Corollary 9.3 Let $\boldsymbol{x}^*$ be a minimum of a max-type function $f(\boldsymbol{x})$ over the set Q as (16). If $f_i \in \mathcal{S}^1_\mu(Q)$ $(i = 1, 2, \ldots, m)$, then

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) + \frac{\mu}{2}\|\boldsymbol{x} - \boldsymbol{x}^*\|_2^2, \quad \forall \boldsymbol{x} \in Q.$$

Proof:
From Lemma 9.1 and Theorem 9.2, we have for $\forall \boldsymbol{x} \in Q$,

$$\begin{aligned} f(\boldsymbol{x}) &\geq f(\boldsymbol{x}^*; \boldsymbol{x}) + \frac{\mu}{2}\|\boldsymbol{x} - \boldsymbol{x}^*\|_2^2 \\ &\geq f(\boldsymbol{x}^*; \boldsymbol{x}^*) + \frac{\mu}{2}\|\boldsymbol{x} - \boldsymbol{x}^*\|_2^2 = f(\boldsymbol{x}^*) + \frac{\mu}{2}\|\boldsymbol{x} - \boldsymbol{x}^*\|_2^2. \end{aligned}$$

∎

Lemma 9.4 Let $f_i \in \mathcal{S}^1_\mu(Q)$ for $(i = 1, 2, \ldots, m)$ with $\mu > 0$ and Q be a closed convex set. Then there is a unique solution $\boldsymbol{x}^*$ for the problem (17).

Proof:
Left for exercise. ∎

Definition 9.5 Let $f_i \in \mathcal{C}^1(Q)$ $(i = 1, 2, \ldots, m)$, Q a closed convex set, $\bar{\boldsymbol{x}} \in Q$, and $\gamma > 0$. Denote by

$$\begin{aligned} \boldsymbol{x}_f(\bar{\boldsymbol{x}}; \gamma) &:= \arg\min_{\boldsymbol{y} \in Q} \left[f(\bar{\boldsymbol{x}}; \boldsymbol{y}) + \frac{\gamma}{2}\|\boldsymbol{y} - \bar{\boldsymbol{x}}\|_2^2 \right], \\ \boldsymbol{g}_f(\bar{\boldsymbol{x}}; \gamma) &:= \gamma(\bar{\boldsymbol{x}} - \boldsymbol{x}_f(\bar{\boldsymbol{x}}; \gamma)). \end{aligned}$$

We call $\boldsymbol{g}_f(\bar{\boldsymbol{x}}; \gamma)$ the *gradient mapping of max-type function* f *on* Q. Observe that due to Lemma 9.4, $\boldsymbol{x}_f(\bar{\boldsymbol{x}}; \gamma)$ exists and it is uniquely defined.

Theorem 9.6 Let $f_i \in \mathcal{S}^{1,1}_{\mu,L}(Q)$ $(i = 1, 2, \ldots, m)$, $\gamma \geq L$, $\gamma > 0$, Q a closed convex set, and $\bar{\boldsymbol{x}} \in Q$. Then

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}_f(\bar{\boldsymbol{x}}; \gamma)) + \langle \boldsymbol{g}_f(\bar{\boldsymbol{x}}; \gamma), \boldsymbol{x} - \bar{\boldsymbol{x}} \rangle + \frac{1}{2\gamma}\|\boldsymbol{g}_f(\bar{\boldsymbol{x}}; \gamma)\|_2^2 + \frac{\mu}{2}\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|_2^2, \quad \forall \boldsymbol{x} \in Q.$$

Proof: Let us use the following notation: $\boldsymbol{x}_f := \boldsymbol{x}_f(\bar{\boldsymbol{x}}; \gamma)$ and $\boldsymbol{g}_f := \boldsymbol{g}_f(\bar{\boldsymbol{x}}; \gamma)$.

From Lemma 9.1 and Corollary 9.3 (taking $f(\boldsymbol{x})$ in there as $f(\bar{\boldsymbol{x}}; \boldsymbol{x}) + \frac{\gamma}{2}\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|_2^2$), we have $\forall \boldsymbol{x} \in Q$,

