Linear Ordinary Differential Equations

Preliminaries: Matrix Norms. Denote by $\mathrm{M}_n(\mathbb{R})$ the set of $n \times n$ matrix with real components, which can be identified the vector space $\mathbb{R}^{n^2}$. In particular, the Euclidean norm of $\mathbb{R}^{n^2}$ induces a norm

$$(1.1) \qquad |X|_{\mathrm{E}} = \sqrt{\mathrm{tr}({}^t XX)} = \sqrt{\sum_{i,j=1}^{n} x_{ij}^2}$$

on $\mathrm{M}_n(\mathbb{R})$. On the other hand, we let

$$(1.2) \qquad |X|_{\mathrm{M}} := \sup\left\{\frac{|X\boldsymbol{v}|}{|\boldsymbol{v}|}\,;\, \boldsymbol{v} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}\right\},$$

where $|\cdot|$ on the right-hand side denotes the Euclidean norm of $\mathbb{R}^n$.

Lemma 1.1. (1) *The map $X \mapsto |X|_{\mathrm{M}}$ is a norm of $\mathrm{M}_n(\mathbb{R})$.*

(2) *For X, $Y \in \mathrm{M}_n(\mathbb{R})$, it holds that $|XY|_{\mathrm{M}} \leqq |X|_{\mathrm{M}}\, |Y|_{\mathrm{M}}$.*

(3) *Let $\lambda = \lambda(X)$ be the maximum eigenvalue of semi-positive definite symmetric matrix ${}^t XX$. Then $|X|_{\mathrm{M}} = \sqrt{\lambda}$ holds.*

(4) $(1/\sqrt{n})|X|_{\mathrm{E}} \leqq |X|_{\mathrm{M}} \leqq |X|_{\mathrm{E}}$.

(5) *The map $|\cdot|_{\mathrm{M}}\colon \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ is continuous with respect to the Euclidean norm.*

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Proof. Since $|X\boldsymbol{v}|/|\boldsymbol{v}|$ is invariant under scalar multiplications to $\boldsymbol{v}$, we have $|X|_{\mathrm{M}} = \sup\{|X\boldsymbol{v}|\,;\, \boldsymbol{v} \in S^{n-1}\}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Here, the $S^{n-1} \ni \boldsymbol{x} \mapsto |A\boldsymbol{x}| \in \mathbb{R}$ is a continuous function defined on a compact space, and so the map takes maximum. Thus, the right-hand side of (1.2) is well-defined. It is easy to verify that $|\cdot|_{\mathrm{M}}$ satisfies the axiom of the norm.

Since $A := {}^t XX$ is positive semi-definite the eigenvalues λ_j $(j = 1, \dots, n)$ are non-negative real numbers. In particular, there exists an orthonormal basis $[\boldsymbol{a}_j]$ of $\mathbb{R}^n$ satisfying $A\boldsymbol{a}_j = \lambda_j \boldsymbol{a}_j$ $(j = 12, \dots, n)$. Let λ be the maximum eigenvalues of A, and write $\boldsymbol{v} = v_1\boldsymbol{a}_1 + \dots + v_n\boldsymbol{a}_n$. Then it holds that

$$\langle X\boldsymbol{v}, X\boldsymbol{v}\rangle = \lambda_1 v_1^2 + \dots + \lambda_n v_n^2 \leqq \lambda \langle \boldsymbol{v}, \boldsymbol{v}\rangle,$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product of $\mathbb{R}^n$. The equality of this inequality holds if and only if $\boldsymbol{v}$ is the λ-eigenvector, proving (3). Noticing the norm (1.1) is invariant under conjugations $X \mapsto {}^t PXP$ $(P \in \mathrm{O}(n))$, we obtain $|X|_{\mathrm{E}} = \sqrt{\lambda_1 + \dots + \lambda_n}$ by diagonalizing ${}^t XX$ by an orthogonal matrix P. Then we obtain (4). Hence two norms $|\cdot|_{\mathrm{E}}$ and $|\cdot|_{\mathrm{M}}$ induce the same topology $\mathrm{M}_n(\mathbb{R})$. In particular, we have (5). □

Preliminaries: Matrix-valued Functions.

