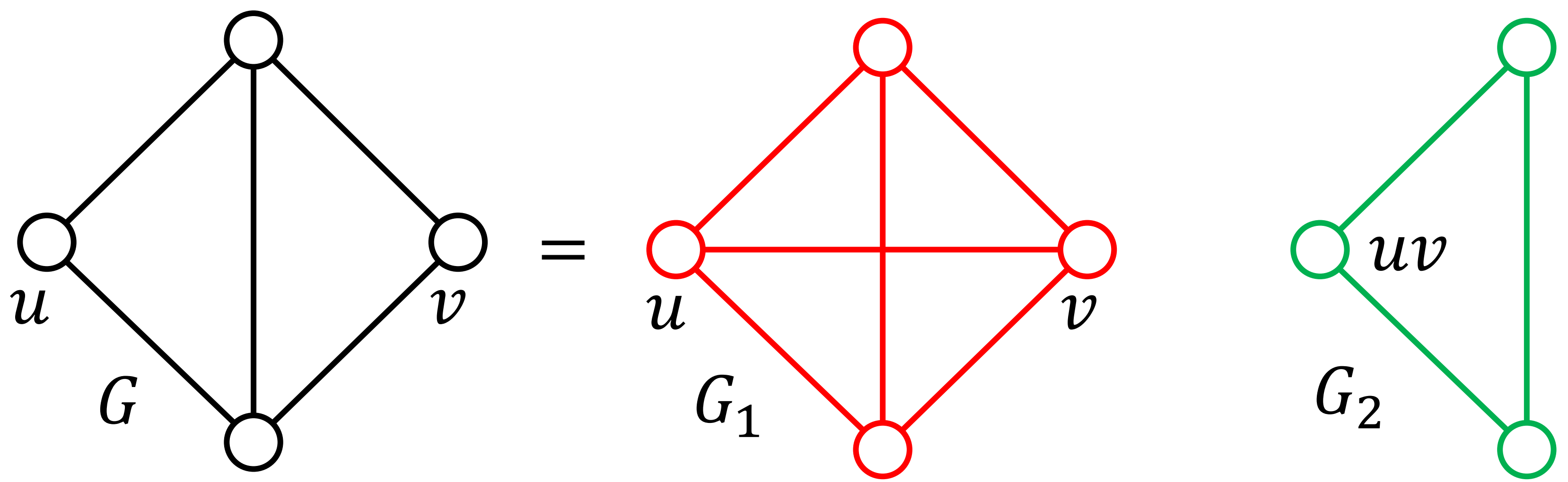
If $k < \chi(G)$ then $P_G(k) \leq 0$. For $k \geq \chi(G)$ there is $P_G(k) > 0$.

The **Four-Color Theorem** for planar graph G states that $P_G(4) > 0$.

It is difficult to compute $P_G(k)$ by inspection, but it can be systematically obtained as a **sum of chromatic functions** of complete graphs.

Theorem. Let $u, v \in V(G)$ be not adjacent, and let G_1 and G_2 be obtained from G by adding the edge uv, and by identifying u and v, respectively. Then $P_G(k) = P_{G_1}(k) + P_{G_2}(k)$.

$k(k-1)(k-2)^2$



$k(k-1)(k-2)(k-3) + k(k-1)(k-2)$

Proof. In a proper coloring of G, u and v may have either the same color or different colors.

The number of proper colorings where u and v have different colors does not change if an edge uv would exist, yielding $P_{G_1}(k)$.

Similarly, the number of proper colorings where u and v have same color does not change if u and v are merged, yielding $P_{G_2}(k)$. ■

Corollary. The chromatic function is a polynomial.

Proof. The procedure of the theorem results in two graphs. In G_1 the number of edges is increased. In G_2 the number of vertices is decreased.

The process is finite. It ends with producing complete graphs, whose chromatic functions are polynomial.

The chromatic function is therefore a finite sum of polynomials, which must be polynomial too. ■

For n-vertex graph G the degree of $P_G(k)$ is n, the coefficient of k^n is 1 and that of k^{n-1} is $|E(G)|$, the sign of the coefficients is alternating, and the free coefficient is zero. (homework)

Example. **Scheduling feasibility**. Lectures scheduling is in order, for which some time slots are given (e.g. campus is open). There is no limit on available rooms.

It is known that some lectures cannot take place in parallel (e.g. some students are registered to both).

Is scheduling feasible? How many schedules there are?

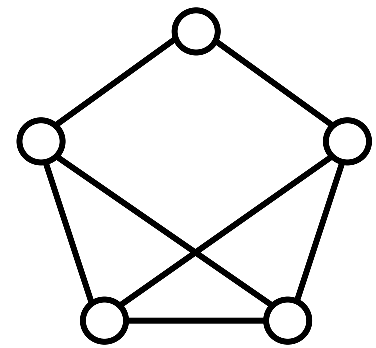
Solution. Define a graph $G(V,E)$ where $v \in V$ corresponds to a lecture, and $e(u,v) \in E$ corresponds to lectures that cannot be scheduled simultaneously.

