Integrability Conditions

Let $\Omega(u,v)$ and $\Lambda(u,v)$ be $n\times n$-matrix valued C^∞-maps defined on a domain $U\subset\mathbb{R}^2$. In this section, we consider an initial value problem of a system of linear partial differential equations

$$\frac{\partial X}{\partial u}=X\Omega,\qquad \frac{\partial X}{\partial v}=X\Lambda,\qquad X(u_0,v_0)=X_0, \tag{2.1}$$

where $(u_0,v_0)\in U$ is a fixed point, X is an $n\times n$-matrix valued unknown, and $X_0\in \mathrm{M}_n(\mathbb{R})$.

Proposition 2.1. *If a matrix-valued C^∞-function $X(u,v)$ defined on $U\subset\mathbb{R}^2$ satisfies (2.1) with $X_0\in \mathrm{GL}(n,\mathbb{R})$, then $X(u,v)\in\mathrm{GL}(n,\mathbb{R})$ for all $(u,v)\in U$. In addition, if Ω and Λ are skew-symmetric and $X_0\in\mathrm{SO}(n)$, then $X\in\mathrm{SO}(n)$ holds on U.*

Proof. Take a smooth path $\gamma\colon[0,1]\to U$ joining (u_0,v_0) and (u,v), and write $\gamma(t)=(u(t),v(t))$[4]. Setting $\widetilde{X}(t):=X\circ\gamma(t)=X\bigl(u(t),v(t)\bigr)$, (2.1) implies

$$\frac{d\widetilde{X}}{dt}=\widetilde{X}\left(\frac{du}{dt}\Omega+\frac{dv}{dt}\Lambda\right),\qquad \widetilde{X}(0)=X_0.$$

Hence, by Proposition 1.3, $\det\widetilde{X}(1)\neq 0$. The latter half of the statement follows from Proposition 1.4. □

Lemma 2.2. *If a matrix-valued C^∞ function $X\colon U\to\mathrm{GL}(n,\mathbb{R})$ satisfies (2.1), it holds that*

$$\Omega_v-\Lambda_u=\Omega\Lambda-\Lambda\Omega. \tag{2.2}$$

Proof. Differentiating the first (resp. second) equation of (2.1) by v (resp. u), we have

$$\begin{aligned} X_{uv}&=X_v\Omega+X\Omega_v=X(\Lambda\Omega+\Omega_v),\\ X_{vu}&=X_u\Lambda+X\Lambda_u=X(\Omega\Lambda+\Lambda_u).\end{aligned}$$

These two matrices coincide Since X is of class C^∞. Hence we have the conclusion. □

The equality (2.2) is called the *integrability condition* or *compatibility condition* of (2.1).

Frobenius' theorem In this section, we shall prove the following

Theorem 2.3. *Let $\Omega(u,v)$ and $\Lambda(u,v)$ be $n\times n$-matrix valued C^∞-functions defined on a simply connected domain $U\subset\mathbb{R}^2$*

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[4]Since U is connected, there exists a *continuous* path $\gamma\colon[0,1]\to U$ joining (u_0,v_0) and (u,v). Then one can find a smooth curve $\tilde\gamma$ joining these points as follows: For each $t\in[0,1]$, there exists a positive number $\rho_t>0$ such that $B_{\rho_t}(\gamma(t))\subset U$. Since $\gamma([0,1])$ is compact, there exists a finite sequence $0=t_0<t_1<\dots<t_N=1$ such that $\gamma([0,1])=\cup_{j=0}^N B_{\rho_{t_j}}(\gamma(t_j))$, where $B_\varepsilon(p)$ denotes a disk of radius ε centered at p. Choose $p_j\in B_{\rho_{t_{j-1}}}(\gamma(t_{j-1}))\cap B_{\rho_{t_j}}(\gamma(t_j))$ $(j=1,\dots,N)$. Then the polygonal line with vertices $\{\gamma(0),p_1,\dots,p_N,\gamma(1)\}$ lies on U and a piecewise linear path joining $\gamma(0)=(u_0,v_0)$ and $\gamma(1)=(u,v)$. Modifying such a path at vertices, we have a smooth path joining $\gamma(0)$ and $\gamma(1)$ (cf. see [2-1, Appendix B-5]).

satisfying (2.2). *Then for each* $(u_0, v_0) \in U$ *and* $X_0 \in \mathrm{M}_n(\mathbb{R})$, *there exists the unique* $n \times n$*-matrix valued function* $X : U \to \mathrm{M}_n(\mathbb{R})$ (2.1). *Moreover,*

- *if* $X_0 \in \mathrm{GL}(n, \mathbb{R})$, $X(u, v) \in \mathrm{GL}(n, \mathbb{R})$ *holds on* U,
- *if* $\operatorname{tr} \Omega = \operatorname{tr} \Lambda = 0$ *holds on* U *and* $X_0 \in \mathrm{SL}(n, \mathbb{R})$, $X(u, v) \in \mathrm{SL}(n, \mathbb{R})$ *holds on* U,
- *if* Ω *and* Λ *are skew-symmetric matrices, and* $X_0 \in \mathrm{SO}(n)$, $X(u, v) \in \mathrm{SO}(n)$ *holds on* U.