Lemma 1.2. *Let X and Y be C^∞-maps defined on a domain $U \subset \mathbb{R}^m$ into $\mathrm{M}_n(\mathbb{R})$. Then*

(1) $\dfrac{\partial}{\partial u_j}(XY) = \dfrac{\partial X}{\partial u_j} Y + X \dfrac{\partial Y}{\partial u_j}$,

(2) $\dfrac{\partial}{\partial u_j}\det X = \operatorname{tr}\left(\widetilde{X}\dfrac{\partial X}{\partial u_j}\right)$, *and*

(3) $\dfrac{\partial}{\partial u_j}X^{-1} = -X^{-1}\dfrac{\partial X}{\partial u_j}X^{-1}$,

where $\widetilde{X}$ *is the cofactor matrix of* X, *and we assume in* (3).

Proposition 1.3. *Assume two* C^∞ *matrix-valued functions* $X(t)$ *and* $\Omega(t)$ *satisfy*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0. \tag{1.3}$$

Then

$$\det X(t) = (\det X_0)\exp\int_{t_0}^{t}\operatorname{tr}\Omega(\tau)\,d\tau \tag{1.4}$$

holds. In particular, if $X_0 \in \mathrm{GL}(n,\mathbb{R})$, [1] *then* $X(t) \in \mathrm{GL}(n,\mathbb{R})$ *for all* t.

Proof. By (2) of Lemma 1.2, we have

$$\begin{aligned}\frac{d}{dt}\det X(t) &= \operatorname{tr}\left(\widetilde{X}(t)\frac{dX(t)}{dt}\right) = \operatorname{tr}\left(\widetilde{X}(t)X(t)\Omega(t)\right)\\ &= \operatorname{tr}\bigl(\det X(t)\Omega(t)\bigr) = \det X(t)\operatorname{tr}\Omega(t).\end{aligned}$$

Here, we used the relation $\widetilde{X}X = X\widetilde{X} = (\det X)\,\mathrm{id}$[2]. Hence $\frac{d}{dt}\bigl(\rho(t)^{-1}\det X(t)\bigr) = 0$, where $\rho(t)$ is the right-hand side of (1.4). □

[1] $\mathrm{GL}(n,\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\,\det A \neq 0\}$: the general linear group.
[2] In this lecture, id denotes the identity matrix.

Proposition 1.4. *Assume* $\Omega(t)$ *in* (1.3) *is skew-symmetric for all* t, *that is,* ${}^t\Omega + \Omega$ *is identically* O. *If* $X_0 \in \mathrm{O}(n)$ *(resp.* $X_0 \in \mathrm{SO}(n)$*)*[3], $X(t) \in \mathrm{O}(n)$ *(resp.* $X(t) \in \mathrm{SO}(n)$*) for all* t.

Proof. By (1) in Lemma 1.2,

$$\begin{aligned}\frac{d}{dt}(X^tX) &= \frac{dX}{dt}{}^tX + X{}^t\left(\frac{dX}{dt}\right)\\ &= X\Omega^tX + X^t\Omega^tX = X(\Omega + {}^t\Omega)^tX = O.\end{aligned}$$

Hence X^tX is constant, that is, if $X_0 \in \mathrm{O}(n)$,

$$X(t)^tX(t) = X(t_0)^tX(t_0) = X_0{}^tX_0 = \mathrm{id}\,.$$

if $X_0 \in \mathrm{O}(n)$, proves the first case of the proposition. Since $\det A = \pm 1$ when $A \in \mathrm{O}(n)$, the second case follows by continuity of $\det X(t)$. □

Preliminaries: Norms of Matrix-Valued functions. Let $I = [a,b]$ be a closed interval, and denote by $C^0(I,\mathrm{M}_n(\mathbb{R}))$ the set of continuous functions $X\colon I \to \mathrm{M}_n(\mathbb{R})$. For any fixed number k, we define

$$||X||_{I,k} := \sup\left\{e^{-kt}|X(t)|_{\mathrm{M}}\,;\,t \in I\right\} \tag{1.5}$$

for $X \in C^0(I,\mathrm{M}_n(\mathbb{R}))$. When $k = 0$, $||\cdot||_{I,0}$ is the uniform norm for continuous functions, which is complete. Similarly, one can prove the following in the same way:

[3] $\mathrm{O}(n) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\,{}^tAA = A^tA = \mathrm{id}\}$: the orthogonal group; $\mathrm{SO}(n) = \{A \in \mathrm{O}(n)\,;\,\det A = 1\}$: the special orthogonal group.