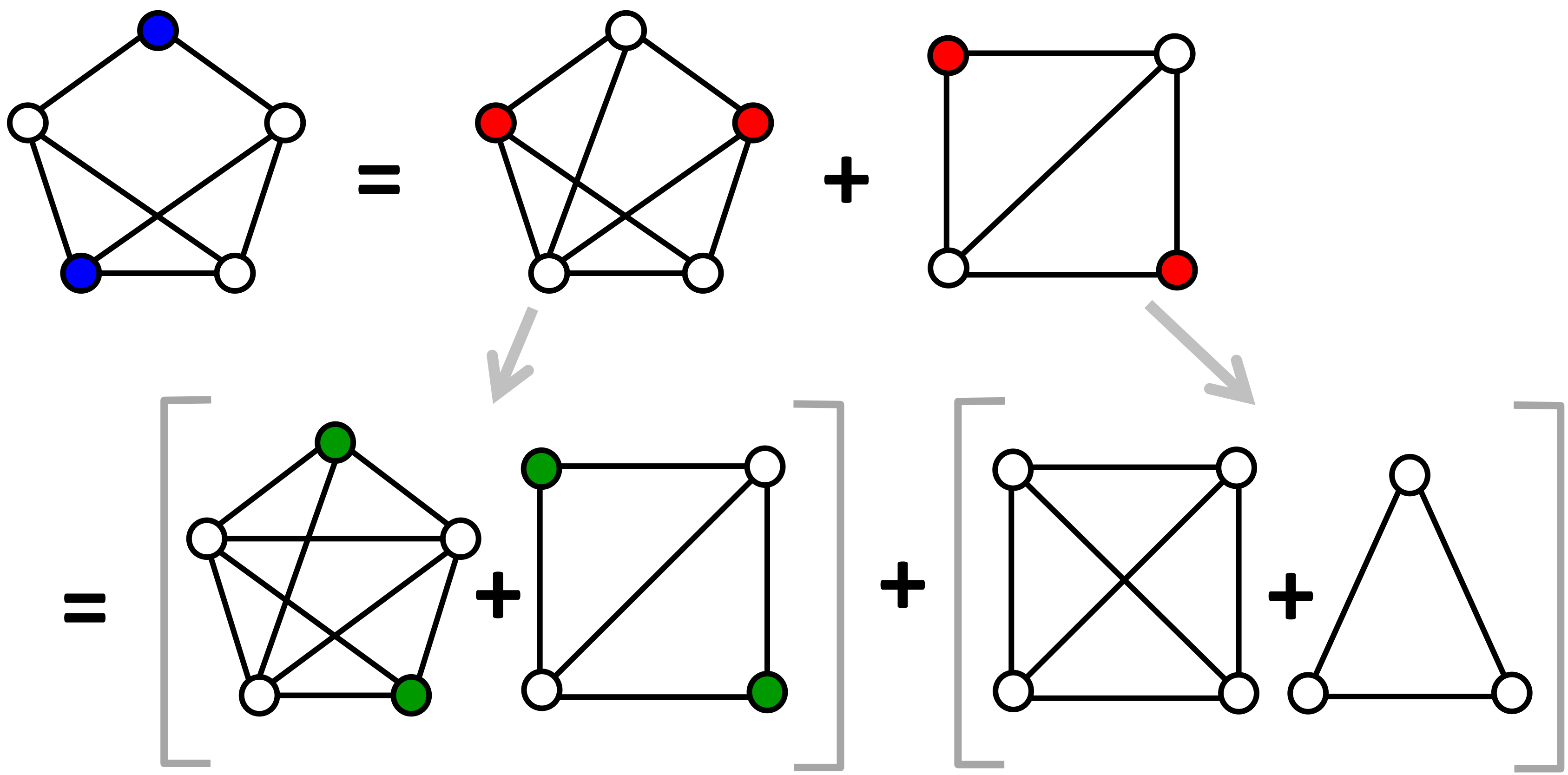
Derive the chromatic polynomial $P_G(k)$, where k is the number of time slots.

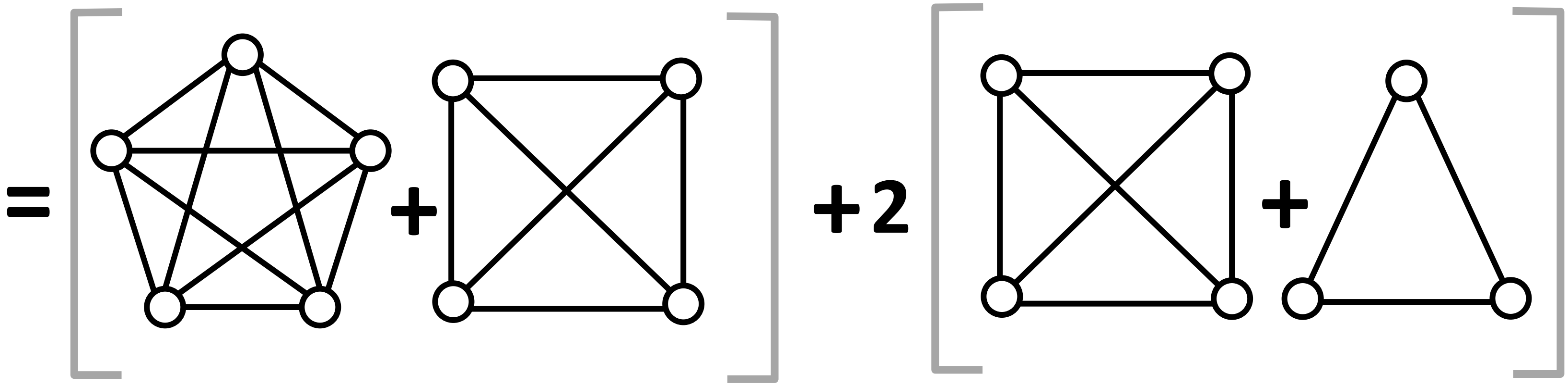
Given k, evaluation of $P_G(k)$ yields the number of distinct schedules. $P_G(k) \leq 0$ indicates non existence of feasible scheduling. ■

Example.