To prove Theorem 2.3, it is sufficient to show for the case $U = \mathbb{R}^2$. In fact, by Lemma 2.4 and Fact 2.5 below, we can replace U with $\mathbb{R}^2$ by an appropriate coordinate change.

Lemma 2.4. *Let* $V \ni (\xi, \eta) \mapsto (u, v) \in U$ *be a diffeomorphism between domains* V, $U \subset \mathbb{R}^2$, *and let* $\Omega = \Omega(u, v)$ *and* $\Lambda = \Lambda(u, v)$ *be matrix-valued functions on* U. *Set*

$$\begin{aligned} \widetilde{\Omega}(\xi, \eta) &:= \Omega\bigl(u(\xi, \eta), v(\xi, \eta)\bigr)\frac{\partial u}{\partial \xi} + \Lambda\bigl(u(\xi, \eta), v(\xi, \eta)\bigr)\frac{\partial v}{\partial \xi}, \\ \widetilde{\Lambda}(\xi, \eta) &:= \Omega\bigl(u(\xi, \eta), v(\xi, \eta)\bigr)\frac{\partial u}{\partial \eta} + \Lambda\bigl(u(\xi, \eta), v(\xi, \eta)\bigr)\frac{\partial v}{\partial \eta}. \end{aligned} \tag{2.3}$$

If a matrix-valued function $X : U \to \mathrm{M}_n(\mathbb{R})$ *satisfies* (2.1), $\widetilde{X}(\xi, \eta) = X(u(\xi, \eta), v(\xi, \eta))$ *satisfies*

$$\frac{\partial \widetilde{X}}{\partial \xi} = \widetilde{X}\widetilde{\Omega}, \quad \frac{\partial \widetilde{X}}{\partial \eta} = \widetilde{X}\widetilde{\Lambda}, \quad \widetilde{X}(\xi_0, \eta_0) = X_0, \tag{2.4}$$

where $\bigl(u(\xi_0, \eta_0), v(\xi_0, \eta_0)\bigr) = (u_0, v_0)$. *Moreover, the integrability condition* (2.2) *of* (2.1) *is equivalent to that of* (2.4).

Proof. The equation (2.1) can be considered as a equality of 1-forms

$$dX = X\Theta, \qquad \Theta := \Omega\, du + \Lambda\, dv,$$

which does not depend on a choice of coordinate systems. If we write

$$\Theta = \Omega\, du + \Lambda\, dv = \widetilde{\Omega}\, d\xi + \widetilde{\Lambda}\, d\eta,$$

Ω, Λ, $\widetilde{\Omega}$ and $\widetilde{\Lambda}$ satisfy (2.3). Here, the integrability condition can be rewritten as

$$d\Theta + \Theta \wedge \Theta = O,$$

which is an equality of 2-forms. This does not depend on coordinates, the conclusion follows. □

Fact 2.5. *A simply connected domain in* $\mathbb{R}^2$ *is diffeomorphic to* $\mathbb{R}^2$.

In fact, the Riemann mapping theorem yields the fact above[5].

Proof of Theorem 2.3. By Lemma 2.4 and Fact 2.5, we may assume $U = \mathbb{R}^2$, $(u_0, v_0) = (0, 0)$ without loss of generality.

<u>*Existence:*</u> By the fundamental theorem of linear ordinary differential equations (Corollary 1.7), there exists the unique C^∞-map $F : \mathbb{R} \to \mathrm{M}_n(\mathbb{R})$ such that

$$\frac{dF}{du}(u) = F(u)\Omega(u, 0) \qquad F(0) = X_0.$$

[5]Identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, a simply connected domain of $U = \mathbb{R}^2$ is conformally equivalent to the unit disc $D := \{z \in \mathbb{C} \mid |z| < 1\}$ or $\mathbb{C}$, because of the Riemann mapping theorem (cf. [2-3]). Though D and $\mathbb{C}$ are not conformally equivalent, D and $\mathbb{R}^2$ are diffeomorphic. Then any simply connected domain is diffeomorphic to $\mathbb{R}^2$.