Lemma 1.5. *The map* $\|\cdot\|_{I,k}\colon C^0(I,\mathrm{M}_n(\mathbb{R}))$ *is a complete norm.*

Linear Ordinary Differential Equations. We prove the fundamental theorem for *linear* ordinary differential equations.

Proposition 1.6. *Let* $\Omega(t)$ *be a* C^∞*-function valued in* $\mathrm{M}_n(\mathbb{R})$ *defined on an interval* I*. Then for each* $t_0\in I$*, there exists the unique matrix-valued* C^∞*-function* $X(t)=X_{t_0,\mathrm{id}}(t)$ *such that*

$$\frac{dX(t)}{dt}=X(t)\Omega(t),\qquad X(t_0)=\mathrm{id}\,. \tag{1.6}$$

Proof. Uniqueness: Assume $X(t)$ and $Y(t)$ satisfy (1.6). Then

$$\begin{aligned}Y(t)-X(t)&=\int_{t_0}^t\bigl(Y'(\tau)-X'(\tau)\bigr)\,d\tau\\&=\int_{t_0}^t\bigl(Y(\tau)-X(\tau)\bigr)\Omega(\tau)\,d\tau\end{aligned}$$

holds. Hence for an arbitrary closed interval $J\subset I$,

$$\begin{aligned}|Y(t)-X(t)|_{\mathrm{M}}&\leqq\left|\int_{t_0}^t\bigl|\bigl(Y(\tau)-X(\tau)\bigr)\Omega(\tau)\bigr|_{\mathrm{M}}\,d\tau\right|\\&\leqq\left|\int_{t_0}^t|Y(\tau)-X(\tau)|_{\mathrm{M}}\,|\Omega(\tau)|_{\mathrm{M}}\,d\tau\right|\\&=\left|\int_{t_0}^t e^{-k\tau}\,|Y(\tau)-X(\tau)|_{\mathrm{M}}\,e^{k\tau}\,|\Omega(\tau)|_{\mathrm{M}}\,d\tau\right|\\&\leqq\|Y-X\|_{J,k}\sup_J|\Omega|_{\mathrm{M}}\left|\int_{t_0}^t e^{k\tau}\,d\tau\right|\\&=\|Y-X\|_{J,k}\frac{\sup_J|\Omega|_{\mathrm{M}}}{|k|}e^{kt}\left|1-e^{-k(t-t_0)}\right|\end{aligned}$$

holds for $t\in J$. Here, setting $J=[t_0,a]$ and $k=2\sup_J|\Omega|_{\mathrm{M}}$, we have

$$\|Y-X\|_{J,k}\leqq\frac{1}{2}\|Y-X\|_{J,k},$$

that is, $\|Y-X\|_{J,k}=0$, proving $Y(t)=X(t)$ for $t\in J$. Similarly, on the interval $J'=[a,t_0]$, we can conclude $Y=X$ on J' setting $k=-2\sup_J|\Omega|_{\mathrm{M}}$. Since J and J' are arbitrary, $Y=X$ holds on I.
Existence: Let $J:=[t_0,a]\subset I$ be a closed interval, and define a sequence $\{X_j\}$ of matrix-valued functions defined on I satisfying $X_0(t)=\mathrm{id}$ and

$$X_{j+1}(t)=\mathrm{id}+\int_{t_0}^t X_j(\tau)\Omega(\tau)\,d\tau\quad(j=0,1,2,\dots). \tag{1.7}$$