$$P_G(k) = k^5 - 7k^4 + ak^3 - bk^2 + ck$$





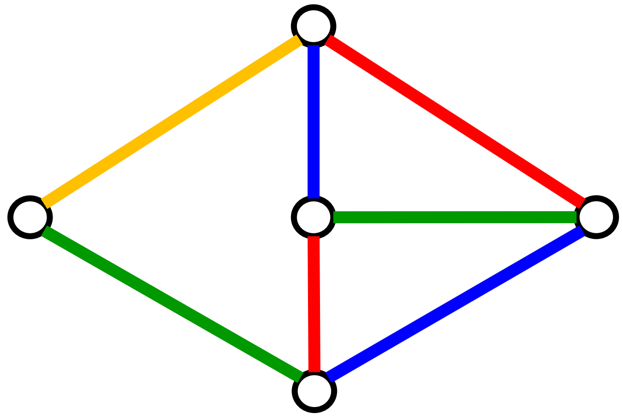


$$P_G(k) = k(k-1)(k-2)\,(k-3)\,(k-4)$$
$$+\,3k(k-1)(k-2)\,(k-3)$$
$$+\,2k(k-1)(k-2)$$
$$= k^5 - 7k^4 + 19k^3 - 23k^2 + 10k$$

Show that if G and H are disjoint (no common vertices) then$P_{G \cup H}(k) = P_G(k)P_H(k)$. (homework)

Show that if $G \cap H$ is complete then $P_{G \cup H}(k)P_{G \cap H}(k) = P_G(k)P_H(k)$. (homework)

Edge Coloring



A k-edge-coloring of a graph G is a labeling $f: E(G) \rightarrow \{1,2, \dots, k\}$.

Edge coloring partitions $E(G)$ into k sets (some possibly empty) $\{E_1, E_2, \dots, E_k\}$.

An edge coloring is proper if adjacent edges have different colors. All coloring henceforth are assumed proper.

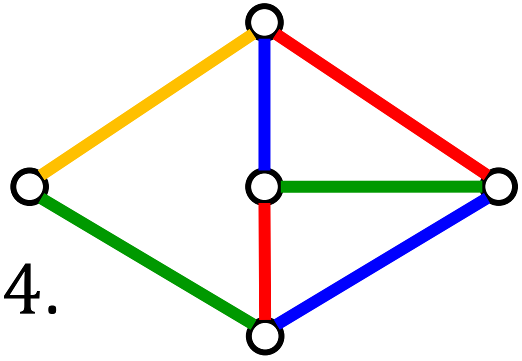
Edge coloring thus partitions $E(G)$ into k sets $\{M_1, M_2, \ldots, M_k\}$ of matchings. (Only loopless graphs admit proper edge coloring).

G is k-edge-colorable if it has k-edge-coloring.

Clearly, G is m-edge-colorable, where $m = |E(G)|$.

The edge chromatic number $\chi'(G)$ is the smallest k such that G has k-edge-coloring. G is called k-edge-chromatic.

Clearly, $\chi'(G) \geq \Delta(G)$.



Not 3-edge-colorable, hence $\chi'(G) = 4$.

Example. Timetabling. m teachers $x_1, x_2, \ldots, x_m$ and n classes $y_1, y_2, \ldots, y_n$ are given. Teacher x_i is required to teach class y_j a lessen of period p_{ij}. Schedule a complete timetable having minimum total duration.

Solution. The scheduling is represented by a **bipartite** graph $H[X, Y]$, $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$ vertices x_i and y_j are connected with p_{ij} **parallel** edges.

The minimum number of colors required for H edge-coloring ensures minimum duration.

- No schedule overlap for a teacher.
- No lesson overlap for a class. ■

Homework.

1. Show that a d-regular graph $G(V,E)$ is d-edge colorable iff E can be partitioned into perfect matchings.
2. Show by an appropriate edge coloring that $\chi'(K_{m,n}) = \max\{m,n\}$.
3. Given graph $G(V,E)$, $|V|=n$, $|E|=m$, show that $\chi'(G) \geq \lceil 2m/n \rceil$.
4. Eight schoolgirls go for a walk in pairs every day. Can they arrange their outgoing so that every girl has a different companion at every day of the week?

Edge Coloring of Bipartite Graphs

Let the subgraph H span G ($V(H) = V(G)$), and $C := \{M_1, M_2, \dots, M_k\}$ be a k-edge-coloring of H.

A color is **available** for an edge $e \in E(G) \backslash E(H)$ if it is available in its two end vertices.

If e is uncolored, any of its available colors can be assigned to extend C to a k-edge-coloring of $H + e$.

For $i \neq j$, each component of $H_{ij} \coloneqq H[M_i \cup M_j]$ is either an even cycle or a path (called ***ij*-path**). (why?)

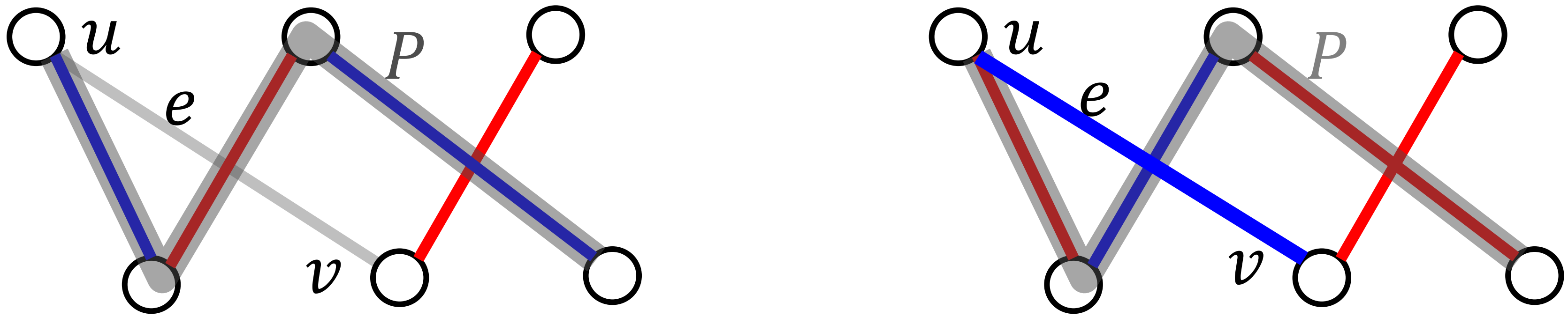
Theorem. If G is bipartite then $\chi'(G) = \Delta(G)$.