$$\begin{aligned} f(\boldsymbol{x}) - \frac{\mu}{2}\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|_2^2 &\geq f(\bar{\boldsymbol{x}}; \boldsymbol{x}) \\ &= f(\bar{\boldsymbol{x}}; \boldsymbol{x}) + \frac{\gamma}{2}\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|_2^2 - \frac{\gamma}{2}\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|_2^2 \\ &\geq f(\bar{\boldsymbol{x}}; \boldsymbol{x}_f) + \frac{\gamma}{2}\|\boldsymbol{x}_f - \bar{\boldsymbol{x}}\|_2^2 + \frac{\gamma}{2}\|\boldsymbol{x} - \boldsymbol{x}_f\|_2^2 - \frac{\gamma}{2}\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|_2^2 \\ &= f(\bar{\boldsymbol{x}}; \boldsymbol{x}_f) + \frac{\gamma}{2}\|\boldsymbol{x}_f - \bar{\boldsymbol{x}}\|_2^2 + \frac{\gamma}{2}\langle \bar{\boldsymbol{x}} - \boldsymbol{x}_f, 2\boldsymbol{x} - \boldsymbol{x}_f - \bar{\boldsymbol{x}} \rangle \\ &= f(\bar{\boldsymbol{x}}; \boldsymbol{x}_f) + \frac{\gamma}{2}\|\boldsymbol{x}_f - \bar{\boldsymbol{x}}\|_2^2 + \frac{\gamma}{2}\langle \bar{\boldsymbol{x}} - \boldsymbol{x}_f, 2(\boldsymbol{x} - \bar{\boldsymbol{x}}) + \bar{\boldsymbol{x}} - \boldsymbol{x}_f \rangle \\ &= f(\bar{\boldsymbol{x}}; \boldsymbol{x}_f) + \frac{\gamma}{2}\|\boldsymbol{x}_f - \bar{\boldsymbol{x}}\|_2^2 + \langle \boldsymbol{g}_f, \boldsymbol{x} - \bar{\boldsymbol{x}} \rangle + \frac{1}{2\gamma}\|\boldsymbol{g}_f\|_2^2 \\ &\geq f(\boldsymbol{x}_f) + \langle \boldsymbol{g}_f, \boldsymbol{x} - \bar{\boldsymbol{x}} \rangle + \frac{1}{2\gamma}\|\boldsymbol{g}_f\|_2^2, \end{aligned}$$

where the last inequality is due to the fact that $\gamma \geq L$. ∎

Now, we are ready to define our estimated sequence. Assume that $f_i \in \mathcal{S}^{1,1}_{\mu,L}(Q)$ $(i = 1, 2, \ldots, m)$ possible with $\mu = 0$ (which means that $f_i \in \mathcal{F}^{1,1}_L(Q)$), $\boldsymbol{x}_0 \in Q$, and $\gamma_0 > 0$. Define

$$\begin{aligned}
\phi_0(\boldsymbol{x}) &:= f(\boldsymbol{x}_0) + \frac{\gamma_0}{2}\|\boldsymbol{x} - \boldsymbol{x}_0\|_2^2, \\
\phi_{k+1}(\boldsymbol{x}) &:= (1-\alpha_k)\phi_k(\boldsymbol{x}) + \alpha_k \left[f(\boldsymbol{x}_f(\boldsymbol{y}_k; L)) + \frac{1}{2L}\|\boldsymbol{g}_f(\boldsymbol{y}_k; L)\|_2^2 + \langle \boldsymbol{g}_f(\boldsymbol{y}_k; L), \boldsymbol{x} - \boldsymbol{y}_k \rangle \right. \\
&\quad \left. + \frac{\mu}{2}\|\boldsymbol{x} - \boldsymbol{y}_k\|_2^2 \right],
\end{aligned}$$

for the sequences $\{\alpha_k\}_{k=0}^{\infty}$ and $\{\boldsymbol{y}_k\}_{k=0}^{\infty}$ which will be defined later.

Similarly to the previous subsection, we can prove that $\{\phi_k(\boldsymbol{x})\}_{k=0}^{\infty}$ can be written in the form

$$\phi_k(\boldsymbol{x}) = \phi_k^* + \frac{\gamma_k}{2}\|\boldsymbol{x} - \boldsymbol{v}_k\|_2^2$$

for $\phi_0^* = f(\boldsymbol{x}_0)$, $\boldsymbol{v}_0 = \boldsymbol{x}_0$:

$$\begin{aligned}
\gamma_{k+1} &= (1-\alpha_k)\gamma_k + \alpha_k \mu \\
\boldsymbol{v}_{k+1} &= \frac{1}{\gamma_{k+1}}[(1-\alpha_k)\gamma_k \boldsymbol{v}_k + \alpha_k \mu \boldsymbol{y}_k - \alpha_k \boldsymbol{g}_f(\boldsymbol{y}_k; L)], \\
\phi_{k+1}^* &= (1-\alpha_k)\phi_k^* + \alpha_k f(\boldsymbol{x}_f(\boldsymbol{y}_k; L)) + \left(\frac{\alpha_k}{2L} - \frac{\alpha_k^2}{2\gamma_{k+1}} \right) \|\boldsymbol{g}_f(\boldsymbol{y}_k; L)\|_2^2 \\
&\quad + \frac{\alpha_k(1-\alpha_k)\gamma_k}{\gamma_{k+1}} \left(\frac{\mu}{2}\|\boldsymbol{y}_k - \boldsymbol{v}_k\|_2^2 + \langle \boldsymbol{g}_f(\boldsymbol{y}_k; L), \boldsymbol{v}_k - \boldsymbol{y}_k \rangle \right).
\end{aligned}$$

