Neutron Transport Theory Lecture Note (7)
-Multigroup Diffusion Theory-

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6. Multigroup Diffusion Theory

6.1 Derivation of the multigroup equation from energy-dependent diffusion theory

Energy-dependent diffusion equation

$$\frac{1}{v}\frac{\partial \phi}{\partial t} - \nabla D \nabla \phi + \Sigma_t \phi(\mathbf{r}, E, t) = \int_0^\infty dE' \Sigma_S(E' \to E)\phi(\mathbf{r}, E', t) + \chi(E)\int_0^\infty dE' \nu\Sigma_f(E')\phi(\mathbf{r}, E', t) + S_{ext}(\mathbf{r}, E, t) \quad \cdots (1)$$

$\chi(E)$: fission neutron spectrum

S_{ext} : external source

We will discretize the neutron energy into energy intervals on groups by breaking the neutron energy range into G energy groups.

Integrating Eq.(1) over the gth energy group characterized by energies $E_g < E < E_{g-1}$

$$\frac{\partial}{\partial t}\int_{E_g}^{E_{g-1}} dE \frac{1}{v}\phi - \nabla \cdot \int_{E_g}^{E_{g-1}} dE D \nabla \phi + \int_{E_g}^{E_{g-1}} dE \Sigma_t \phi = \int_{E_g}^{E_{g-1}} dE \int_0^\infty dE' \Sigma_s(E' \to E)\phi(\mathbf{r}, E', t) + \int_{E_g}^{E_{g-1}} dE S \quad \cdots (2)$$

We will make some formal definitions.

Neutron flux in group g $\quad \phi_g(\mathbf{r}, t) \equiv \int_{E_g}^{E_{g-1}} dE \phi(\mathbf{r}, E, t) \quad \cdots (3)$

Total cross section for group g $\quad \Sigma_{tg} \equiv \frac{1}{\phi_g}\int_{E_g}^{E_{g-1}} dE \Sigma_t(E)\phi(\mathbf{r}, E, t) \quad \cdots (4)$

Neutron speed characterizing group g $\quad \frac{1}{v_g} \equiv \frac{1}{\phi_g}\int_{E_g}^{E_{g-1}} dE \frac{1}{v}\phi(\mathbf{r}, E, t) \quad \cdots (5)$

By integration of scattering term over E'

$$\int_{E_g}^{E_{g-1}} dE \int_0^{\infty} dE' \, \Sigma_S(E' \to E)\phi(\mathbf{r}, E', t)$$

$$= \sum_{g'=1}^{G} \int_{E_g}^{E_{g-1}} dE \int_{E_{g'}}^{E_{g'-1}} dE' \Sigma_S(E' \to E)\phi(\mathbf{r}, E', t) \qquad \cdots (6)$$

G : Total number of energy groups

We define the group-transfer cross section as

$$\Sigma_{sg',g} \equiv \frac{1}{\phi_{g'}} \int_{E_g}^{E_{g-1}} dE \int_{E_{g'}}^{E_{g'-1}} dE' \Sigma_S(E' \to E)\phi(\mathbf{r}, E', t) \qquad \cdots (7)$$

By similar procedure for the fission term

$$\int_{E_g}^{E_{g-1}} dE S_f(\mathbf{r}, E, t) = \int_{E_g}^{E_{g-1}} dE \chi(E) \left[\sum_{g'=1}^{G} \int_{E_{g'}}^{E_{g'-1}} dE' \nu(E') \Sigma_f(E') \phi(\mathbf{r}, E', t) \right] \qquad \cdots (8)$$

Then we define the fission cross section for group g' as

$$\nu_{g'} \Sigma_{fg'} \equiv \frac{1}{\phi_{g'}} \int_{E_{g'}}^{E_{g'-1}} dE' \nu(E') \Sigma_f(E') \phi(\mathbf{r}, E', t) \qquad \cdots (9)$$

while defining

$$\chi_g \equiv \int_{E_g}^{E_{g-1}} dE \chi(E) \qquad \cdots (10)$$

If energy-dependent form of Fick's law is postulated that

$$\mathbf{J}(\mathbf{r}, E) = -D(E) \nabla \phi(\mathbf{r}, E) \qquad \cdots (11)$$

then the group current can be expressed by integrating this equation over group. The result is

$$\mathbf{J}_g(\mathbf{r}) = -D_g \nabla \phi_g(\mathbf{r}) \qquad \cdots (12)$$

where the group diffusion coefficient D_g is defined by

$$D_g \equiv \frac{\int_{E_g}^{E_{g-1}} dE D(E) \nabla \phi(\mathbf{r}, E)}{\nabla \phi_g(\mathbf{r})} \qquad \cdots (13)$$

If we use these definitions to rewrite Eq.(1), we arrive at the multigroup diffusion equations

$$\frac{1}{v_g}\frac{\partial \phi_g}{\partial t} - \nabla D_g \nabla \phi_g + \Sigma_{tg}\phi_g(\mathbf{r}, t)$$

$$= \sum_{g'=1}^{G} \Sigma_{sg',g}\phi_{g'} + \chi_g \sum_{g'=1}^{G} \nu_{g'}\Sigma_{fg'}\phi_{g'} + S_g \qquad \cdots (14)$$

v_g, D_g, Σ_{tg}, $\Sigma_{sg',g}$, χ_g, ν_g, Σ_{fg}, S_g are called group constants.