Proof. By induction on $m = |E(G)|$. Let $e = uv \in E(G)$. Assume that $H = G \backslash e$ has a Δ−edge−coloring $\{M_1, M_2, \dots, M_\Delta\}$.

If a color is available for e we are done. Otherwise, each of the Δ colors is represented either at u or at v.

Since the degrees of u and v in $G \backslash e$ are $\Delta - 1$ at most, there are colors $i \neq j$, where i is available at u and exists in v, and j is available at v and exists in u.

Consider the subgraph $H_{ij} = H[M_i \cup M_j]$. Because u has a degree one in H_{ij}, the component containing u is an ij-path P.



P cannot terminate at v. If it did, it would started from u with color j and end at v with color i, hence comprising **even** number of edges.

$P + e$ would then be an **odd** cycle in G, impossible for a bipartite graph.

Interchanging the colors of P, a new Δ-edge-colorable H is obtained, where color j is available at both u and v.

Assigning color j to e obtains a Δ-edge-coloring of G. ■

Upper Bound of $\chi'(G)$

Clearly, $\chi'(G) \geq \Delta(G)$, and for bipartite graphs there is $\chi'(G) = \Delta(G)$.

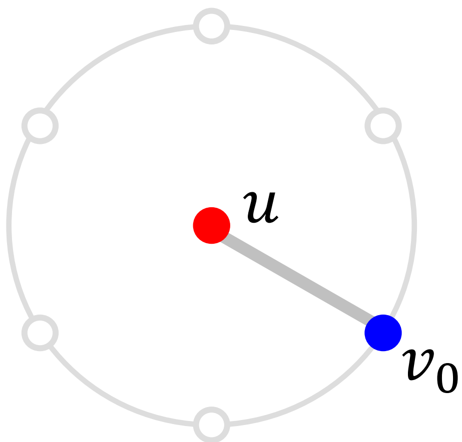
What can be said about an upper bound? Surprisingly, it is very tight.

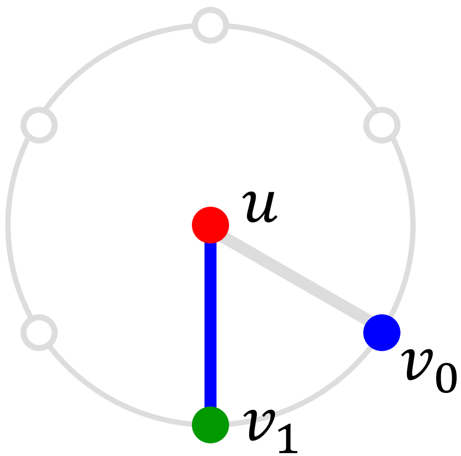
Theorem. (Vizing 1964, Gupta 1966). Let G be a simple graph (no parallel edges, loopless). Then $\chi'(G) \leq \Delta(G) + 1$.

Proof. Let G' be a proper subgraph of G, edge-colored with $\Delta(G) + 1$ colors, but uv could not be colored. We present a recoloring procedure to include uv.

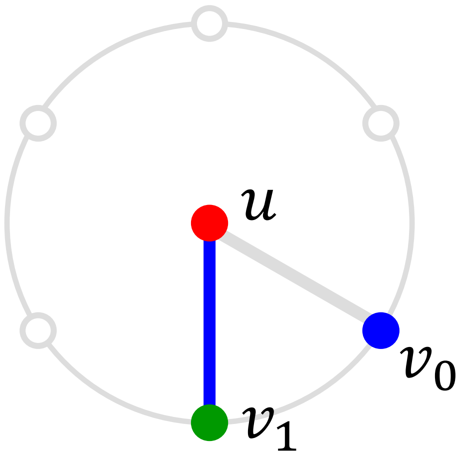
Since more than $\Delta(G)$ colors are used, every vertex has a missing color.

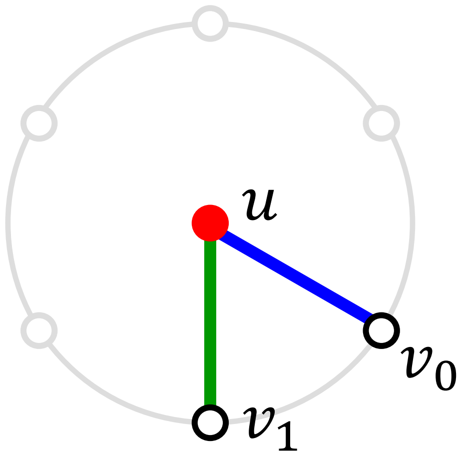
Let $\boldsymbol{a_0}$ be missing at u and $\boldsymbol{a_1}$ be missing at v_0. ($\boldsymbol{a_0}$ must be presented at v_0 and $\boldsymbol{a_1}$ at u.)





Let v_1 be a neighbor of u such that uv_1 is colored $\boldsymbol{a_1}$. Some color $\boldsymbol{a_2}$ must be missing at v_1 since $\Delta(G)+1$ colors are used.