For each $u \in \mathbb{R}$, we denote by $G^u(v)$ the unique solution of the ordinary differential equation

$$\frac{dG^u}{dv}(v) = G^u(v)\Lambda(u,v), \qquad G^u(0) = F(u)$$

in v. Then the function $X(u,v) := G^u(v)$ is the desired one. In fact, the solution of a ordinary differential equation depends smoothly on the initial value, $X(u,v)$ is a matrix-valued C^∞ function defined on $\mathbb{R}^2$. By definition of $G^u(v)$, we have

$$\frac{\partial X}{\partial v}(u,v) = \frac{dG^u}{dv}(v) = G^u(v)\Lambda(u,v) = X(u,v)\Lambda(u,v). \tag{2.5}$$

Since X is C^∞, $X_{uv} = X_{vu}$ holds. Then by the integrability condition (2.2), it holds that

$$\begin{aligned}\frac{\partial}{\partial v}\left(\frac{\partial X}{\partial u} - X\Omega\right) &= \frac{\partial}{\partial u}\frac{\partial X}{\partial v} - \frac{\partial X}{\partial v}\Omega - X\frac{\partial \Omega}{\partial v} \\ &= \frac{\partial}{\partial u}(X\Lambda) - \frac{\partial X}{\partial v}\Omega - X\frac{\partial \Omega}{\partial v} \\ &= \frac{\partial X}{\partial u}\Lambda + X\frac{\partial \Lambda}{\partial u} - \frac{\partial X}{\partial v}\Omega - X\frac{\partial \Omega}{\partial v} \\ &= X(\Lambda_u - \Omega_v) + \frac{\partial X}{\partial u}\Lambda - \frac{\partial X}{\partial v}\Omega \\ &= X(\Lambda_u - \Omega_v - \Lambda\Omega) + \frac{\partial X}{\partial u}\Lambda \\ &= -X\Omega\Lambda + \frac{\partial X}{\partial u}\Lambda \\ &= \left(\frac{\partial X}{\partial u} - X\Omega\right)\Lambda.\end{aligned}$$

That is, for each fixed u, the map $H(v) := X_u(u,v) - X\Omega$ satisfies an ordinary differential equation in v as follows:

$$\frac{dH}{dv}(u,v) = H(u,v)\Lambda(u,v).$$

Letting $v = 0$, we have

$$\begin{aligned}H(u,0) &= X_u(u,0) - X(u,0)\Omega(u,0) \\ &= (G^u)_u(u,0) - G^u(0)\Omega(u,0) \\ &= F'(u) - F(u)\Omega(u,0) = O\end{aligned}$$

and then, by uniqueness of the solutions of initial value problems for ordinary differential equations, $H(u,v) = 0$ holds. Since (u,v) is arbitrarily taken, we have

$$\frac{\partial X}{\partial u}(u,v) = X(u,v)\Omega(u,v),$$

that is, $X(u,v)$ is the solution of (2.1).

Uniqueness: Let X and $\hat{X}$ be matrix-valued functions satisfying (2.1). Then $\hat{X} - X$ is a solution of (2.1) with $X_0 = O$ since (2.1) is linear. Hence, to show the uniqueness, it is sufficient to show that the solution X of (2.1) with initial condition $X_0 = O$ is the constant function $X(u,v) = O$.

Let X be such a solution of (2.1). Here, $X(0,0) = O$ as we have set $(u_0, v_0) = (0,0)$. For an arbitrary $(u,v) \in \mathbb{R}^2$, let $F(t) := X(tu, tv)$. Then

$$\begin{aligned}\frac{d}{dt}F(t) &= uX_u(tu,tv) + vX_v(tu,tv) \\ &= X(tu,tv)(u\Omega(tu,tv) + v\Lambda(tu,tv)) = F(t)\omega(t)\end{aligned} \tag{2.6}$$

holds, where $\omega(t) = u\Omega(tu,tv) + v\Lambda(tu,tv)$. Then the ordinary differential equation (2.6) for $F(t)$ in t, the uniqueness of solutions of ordinary differential equations yields $F(t) = O$ since $F(0) = X(0,0) = O$. In particular, we have $X(u,v) = F(1) = O$. Since (u,v) has been taken arbitrarily, $X(u,v) = 0$ holds for all $(u,v) \in \mathbb{R}^2$. Hence we have the uniqueness. □

Application: Poincaré's lemma.