Let $k:=2\sup_J|\Omega|_{\mathrm{M}}$. Then

$$\begin{aligned}|X_{j+1}(t)-X_j(t)|_{\mathrm{M}}&\leqq\int_{t_0}^t|X_j(\tau)-X_{j-1}(\tau)|_{\mathrm{M}}|\Omega(\tau)|_{\mathrm{M}}\,d\tau\\&\leqq e^{kt}\|X_j-X_{j-1}\|_{J,k}\frac{\sup_J|\Omega|_{\mathrm{M}}}{k}=\frac{e^{kt}}{2}\|X_j-X_{j-1}\|_{J,k},\end{aligned}$$

and hence $\|X_{j+1}-X_j\|_{J,k}\leqq\frac{1}{2}\|X_j-X_{j-1}\|_{J,k}$, that is, $\{X_j\}$ is a Cauchy sequence with respect to $\|\cdot\|_{J,k}$. Thus, by completeness

(Lemma 1.5), it converges to some $X \in C^0(J, \mathrm{M}_n(\mathbb{R}))$. By (1.7), the limit X satisfies

$$X(t_0) = \mathrm{id}, \qquad X(t) = \mathrm{id} + \int_{t_0}^{t} X(\tau)\Omega(\tau)\,d\tau.$$

Applying the fundamental theorem of calculus, we can see that X satisfies $X'(t) = X(t)\Omega(t)$ ($' = d/dt$). Since J can be taken arbitrarily, existence of the solution on $I \cap \{t \geqq t_0\}$ is proved. Existence of $I \cap \{t \leqq t_0\}$ can be proved in the same way. So far, existence of a differentiable function $X(t)$ satisfying (1.6) is obtained.

Finally, we shall prove that X is of class C^∞. Since $X'(t) = X(t)\Omega(t)$, the derivative X' of X is continuous. Hence X is of class C^1, and so is $X(t)\Omega(t)$. Thus we have that $X'(t)$ is of class C^1, and then X is of class C^2. Iterating this argument, we can prove that $X(t)$ is of class C^r for arbitrary r. □

Corollary 1.7. *Let $\Omega(t)$ be a matrix-valued C^∞-function defined on an interval I. Then for each $t_0 \in I$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0,X_0}(t)$ defined on I such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0. \tag{1.8}$$

In particular, $X_{t_0,X_0}(t)$ is of class C^∞ in X_0 and t.

Proof. We rewrite $X(t)$ in Proposition 1.6 as $Y(t) = X_{t_0,\mathrm{id}}(t)$. Then the function

$$X(t) := X_0 Y(t) = X_0 X_{t_0,\mathrm{id}}(t), \tag{1.9}$$

is desired one. Conversely, assume $X(t)$ satisfies the conclusion. Noticing $Y(t)$ is a regular matrix for all t because of Proposition 1.3,

$$W(t) := X(t)Y(t)^{-1}$$

satisfies

$$\begin{aligned}\frac{dW}{dt} &= \frac{dX}{dt}Y^{-1} - X(t)Y^{-1}\frac{dY}{dt}Y^{-1}\\ &= X\Omega Y^{-1} - XY^{-1}Y\Omega Y^{-1} = O.\end{aligned}$$

Hence

$$W(t) = W(t_0) = X(t_0)Y(t_0)^{-1} = X_0.$$

Hence the uniqueness is obtained. The final part is obvious by the expression (1.9). □

Proposition 1.8. *Let $\Omega(t)$ and $B(t)$ be a matrix-valued C^∞-functions defined on I. Then for each $t_0 \in I$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function defined on I satisfying*

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0. \tag{1.10}$$

Proof. Rewrite X in Proposition 1.6 as $Y(t) := X_{t_0,\mathrm{id}}(t)$. Then

$$X(t) = \left(X_0 + \int_{t_0}^{t} B(\tau)Y^{-1}(\tau)\,d\tau\right)Y(t) \tag{1.11}$$

satisfies (1.10). Conversely, if X satisfies (1.10), $W := XY^{-1}$ satisfies

$$X' = W'Y + WY' = W'Y + WY\Omega, \quad X\Omega + B = WY\Omega + B,$$

and then we have $W' = BY^{-1}$. Since $W(t_0) = X_0$,

$$W = X_0 + \int_{t_0}^{t} B(\tau)Y^{-1}(\tau)\,d\tau.$$

Thus we obtain (1.11). □

Theorem 1.9. *Let I and U be an interval and a domain in $\mathbb{R}^m$, respectively, and let $\Omega(t,\boldsymbol{\alpha})$ and $B(t,\boldsymbol{\alpha})$ be matrix-valued C^∞-functions defined on $I \times U$ ($\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_m)$). Then for each $t_0 \in I$, $\boldsymbol{\alpha} \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0,X_0,\boldsymbol{\alpha}}(t)$ defined on I such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t,\boldsymbol{\alpha}) + B(t,\boldsymbol{\alpha}), \qquad X(t_0) = X_0. \tag{1.12}$$