Suppose $\boldsymbol{a_2}$ does not appear on u. We could recolor uv_1 with $\boldsymbol{a_2}$, free $\boldsymbol{a_1}$ from u, and then color uv with $\boldsymbol{a_1}$.

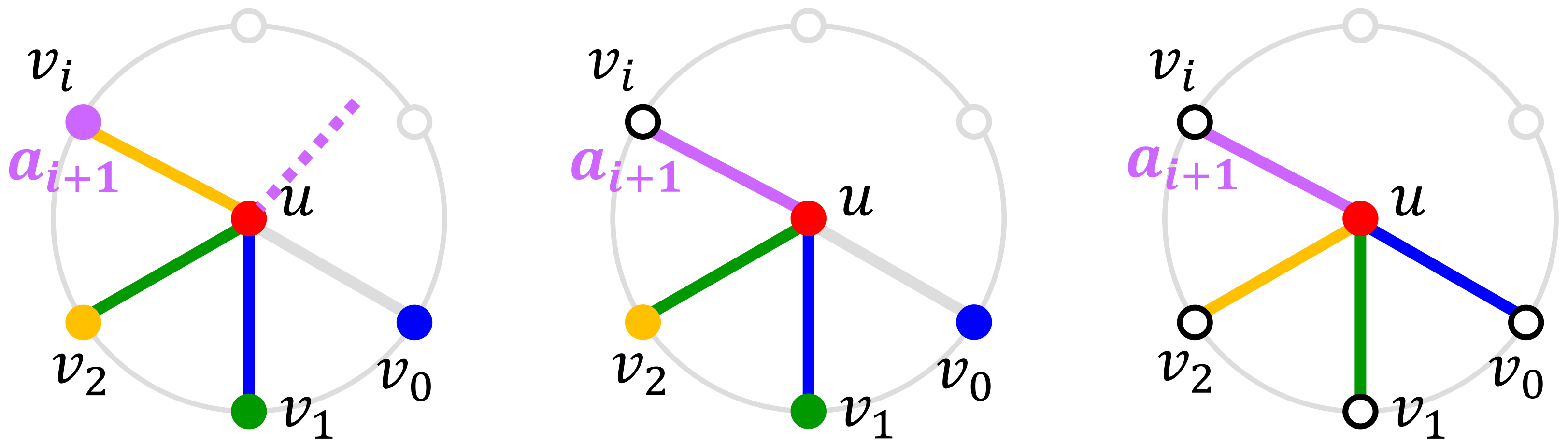
So we suppose that $\boldsymbol{a_2}$ appears on u.

The process continues for $i \geq 2$.

Finding a new color a_i that appears at u, let v_i be the neighbor of u such that the edge uv_i is colored a_i.

At v_i some color a_{i+1} must be missing.

If a_{i+1} is missing at u, we **downshift** color a_j from uv_j to uv_{j-1} for $1 \leq j \leq i$.

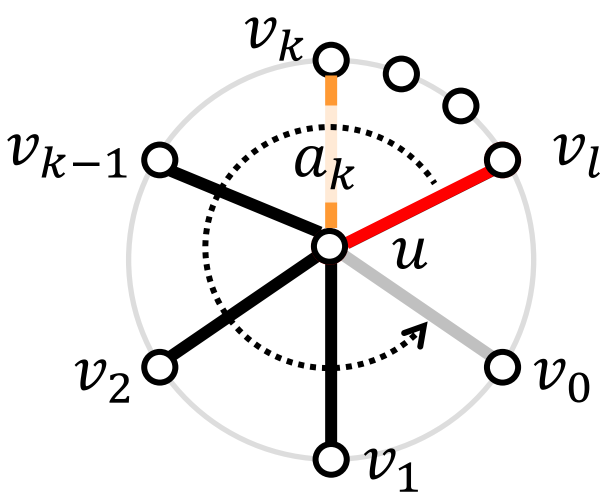


We are finished, unless a_{i+1} appears at u, in which case the process continues to v_{j+1} and a color a_{i+2}.

There are only $\Delta(G) + 1$ colors, hence the repetitive selection of a_{i+1} must eventually repeat a color.

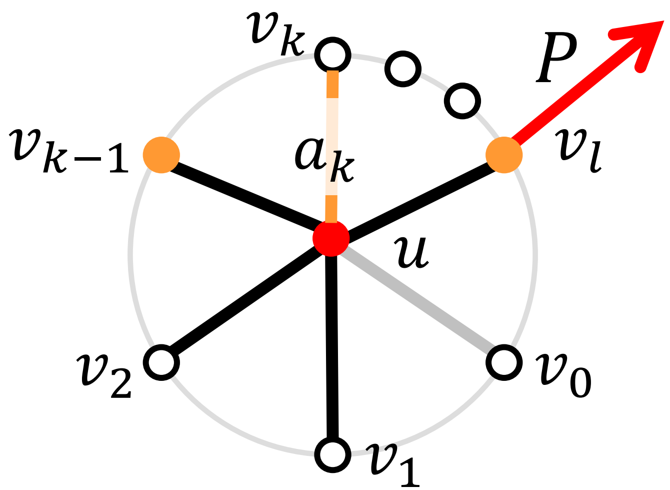
Let $a_1, \dots, a_l$ be the shortest non repetitive color list such that a_{l+1} is missing at v_l and repeats one of $a_1, \dots, a_{l-1}$.

Let $a_{l+1} = \boldsymbol{a_k}$ for some $1 \leq k \leq l-1$. This color was missing at v_{k-1} and appeared on uv_k.