Theorem 2.6 (Poincaré's lemma)**.** *If a differential* 1*-form*

$$\omega = \alpha(u,v)\,du + \beta(u,v)\,dv$$

defined on a simply connected domain $U \subset \mathbb{R}^2$ *is closed, that is,* $d\omega = 0$ *holds, then there exists a* C^∞*-function* f *on* U *such that* $df = \omega$*. Such a function* f *is unique up to additive constants.*

Proof. Since $d\omega = (\beta_u - \alpha_v)\,du \wedge dv$, the assumption is equivalent to

$$\beta_u - \alpha_v = 0. \tag{2.7}$$

Consider a system of linear partial differential equations with unknown a 1×1-matrix valued function (i.e. a real-valued function) $\xi(u,v)$ as

$$\frac{\partial \xi}{\partial u} = \xi\alpha, \qquad \frac{\partial \xi}{\partial v} = \xi\beta, \qquad \xi(u_0,v_0) = 1. \tag{2.8}$$

Then it satisfies (2.2) because of (2.7). Hence by Theorem 2.3, there exists a smooth function $\xi(u,v)$ satisfying (2.8). In particular, Proposition 1.3 yields $\xi = \det\xi$ never vanishes. Since $\xi(u_0,v_0) = 1 > 0$, this means that $\xi > 0$ holds on U. Letting $f := \log\xi$, we have the function f satisfying $df = \omega$.

Next, we show the uniqueness: if two functions f and g satisfy $df = dg = \omega$, it holds that $d(f-g) = 0$. Hence by connectivity of U, $f - g$ must be constant. □

Application: Conjugation of Harmonic functions. In this paragraph, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. It is well-known that a function

$$f\colon U \ni u + \mathrm{i}v \longmapsto \xi(u,v) + \mathrm{i}\eta(u,v) \in \mathbb{C} \qquad (\mathrm{i} = \sqrt{-1}) \tag{2.9}$$

defined on a domain $U \subset \mathbb{C}$ is *holomorphic* if and only if it satisfies the following relation, called the *Cauchy-Riemann equations*:

$$\frac{\partial \xi}{\partial u} = \frac{\partial \eta}{\partial v}, \qquad \frac{\partial \xi}{\partial v} = -\frac{\partial \eta}{\partial u}. \tag{2.10}$$

Definition 2.7. A function $f\colon U \to \mathbb{R}$ defined on a domain $U \subset \mathbb{R}^2$ is said to be *harmonic* if it satisfies

$$\Delta f = f_{uu} + f_{vv} = 0.$$

The operator Δ is called the *Laplacian*.

Proposition 2.8. *If function* f *in* (2.9) *is holomorphic,* $\xi(u,v)$ *and* $\eta(u,v)$ *are harmonic functions.*

Proof. By (2.10), we have

$$\xi_{uu} = (\xi_u)_u = (\eta_v)_u = \eta_{vu} = \eta_{uv} = (\eta_u)_v = (-\xi_v)_v = -\xi_{vv}.$$

Hence $\Delta\xi = 0$. Similarly,

$$\eta_{uu} = (-\xi_v)_u = -\xi_{vu} = -\xi_{uv} = -(\xi_u)_v = -(\eta_v)_v = -\eta_{vv}.$$

Thus $\Delta\eta = 0$. □

Theorem 2.9. *Let $U \subset \mathbb{C} = \mathbb{R}^2$ be a simply connected domain and $\xi(u, v)$ a C^∞-function harmonic on U*[6]*. Then there exists a C^∞ harmonic function η on U such that $\xi(u, v) + \mathrm{i}\eta(u, v)$ is holomorphic on U.*

Proof. Let $\alpha := -\xi_v\,du + \xi_u\,dv$. Then by the assumption,

$$d\alpha = (\xi_{vv} + \xi_{uu})\,du \wedge dv = 0$$

holds, that is, α is a closed 1-form. Hence by simple connectivity of U and the Poincaré's lemma (Theorem 2.6), there exists a function η such that $d\eta = \eta_u\,du + \eta_v\,dv = \alpha$. Such a function η satisfies (2.10) for given ξ. Hence $\xi + \mathrm{i}\eta$ is holomorphic in $u + \mathrm{i}v$. □

Example 2.10. A function $\xi(u, v) = e^u \cos v$ is harmonic. Set

$$\alpha := -\xi_v\,du + \xi_u\,dv = e^u \sin v\,du + e^u \cos v\,dv.$$

Then $\eta(u, v) = e^u \sin v$ satisfies $d\eta = \alpha$. Hence

$$\xi + \mathrm{i}\eta = e^u(\cos v + \mathrm{i}\sin v) = e^{u+\mathrm{i}v}$$

is holomorphic in $u + \mathrm{i}v$.

