Cliques and Independent Sets

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Shannon Capacity

A message consisting of signals belonging to a certain finite alphabet A is transmitted over a noisy channel.

The message is a sequence of words of k chars each.

Some pairs of chars are so similar that they can be confounded by the receiver due to noise.

What is the largest number of distinct words that can be used in messages without a confusion at the receiver?

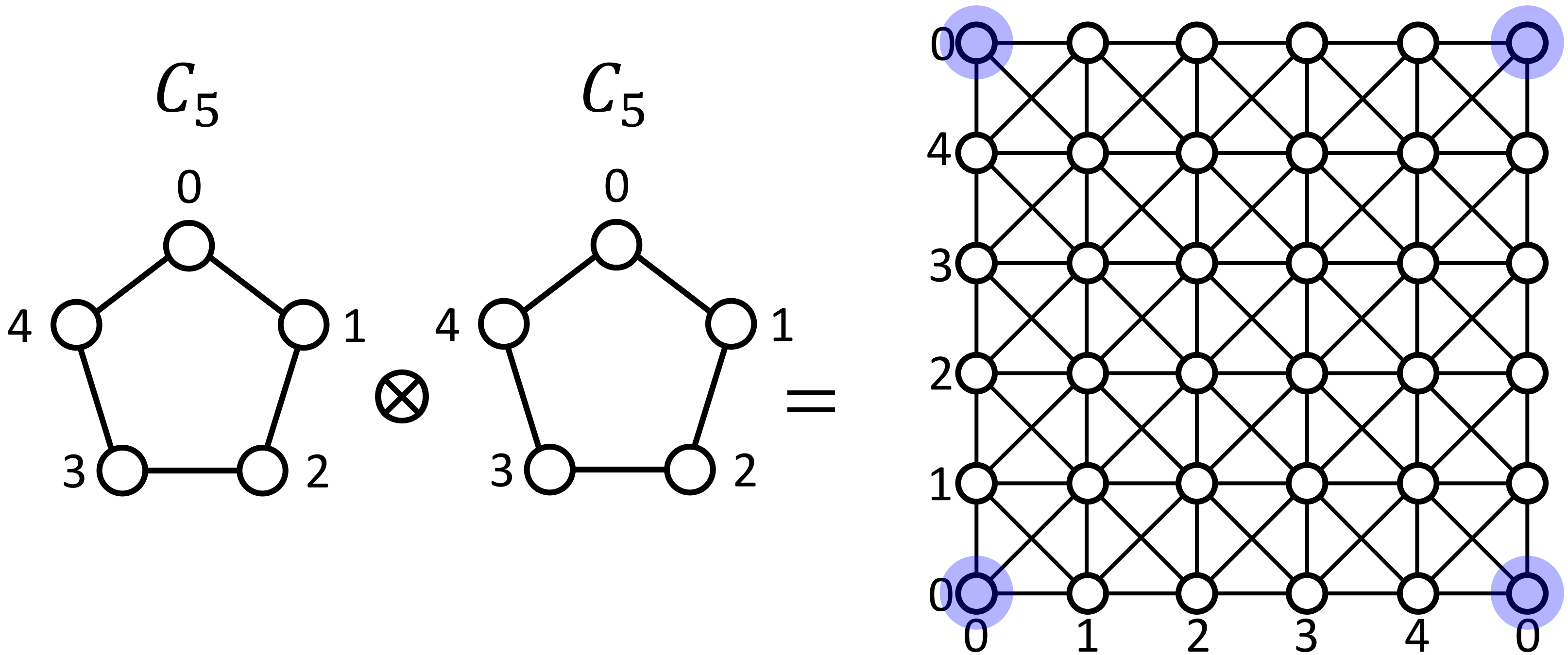
Example.

$A = \{0, 1, 2, 3, 4\}$ and $k = 2$.

If the noise results errors of type $i + 1$ and $i - 1$ (mod 5), the message 00, 12, 24, 31, 43, can be safely transmitted.

Definition. The **strong product $\boldsymbol{G \otimes H}$** of two graphs G and H is defined by the vertex set $V(G) \times V(H)$.

Two vertices ux and vy are adjacent (in $E(G \otimes H)$) iff:
$uv \in E(G)$ and $x = y$, or
$xy \in E(H)$ and $u = v$, or
$uv \in E(G)$ and $xy \in E(H)$.



The strong product is embedded on a torus.

Let G be the graph with vertex set A. An edge uv is defined if u and v represent signals that might be confused with each other.

G^k is a strong product of k copies of G, representing words of length k over A.

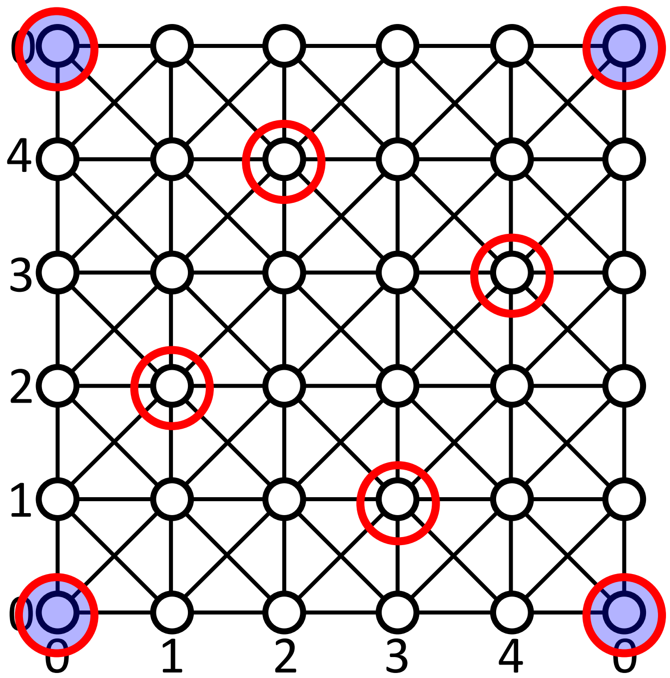
Q: What are the edges of G^k ?

A: Two distinct words $(u_1, u_2, \dots, u_k)$ and $(v_1, v_2, \dots, v_k)$ are connected with an edge iff $u_i = v_i$ or $u_i v_i \in E(G)$ for all $1 \leq i \leq k$.

Edges of G^k correspond to words that might be confused with each other.

The largest number of distinct words equals the size of maximum independent set $\alpha\left(G^k\right)$.

In this example $k = 2$ and $\alpha({C_5}^2) = 5$.



Shannon proposed in information theory (1956) the parameter

$$\theta(G) = \lim_{k \to \infty} \sqrt[k]{\alpha(G^k)},$$

known as **Shannon capacity of G**, as a measure of the capacity for error-free transmission over a noise channel whose associated graph is G.