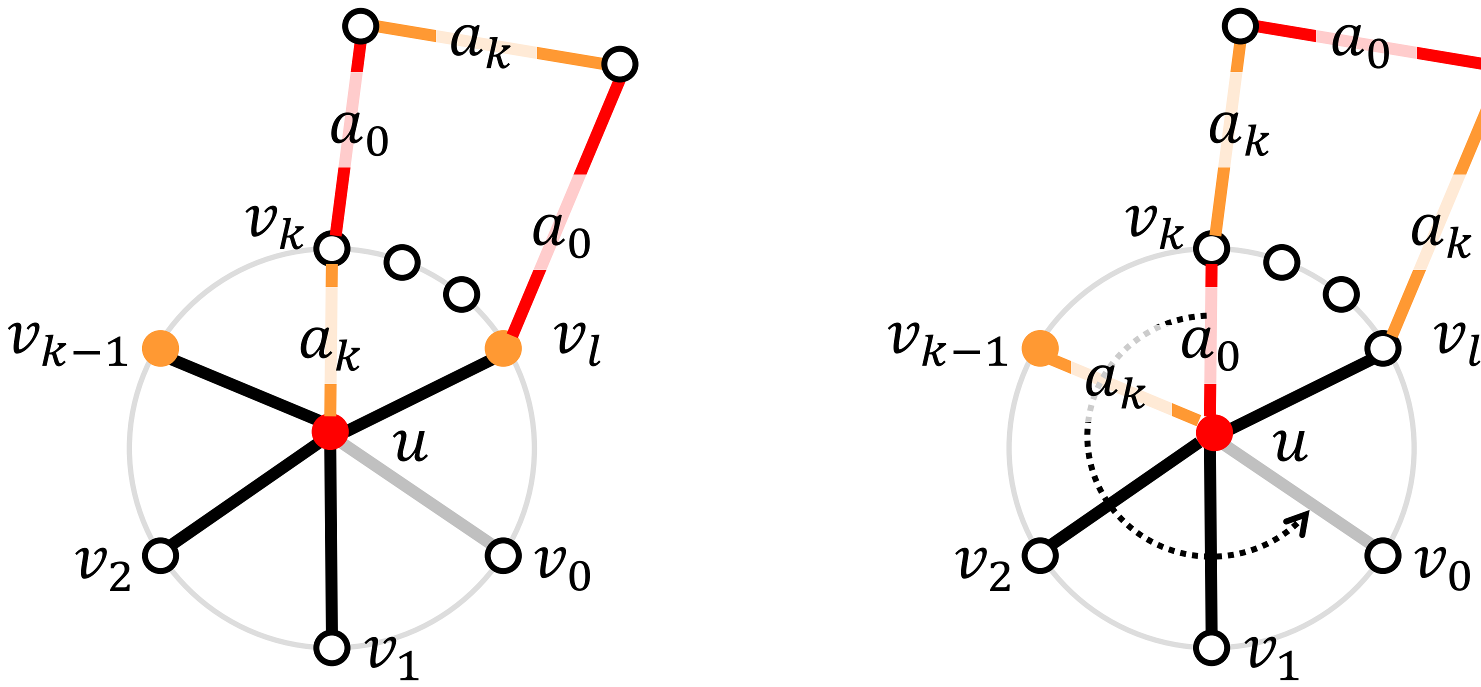
If v_l is missing $\boldsymbol{a_0}$, we use $\boldsymbol{a_0}$ on uv_l and downshift colors from v_l and to complete the augmentation.

Hence we assume that $\boldsymbol{a_0}$ appears at v_l and $\boldsymbol{a_k}$ does not.



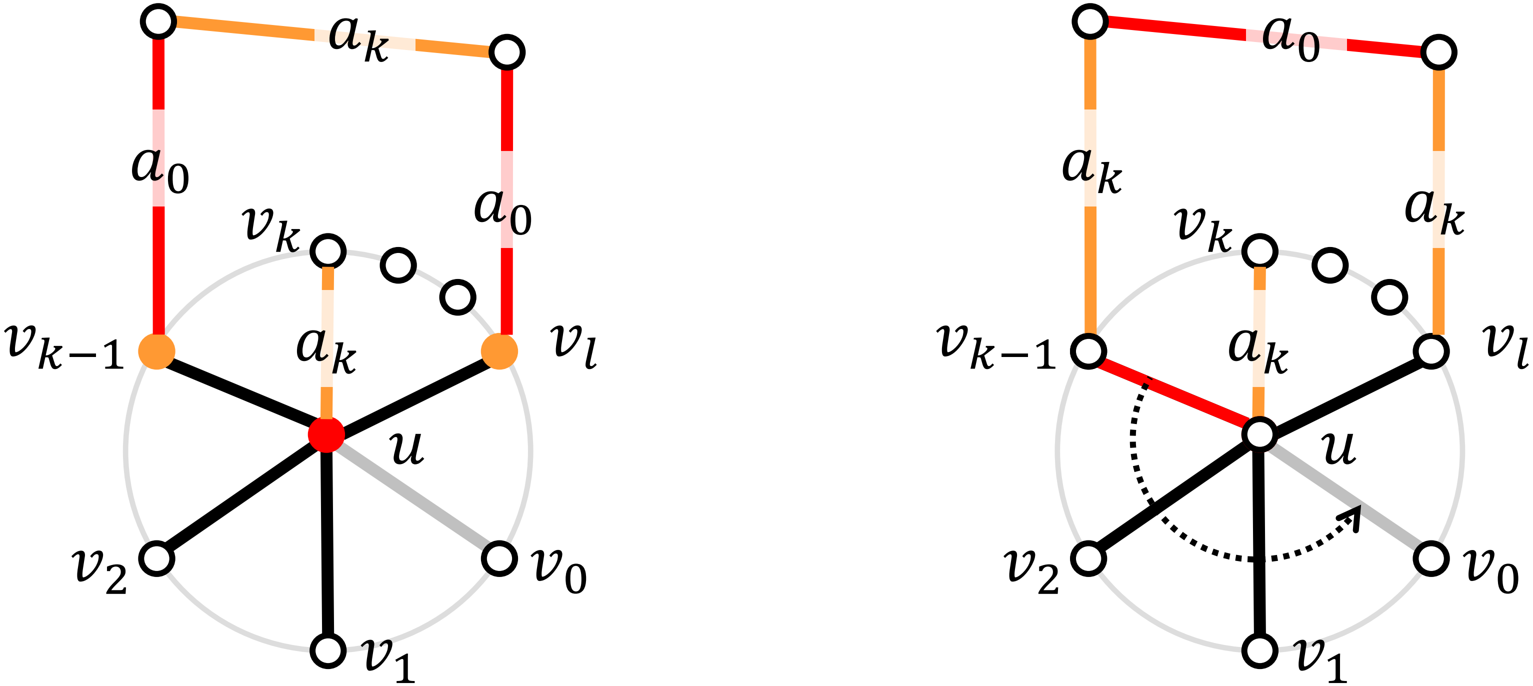
Let P be the longest alternating path of edges colored $\boldsymbol{a_0}$ and $\boldsymbol{a_k}$ that begins at v_l (with $\boldsymbol{a_0}$). P is unique. (why?)