6.2 Two group diffusion theory

In the case of two energy groups, the groups are chosen to characterize fast neutrons and thermal neutrons.

The cut off energy for the thermal group is chosen sufficiently high such that up scattering out of the thermal group can be ignored. (～1eV)

Neutron flux for each group

$$\text{Fast group} \quad \phi_1(\mathbf{r}, t) = \int_{E_1}^{E_0} dE\, \phi(\mathbf{r}, E, t) \equiv \text{fast flux}$$

$$\text{Thermal group} \quad \phi_2(\mathbf{r}, t) = \int_{E_2}^{E_1} dE\, \phi(\mathbf{r}, E, t) \equiv \text{thermal flux} \qquad \cdots (15)$$

E_0 : upper limit of the fast group (10～20MeV)

E_1 : boundary between the fast group and the thermal group (～1eV)

E_2 : lower limit of the thermal group (～0eV)

Since essentially all fission neutrons are born in the fast group

$$\chi_1 = \int_{E_1}^{E_0} dE\chi(E) = 1, \qquad \chi_2 = \int_{E_2}^{E_1} dE\chi(E) = 0 \qquad \cdots (16)$$

$\chi(E)$: fission spectrum

The fission sources S_{f1}, S_{f2}

$$S_{f1} = \nu_1\Sigma_{f1}\phi_1 + \nu_2\Sigma_{f2}\phi_2 \quad \text{(fast)}$$

$$S_{f2} = 0 \quad \text{(thermal)} \qquad \cdots (17)$$

We define the removal cross section Σ_{Rg} as

$$\Sigma_{Rg} \equiv \Sigma_{tg} - \Sigma_{sg,g} \qquad \cdots (18)$$

Since no up scattering out of the thermal group,

$$\int_{E_2}^{E_1} dE\Sigma_s(E' \to E) = \Sigma_s(E'), \qquad E_2 \le E' \le E_1 \qquad \cdots(19)$$

Hence

$$\Sigma_{s2,2} = \frac{1}{\phi_2}\int_{E_2}^{E_1} dE \int_{E_2}^{E_1} dE'\Sigma_s(E' \to E)\phi(\mathbf{r}, E')$$
$$= \frac{1}{\phi_2}\int_{E_2}^{E_1} dE'\,\Sigma_s(E')\phi(\mathbf{r}, E') = \Sigma_{s2} \qquad \cdots(20)$$

Thus the removal cross section for the thermal group is

$$\Sigma_{R2} = \Sigma_{t2} - \Sigma_{s22} = \Sigma_{t2}$$
$$- \Sigma_{s2} = \Sigma_{a2} \qquad \cdots(21)$$

Other group constants are defined as in the previous section.

The group constants are calculated by first performing a fine spectrum calculation for the group of interest and then averaging the appropriate cross section data over this spectrum. (cf. Eq.(3)～(13))

By using these group constants, two-group diffusion equations for critical reactor are

$$-\nabla D_1\nabla\phi_1 + \Sigma_{R1}\phi_1 = \nu_1\Sigma_{f1}\phi_1 + \nu_2\Sigma_{f2}\phi_2$$
$$-\nabla D_2\nabla\phi_2 + \Sigma_{a2}\phi_2 = \Sigma_{s12}\phi_1 \qquad \cdots(22)$$

6.3 Modified one-group diffusion theory

If we can ignore thermal neutron leakage from the core,

$$\frac{-D_2\nabla^2\phi_2}{\Sigma_{a2}\phi_2} = \frac{D_2B^2}{\Sigma_{a2}} = {L_2}^2B^2 \ll 1 \qquad \cdots(23)$$

Ex. large sized light water reactor

from Eq.(22)

$$\phi_2(\mathbf{r}) = \frac{\Sigma_{s12}}{\Sigma_{a2}}\phi_1(\mathbf{r}) \qquad \cdots(24)$$

Hence we find the modified one-group diffusion model from Eq.(22)

$$-\nabla D_1\nabla\phi_1 + \Sigma_{R1}\phi_1 = \nu_1\Sigma_{f1}\phi_1 + \nu_2\Sigma_{f2}\left(\frac{\Sigma_{s12}}{\Sigma_{a2}}\right)\phi_1 \qquad \cdots(25)$$