Definition 2.11. The harmonic function η in Theorem 2.9 is called the *conjugate* harmonic function of ξ.

[6] The theorem holds under the assumption of C^2-differentiablity.

The fundamental theorem for Surfaces. Let $p\colon U \to \mathbb{R}^3$ be a parametrization of a *regular surface* defined on a domain $U \subset \mathbb{R}^2$. That is, $p = p(u, v)$ is a C^∞-map such that p_u and p_v are linearly independent at each point on U. Then $\nu := (p_u \times p_v)/|p_u \times p_v|$ is the *unit normal vector field* to the surface. The matrix-valued function $\mathcal{F} := (p_u, p_v, \nu)\colon U \to \mathrm{M}_3(\mathbb{R})$ is called the *Gauss frame* of p. We set

$$\begin{aligned} ds^2 &:= E\,du^2 + 2F\,du\,dv + G\,dv^2, \\ I\!I &:= L\,du^2 + 2M\,du\,dv + N\,dv^2, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} E &= p_u \cdot p_u & F &= p_u \cdot p_v & G &= p_v \cdot p_v \\ L &= p_{uu} \cdot \nu & M &= p_{uv} \cdot \nu & N &= p_{vv} \cdot \nu. \end{aligned}$$

We call ds^2 (resp. $I\!I$) the *first* (resp. *second*) *fundamental form.* Note that linear independence of p_u and p_v implies

$$E > 0, \qquad G > 0 \qquad \text{and} \qquad EG - F^2 > 0. \tag{2.12}$$

Set

$$\begin{aligned} \Gamma_{11}^1 &:= \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, \\ \Gamma_{11}^2 &:= \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}, \\ \Gamma_{12}^1 &= \Gamma_{21}^1 := \frac{GE_v - FG_u}{2(EG - F^2)}, \end{aligned} \tag{2.13}$$

$$\begin{aligned}\Gamma_{12}^2 &= \Gamma_{21}^2 := \frac{EG_u - FE_v}{2(EG-F^2)},\\ \Gamma_{22}^1 &:= \frac{2GF_v - GG_u - FG_v}{2(EG-F^2)},\\ \Gamma_{22}^2 &:= \frac{EG_v - 2FF_v + FG_u}{2(EG-F^2)}.\end{aligned}$$

and

$$A = \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix} := \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}. \tag{2.14}$$

The functions Γ_{ij}^k and the matrix A are called the *Christoffel symbols* and the *Weingarten matrix*. We state the following the *fundamental theorem for surfaces*, and give a proof (for a special case) in the following section.

Theorem 2.12 (The Fundamental Theorem for Surfaces)**.** *Let $p\colon U \ni (u,v) \mapsto p(u,v) \in \mathbb{R}^3$ be a parametrization of a regular surface defined on a domain $U \subset \mathbb{R}^2$. Then the Gauss frame $\mathcal{F} := \{p_u, p_v, \nu\}$ satisfies the equations*

$$\frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\Omega, \qquad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\Lambda, \tag{2.15}$$

$$\Omega := \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & -A_1^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & -A_1^2 \\ L & M & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} \Gamma_{21}^1 & \Gamma_{22}^1 & -A_2^1 \\ \Gamma_{21}^2 & \Gamma_{22}^2 & -A_2^2 \\ M & N & 0 \end{pmatrix},$$

where Γ_{jk}^i $(i,j,k = 1,2)$, A_l^k and L, M, N are the Christoffel symbols, the entries of the Weingarten matrix and the entries of the second fundamental form, respectively.

Theorem 2.13. *Let $U \subset \mathbb{R}^2$ be a simply connected domain, E, F, G, L, M, N C^∞-functions satisfying* (2.12)*, and Γ_{ij}^k, A_i^j the functions defined by* (2.13) *and* (2.14)*, respectively. If Ω and Λ satisfies*

$$\Omega_v - \Lambda_u = \Omega\Lambda - \Lambda\Omega,$$

there exists a parameterization $p\colon U \to \mathbb{R}^3$ of regular surface whose fundamental forms are given by (2.11)*. Moreover, such a surface is unique up to orientation preserving isometries of $\mathbb{R}^3$.*

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Exercises

2-1 Let $\xi(u,v) = \log\sqrt{u^2+v^2}$ be a function defined on $U = \mathbb{R}^2 \setminus \{(0,0)\}$

(1) Show that ξ is harmonic on U.

(2) Find the conjugate harmonic function η of ξ on

$$V = \mathbb{R}^2 \setminus \{(u,0) \mid u \leqq 0\} \subset U.$$

(3) Show that there exists no conjugate harmonic function of ξ defined on U.