Depending on the opposite end of P, recoloring can take place to complete the augmentation. There are three possibilities of P end: at v_k, at v_{k-1}, and elsewhere.



If P ends at v_k, it is with $\boldsymbol{a_0}$ since uv_k is colored with $\boldsymbol{a_k}$.

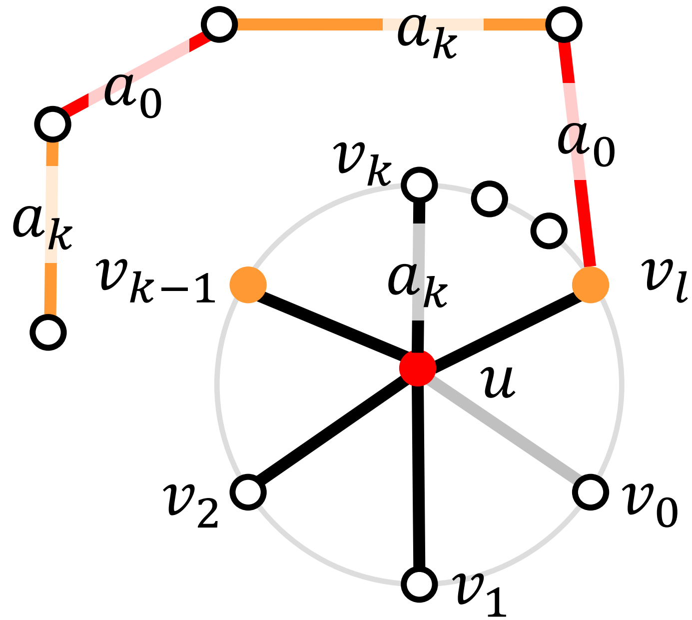
Downshifting colors from v_k, interchanging colors along P, and coloring uv_k with $\boldsymbol{a_0}$, completes the edge coloring augmentation.



If P ends at v_{k-1}, it is with $\boldsymbol{a_0}$ since $\boldsymbol{a_k}$ is missing from v_{k-1}.

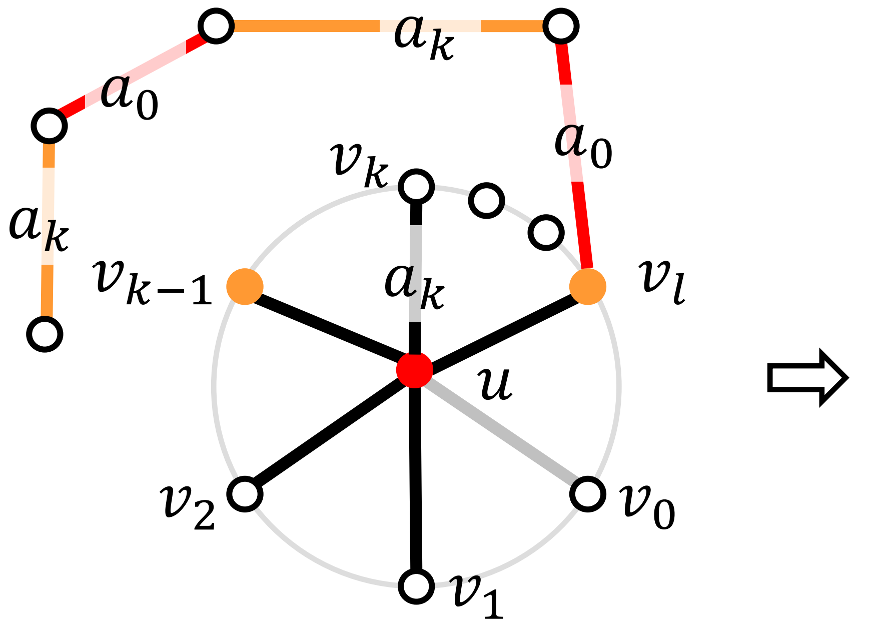
Downshifting from v_{k-1}, interchanging colors along P, and coloring uv_{k-1} with $\boldsymbol{a_0}$, completes the edge coloring augmentation.

Finally, suppose that P neither ends at v_k nor v_{k-1}, so it ends at some vertex outside $\{u, v_k, v_{k-1}\}$.



P ends with edge colored either by $\boldsymbol{a_0}$ or $\boldsymbol{a_k}$, so either of $\boldsymbol{a_0}$ or $\boldsymbol{a_k}$ is missing from the far end of P, as otherwise P would not be longest.

