Neutorn Transport Theory Lecture Note (3)
-Neutron Transport Equation -

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3. Neutron Transport Equation
3.1 Neutron Density and Flux

Definition

Angular neutron density $n(\mathbf{r},E,\hat{\Omega},t)d^3rdEd\hat{\Omega}$

$$\equiv \begin{bmatrix} \text{Expected number of neutrons in } d^3\mathbf{r} \text{ about } \mathbf{r}, \\ \text{energy } dE \text{ about } E, \text{moving in direction } \hat{\Omega} \\ \text{in solid angle } d\hat{\Omega} \text{ at time } t \end{bmatrix}$$

Angular neutron flux

$$\varphi(\mathbf{r},E,\hat{\Omega},t) \equiv vn(\mathbf{r},E,\hat{\Omega},t)$$

where v is the neutron speed

$$v = \sqrt{\frac{2E}{m}}$$

Angular current density

$$\mathbf{j}(\mathbf{r},E,\hat{\Omega},t) \equiv v\hat{\Omega}n(\mathbf{r},E,\hat{\Omega},t) = \hat{\Omega}\varphi(\mathbf{r},E,\hat{\Omega},t)$$

Neutron density

$$N(\mathbf{r},E,t) = \int_{4\pi} d\hat{\Omega}n(r,E,\hat{\Omega},t)$$

and

$$N(\mathbf{r},t) = \int_0^\infty dEN(\mathbf{r},E,t) = \int_0^\infty dE\int_{4\pi} d\hat{\Omega}n(\mathbf{r},E,\hat{\Omega},t)$$

Neutron flux

$$\phi(\mathbf{r},E,t) = \int_{4\pi} d\hat{\Omega}\varphi(\mathbf{r},E,\hat{\Omega},t)$$

and

$$\phi(\mathbf{r},t) = \int_0^\infty dE\phi(\mathbf{r},E,t) = \int_0^\infty dE\int_{4\pi} d\hat{\Omega}\varphi(\mathbf{r},E,\hat{\Omega},t)$$

unit 〔$cm^{-2}\cdot s^{-1}$〕

Neutron current density

$$\mathbf{J}(\mathbf{r},E,t) \equiv \int_{4\pi} d\hat{\Omega}\mathbf{j}(\mathbf{r},E,\hat{\Omega},t)$$

and

$$\mathbf{J}(\mathbf{r},t) \equiv \int_0^\infty dE\mathbf{J}(\mathbf{r},E,t) = \int_0^\infty dE\int_{4\pi} d\hat{\Omega}\mathbf{j}(\mathbf{r},E,\hat{\Omega},t)$$

3.2 Neutron Transport Equation

Consider arbitrary volume V. The number of neutrons in V with energy E in dE and traveling in a direction $\hat{\Omega}$ in $d\hat{\Omega}$ within this volume is,

$$\left[\int_V n(\mathbf{r},E,\hat{\Omega},t)d^3r\right]dEd\hat{\Omega} \ .$$

The time rate of change of this number is given by a balance relation.

$$\frac{\partial}{\partial t}\left[\int_V n(\mathbf{r},E,\hat{\Omega},t)d^3r\right]dEd\hat{\Omega} = (\text{gain in } V) - (\text{loss from } V)$$

If we assume that the arbitrary volume is chosen not to depend on time,

$$\frac{\partial}{\partial t}\left[\int_V n(\mathbf{r},E,\hat{\Omega},t)d^3r\right]dEd\hat{\Omega} = \left[\int_V \frac{\partial n}{\partial t}d^3r\right]dEd\hat{\Omega}$$

Gain mechanisms :

(1) Any neutron source in V (e.g. fission)

(2) Neutrons streaming into V through the surface of the V

(3) Neutrons of different E', $\hat{\Omega}'$ suffering a scattering collision in V that changes E', $\hat{\Omega}'$ into the E, $\hat{\Omega}$ of interest

Loss mechanism :

(4) Neutrons leaking out through the surface of the V

(5) Neutrons suffering a collision (absorption and scattering)

Neutron balance relationship in a unit volume

(Neutron transport equation)

$$\frac{\partial n}{\partial t} + v\hat{\Omega}\cdot\nabla n + v\Sigma_t n(\mathbf{r}, E, \hat{\Omega}, t)$$

$$= \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' v' \Sigma_s\left(E' \to E, \hat{\Omega}' \to \hat{\Omega}\right) n\left(\mathbf{r}, E', \hat{\Omega}', t\right) + S(\mathbf{r}, E, \hat{\Omega}, t) \quad ①$$

where,

$\Sigma_s(E' \to E, \hat{\Omega}' \to \hat{\Omega}) \equiv N\sigma_s(E' \to E, \hat{\Omega}' \to \hat{\Omega})$

$\sigma_s\left(E' \to E, \hat{\Omega}' \to \hat{\Omega}\right)$: Cross section that characterize scattering from an incident energy E', direction $\hat{\Omega}'$ to find energy E and direction $\hat{\Omega}$. (double differential scattering cross section)

$$S(\mathbf{r}, E, \hat{\Omega}, t) d^3r dE d\hat{\Omega} \equiv \begin{bmatrix} \text{rate of source neutrons appearing} \\ \text{in } d^3r \text{ about } r, dE \text{ about } E, \text{and } d\hat{\Omega} \\ \text{about } \hat{\Omega} \end{bmatrix}$$

(neutron source)

3.3 Simplifications to the Neutron Transport Equation

- The one-speed approximation : Assuming that one can characterize the neutrons by a single energy or speed.

  One-speed neutron transport equation

$$\frac{1}{v}\frac{\partial \varphi}{\partial t} + \hat{\Omega}\cdot\nabla\varphi + \Sigma_t\varphi\left(\mathbf{r}, \hat{\Omega}, t\right) = \int_{4\pi} d\hat{\Omega}' \Sigma_s\left(\hat{\Omega}' \to \hat{\Omega}\right)\varphi\left(\mathbf{r}, \hat{\Omega}', t\right) + S(\mathbf{r}, \hat{\Omega}, t)$$

- Isotropic neutron sources

$$S\left(\mathbf{r}, \hat{\Omega}, t\right) = \frac{1}{4\pi} S(\mathbf{r}, t)$$

- Isotropic scattering

$$\Sigma_s\left(\hat{\Omega}' \to \hat{\Omega}\right) = \frac{1}{4\pi}\Sigma_s$$

The one-speed neutron transport equation with isotropic scattering and sources

$$\frac{1}{v}\frac{\partial \varphi}{\partial t} + \hat{\Omega}\cdot\nabla\varphi + \Sigma_t\varphi\left(\mathbf{r}, \hat{\Omega}, t\right) = \frac{\Sigma_s}{4\pi}\int_{4\pi} d\hat{\Omega}'\varphi\left(\mathbf{r}, \hat{\Omega}', t\right) + \frac{S(\mathbf{r}, t)}{4\pi}$$

In a steady state

$$\hat{\Omega} \cdot \nabla\varphi + \Sigma_t \varphi(\mathbf{r}, \hat{\Omega}) = \frac{\Sigma_s}{4\pi} \int_{4\pi} d\,\hat{\Omega}' \varphi(\mathbf{r}, \hat{\Omega}') + \frac{S(\mathbf{r})}{4\pi}$$

In a one-dimensional plane geometry

$$\mu \frac{\partial\varphi}{\partial x} + \Sigma_t \varphi(x, \mu) = \frac{\Sigma_s}{2} \int_{-1}^{+1} d\mu' \varphi(x, \mu') + \frac{S(x)}{2}$$

where

$$\mu = \cos\theta$$