Moreover,

$$I \times I \times \mathrm{M}_n(\mathbb{R}) \times U \ni (t, t_0, X_0, \boldsymbol{\alpha}) \mapsto X_{t_0,X_0,\boldsymbol{\alpha}}(t) \in \mathrm{M}_n(\mathbb{R})$$

is C^∞-map.

Proof. Let $\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}}) := \Omega(t+t_0,\boldsymbol{\alpha})$ and $\widetilde{B}(t,\tilde{\boldsymbol{\alpha}}) = B(t+t_0,\boldsymbol{\alpha})$, and let $\widetilde{X}(t) := X(t+t_0)$. Then (1.12) is equivalent to

$$\frac{d\widetilde{X}(t)}{dt} = \widetilde{X}(t)\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}}) + \widetilde{B}(t,\tilde{\boldsymbol{\alpha}}), \quad \widetilde{X}(0) = X_0, \tag{1.13}$$

where $\tilde{\boldsymbol{\alpha}} := (t_0, \alpha_1, \dots, \alpha_m)$. There exists the unique solution $\widetilde{X}(t) = \widetilde{X}_{\mathrm{id},X_0,\tilde{\boldsymbol{\alpha}}}(t)$ of (1.13) for each $\tilde{\boldsymbol{\alpha}}$ because of Proposition 1.8. So it is sufficient to show differentiability with respect

to the parameter $\tilde{\boldsymbol{\alpha}}$. We set $Z = Z(t)$ as the unique solution of

$$\frac{dZ}{dt} = Z\widetilde{\Omega} + \widetilde{X}\frac{\partial\widetilde{\Omega}}{\partial\alpha_j} + \frac{\partial\widetilde{B}}{\partial\alpha_j}, \qquad Z(0) = O. \tag{1.14}$$

Then it holds that $Z = \partial\widetilde{X}/\partial\alpha_j$ (Problem 1-1). In particular, by the proof of Proposition 1.8, it holds that

$$Z = \frac{\partial\widetilde{X}}{\partial\alpha_j} = \left(\int_0^t \left(\widetilde{X}(\tau)\frac{\partial\widetilde{\Omega}(\tau,\tilde{\boldsymbol{\alpha}})}{\partial\alpha_j} + \frac{\partial\widetilde{B}(\tau,\tilde{\boldsymbol{\alpha}})}{\partial\alpha_j}\right) Y^{-1}(\tau)d\tau\right) Y(t).$$

Here, $Y(t)$ is the unique matrix-valued C^∞-function satisfying $Y'(t) = Y(t)\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}})$, and $Y(0) = \mathrm{id}$. Hence $\widetilde{X}$ is a C^∞-function in $(t,\tilde{\boldsymbol{\alpha}})$. □

Fundamental Theorem for Space Curves. As an application, we prove the fundamental theorem for space curves. A C^∞-map $\gamma\colon I \to \mathbb{R}^3$ defined on an interval $I \in \mathbb{R}$ into $\mathbb{R}^3$ is said to be a *regular curve* if $\dot\gamma \neq \mathbf{0}$ holds on I. For a regular curve $\gamma(t)$, there exists a parameter change $t = t(s)$ such that $\tilde\gamma(s) := \gamma(t(s))$ satisfies $|\tilde\gamma'(s)| = 1$. Such a parameter s is called the *arc-length parameter*.