Now, $\phi_0^* \geq f(\boldsymbol{x}_0)$. Assuming that $\phi_k^* \geq f(\boldsymbol{x}_k)$,

$$\begin{aligned}
\phi_{k+1}^* &\geq (1-\alpha_k) f(\boldsymbol{x}_k) + \alpha_k f(\boldsymbol{x}_f(\boldsymbol{y}_k; L)) + \left(\frac{\alpha_k}{2L} - \frac{\alpha_k^2}{2\gamma_{k+1}} \right) \|\boldsymbol{g}_f(\boldsymbol{y}_k; L)\|_2^2 \\
&\quad + \frac{\alpha_k(1-\alpha_k)\gamma_k}{\gamma_{k+1}} \langle \boldsymbol{g}_f(\boldsymbol{y}_k; L), \boldsymbol{v}_k - \boldsymbol{y}_k \rangle \\
&\geq f(\boldsymbol{x}_f(\boldsymbol{y}_k; L)) + \left(\frac{1}{2L} - \frac{\alpha_k^2}{2\gamma_{k+1}} \right) \|\boldsymbol{g}_f(\boldsymbol{y}_k; L)\|_2^2 \\
&\quad + (1-\alpha_k) \left\langle \boldsymbol{g}_f(\boldsymbol{y}_k; L), \frac{\alpha_k \gamma_k}{\gamma_{k+1}}(\boldsymbol{v}_k - \boldsymbol{y}_k) + \boldsymbol{x}_k - \boldsymbol{y}_k \right\rangle + \frac{(1-\alpha_k)\mu}{2}\|\boldsymbol{x}_k - \boldsymbol{y}_k\|_2^2,
\end{aligned}$$

where the last inequality follows from Theorem 9.6.

Therefore, if we choose

$$\begin{aligned}
\boldsymbol{x}_{k+1} &= \boldsymbol{x}_f(\boldsymbol{y}_k; L), \\
L\alpha_k^2 &= (1-\alpha_k)\gamma_k + \alpha_k \mu, \\
\gamma_{k+1} &:= L\alpha_k^2, \\
\boldsymbol{y}_k &= \frac{1}{\gamma_k + \alpha_k \mu}(\alpha_k \gamma_k \boldsymbol{v}_k + \gamma_{k+1} \boldsymbol{x}_k),
\end{aligned}$$

we obtain $\phi_{k+1}^* \geq f(\boldsymbol{x}_{k+1})$ as desired.

Hereafter, we assume that $L > \mu$ to exclude the trivial case $L = \mu$ with finished in one iteration.

Constant Step Scheme for the Optimal Gradient Method for the Min-Max Problem	
Step 0:	Choose $\boldsymbol{x}_0 \in Q$, $\alpha_0 \in (0,1)$ such that $\frac{\alpha_0(\alpha_0 L-\mu)}{1-\alpha_0} > 0$, $\mu \leq \frac{\alpha_0(\alpha_0 L-\mu)}{1-\alpha_0} \leq L$, set $\boldsymbol{y}_0 := \boldsymbol{x}_0$, $k := 0$.
Step 1:	Compute $f_i(\boldsymbol{y}_k)$ and $\boldsymbol{\nabla f}_i(\boldsymbol{y}_k)$ $(i = 1,2,\ldots,m)$.
Step 2:	Set $\boldsymbol{x}_{k+1} := \boldsymbol{x}_f(\boldsymbol{y}_k; L) := \arg\min_{\boldsymbol{x}\in Q} \left[\max_{i=1,2,\ldots,m} f_i(\boldsymbol{y}_k) + \langle \boldsymbol{\nabla f}_i(\boldsymbol{y}_k), \boldsymbol{x} - \boldsymbol{y}_k \rangle + \frac{\alpha_k(\alpha_k L-\mu)}{2(1-\alpha_k)} \|\boldsymbol{x} - \boldsymbol{y}_k\|_2^2 \right]$.
Step 3:	Compute $\alpha_{k+1} \in (0,1)$ from the equation $\alpha_{k+1}^2 = (1-\alpha_{k+1})\alpha_k^2 + \frac{\mu}{L}\alpha_{k+1}$.
Step 4:	Set $\beta_k := \frac{\alpha_k(1-\alpha_k)}{\alpha_k^2+\alpha_{k+1}}$.
Step 5:	Set $\boldsymbol{y}_{k+1} := \boldsymbol{x}_{k+1} + \beta_k(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k)$, $k := k+1$ and go to Step 1.

The rate of converge of this method is exactly the same as Theorem 8.6 for $\gamma_0 := \alpha_0(\alpha_0 L - \mu)/(1-\alpha_0)$, but we need to solve a convex program in Step 2 for each iteration, and it can turn the method computationally expensive.

9.1 Exercises

1. Prove Lemma 9.4.