Notice that the vertices along P can also touch any of $N(u) - \{v_k, v_{k-1}\}$, since $\boldsymbol{a_o}$ has been assumed to present at these (otherwise coloring downshift had been possible).



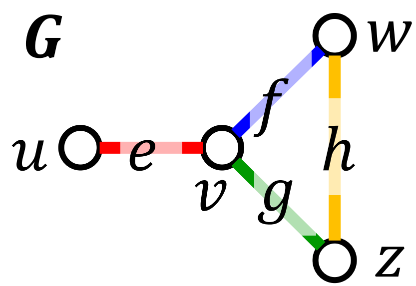
We downshift from v_l, assign color $\boldsymbol{a_o}$ to uv_l, and interchange colors along P. ■

Line Graphs

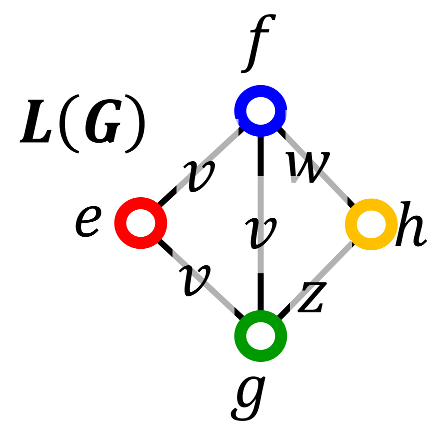
Many questions about vertices have natural analogues involving edges.

Independent sets have no pairs of adjacent vertices; matchings have no adjacent edges.

Vertex coloring partitions the vertices into independent sets; edges can be partitioned into matching.



Definition. The **line graph** $L(G)$ is defined by $V\big(L(G)\big) \equiv E(G)$ and $ef \in E\big(L(G)\big)$ if $e = uv$ and $f = vw$, where $u, v, w \in V(G)$.



Line Graphs Characterization

Theorem. (Krausz 1943) A simple graph G is the line graph of some simple graph H iff $V(G)$ has a partition into **cliques** using each vertex of G at most twice.

Proof. **Necessity** follows from the fact that the edges adjacent at a vertex of H are represented in $L(H)$ by vertices connected in a clique.

Since an edge connects two vertices, those vertices imply two cliques at most.

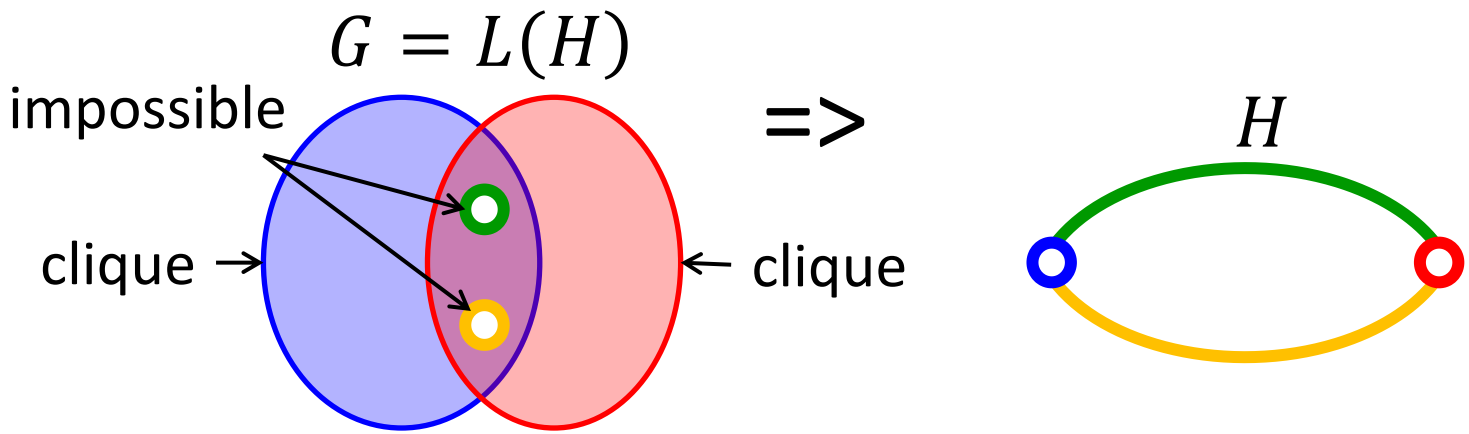
For **sufficiency**, suppose $V(G)$ has such a partition, using cliques $S_1, \dots, S_k$. We shall construct H satisfying $G = L(H)$. Assume that G has no isolated varices.

Let $v_1, \dots, v_l$ be the vertices of G that appear in a single clique of $S_1, \dots, S_k$ (if such one exist). We define a vertex in H for each set of $\boldsymbol{A} = \{S_1, \dots, S_k, v_1, \dots, v_l\}$.

Edges of H are defined such that H vertices are adjacent if the corresponding sets (cliques) of $\boldsymbol{A}$ intersect.

By its construction, each vertex of G appears in exactly two sets of $\boldsymbol{A}$.

Also, two vertices cannot both appear in two sets of $\boldsymbol{A}$, as otherwise a clique was split among the sets.



Hence there are no parallel edges in H, so it is simple, and there is one edge in H for each vertex of G.

Adjacent vertices in G appear together in some S_i and the corresponding edges of H share the vertex corresponding to S_i. Hence $G = L(H)$. ■

Krausz's theorem does not directly yield an efficient test for line graph, which the following does.

Theorem. (Bieneke 1968) A simple graph G is the line graph of some simple graph iff G does not contain any of the following subgraphs as an induced subgraph.