Let $\gamma(s)$ be a regular curve in $\mathbb{R}^3$ parametrized by the arc-length satisfying $\gamma''(s) \neq \mathbf{0}$ for all s. Then

$$\boldsymbol{e}(s) := \gamma'(s), \qquad \boldsymbol{n}(s) := \frac{\gamma''(s)}{|\gamma''(s)|}, \qquad \boldsymbol{b}(s) := \boldsymbol{e}(s) \times \boldsymbol{n}(s)$$

forms a positively oriented orthonormal basis $\{\boldsymbol{e},\boldsymbol{n},\boldsymbol{b}\}$ of $\mathbb{R}^3$ for each s. Regarding each vector as column vector, we have the matrix-valued function

$$\mathcal{F}(s) := (\boldsymbol{e}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)) \in \mathrm{SO}(3). \tag{1.15}$$

in s, which is called the *Frenet frame* associated to the curve γ. Under the situation above, we set

$$\kappa(s) := |\gamma''(s)| > 0, \qquad \tau(s) := -\left\langle \boldsymbol{b}'(s), \boldsymbol{n}(s)\right\rangle,$$

which is called the *curvature* and *torsion*, respectively, of γ. Using these quantities, the Frenet frame satisfies

$$\frac{d\mathcal{F}}{ds} = \mathcal{F}\Omega, \qquad \Omega = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}. \tag{1.16}$$

Proposition 1.10. *The curvature and the torsion are invariant under the transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ of $\mathbb{R}^3$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$). Conversely, two curves $\gamma_1(s)$, $\gamma_2(s)$ parametrized by arc-length parameter have common curvature and torsion, there exist $A \in \mathrm{SO}(3)$ and $\boldsymbol{b} \in \mathbb{R}^3$ such that $\gamma_2 = A\gamma_1 + \boldsymbol{b}$.*

Proof. Let κ, τ and $\mathcal{F}_1$ be the curvature, torsion and the Frenet frame of γ_1, respectively. Then the Frenet frame of $\gamma_2 = A\gamma_1 + \boldsymbol{b}$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$) is $\mathcal{F}_2 = A\mathcal{F}_1$. Hence both $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy (1.16), and then γ_1 and γ_2 have common curvature and torsion.

Conversely, assume γ_1 andγ_2 have common curvature and torsion. Then the frenet frame $\mathcal{F}_1$, $\mathcal{F}_2$ both satisfy (1.16). Let $\mathcal{F}$ be the unique solution of (1.16) with $\mathcal{F}(t_0) = \mathrm{id}$. Then by the proof of Corollary 1.7, we have $\mathcal{F}_j(t) = \mathcal{F}_j(t_0)\mathcal{F}(t)$ ($j = 1, 2$). In particular, since $\mathcal{F}_j \in \mathrm{SO}(3)$, $\mathcal{F}_2(t) = A\mathcal{F}_1(t)$ ($A := \mathcal{F}_2(t_0)\mathcal{F}_1(t_0)^{-1} \in \mathrm{SO}(3)$). Comparing the first column of these, $\gamma_2'(s) = A\gamma_1'(t)$ holds. Integrating this, the conclusion follows. □

Theorem 1.11 (The fundamental theorem for space curves)**.** *For given C^∞-functions $\kappa(s)$ and $\tau(s)$ defined on I such that $\kappa(s) > 0$ on I. Then there exists a space curve $\gamma(s)$ parametrized by arc-length whose curvature and torsion are κ and τ, respectively. Moreover, such a curve is unique up to transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$) of $\mathbb{R}^3$.*

Proof. We have already shown the uniqueness in Proposition 1.10. We shall prove the existence: Let $\Omega(s)$ be as in (1.16), and $\mathcal{F}(s)$ the solution of (1.16) with $\mathcal{F}(s_0) = \mathrm{id}$. Since Ω is skew-symmetric, $\mathcal{F}(s) \in \mathrm{SO}(3)$ by Proposition 1.4. Denoting the column vectors of $\mathcal{F}$ by $\boldsymbol{e}$, $\boldsymbol{n}$, $\boldsymbol{b}$, and let

$$\gamma(s) := \int_{s_0}^{s} \boldsymbol{e}(\sigma)\, d\sigma.$$

Then $\mathcal{F}$ is the frenet frame of γ, and κ, and τ are the curvature and torsion of γ, respectively (Problem 1-2). □

Exercises

1-1 Verify that Z in (1.14) coincides with $\partial\widetilde{X}/\partial\alpha_j$.

1-2 Complete the proof of Theorem 1.11.