ICT.H409

Optics in Information Processing IV

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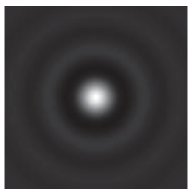
Impulse response (PSF) and transfer function of a lens system



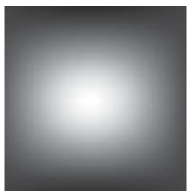


Image formation by a lens system

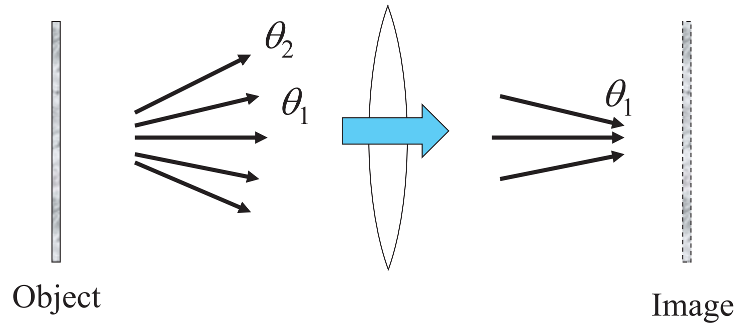
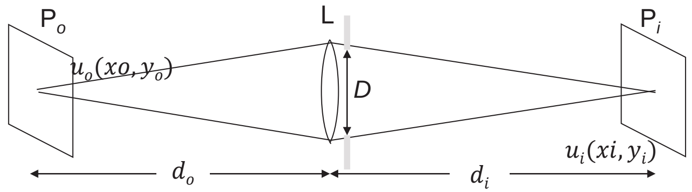


Image formation by a lens system

- In the image formation by a lens system, high-frequency component is blocked by the lens aperture.
- Then it works as low-pass filtering, and the resolution is limited.

- In most cases, it can be modeled as a linear shift-invariant system, and characterized by the impulse response, called "point spread function (PSF)."

Impulse response of a lens system

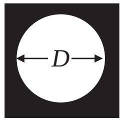


$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$$ Lens formula

$u_o(xo, y_o)$, $u_i(x_i, y_i)$:
Complex amplitude at P_o, P_i planes

$$M = -\frac{d_i}{d_o}$$ Magnification

$P(x, y)$
Pupil function



The input-output relationship becomes

$$u_i(x_i, y_i) = A \int\int_{-\infty}^{\infty} h(x_i, y_i; x_o, y_o)\, u_o(x_o, y_o)\, dx_o dy_o$$

$$h(x_i, y_i; x_o, y_o) = \frac{1}{\lambda^2 d_i d_o} \int\int_{-\infty}^{\infty} P(x, y) \exp\{-j \frac{2\pi}{\lambda d_i}[(x_i - Mx_o)x + (y_i - My_o)y]\}\, dxdy$$

Substituting

$\tilde{x}_o = Mx_o$	$d\tilde{x}_o = Mdx_o$	$\tilde{x} = \frac{x}{\lambda d_i}$	$dx = \lambda d_i\, d\tilde{x}$	
$\tilde{y}_o = My_o$	$d\tilde{y}_o = Mdy_o$	$\tilde{y} = \frac{y}{\lambda d_i}$	$dy = \lambda d_i d\tilde{y}$	$u_{oM}(\tilde{x}_o, \tilde{y}_o) = \frac{1}{M} u_o\left(\frac{\tilde{x}_o}{M}, \frac{\tilde{y}_o}{M}\right)$

$$h(x_i - \tilde{x}_o, y_i - \tilde{y}_o) = \frac{d_i}{d_o} \int\int_{-\infty}^{\infty} P(\lambda d_i \tilde{x}, \lambda d_i \tilde{y}) \exp\{-j2\pi[(x_i - \tilde{x}_o)\tilde{x} + (y_i - \tilde{y}_o)\tilde{y}]\}\, d\tilde{x} d\tilde{y}$$

$$h(x_i, y_i) = M\, \mathcal{F}\{P(\lambda d_i x, \lambda d_i y)\} = MP_f\left(\frac{x_i}{\lambda d_i}, \frac{y_i}{\lambda d_i}\right)$$

$$u_i(x_i, y_i) = \frac{1}{M} \int\int_{-\infty}^{\infty} h(x_i - \tilde{x}_o, y_i - \tilde{y}_o)\, u_{oM}(\tilde{x}_o, \tilde{y}_o)\, d\tilde{x}_o d\tilde{y}_o$$

Let us define $h_C(x_i, y_i) = \frac{1}{M} h(x_i, y_i) = \mathcal{F}\{P(\lambda d_i x, \lambda d_i y)\} = P_f\left(\frac{x_i}{\lambda d_i}, \frac{y_i}{\lambda d_i}\right)$

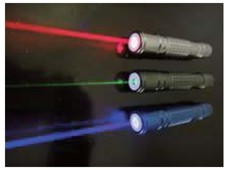
$$u_i(x_i, y_i) = \int\int_{-\infty}^{\infty} h_C(x_i - \tilde{x}_o, y_i - \tilde{y}_o)\, u_{oM}(\tilde{x}_o, \tilde{y}_o)\, d\tilde{x}_o d\tilde{y}_o = h_C(x_i, y_i) * u_{oM}(x_i, y_i)$$

Convolution

Coherent imaging case

Impulse response: $h_C(x_i, y_i)$

Coherent case



The point spread function (impulse response) of an imaging lens system is the Fourier transform of a pupil function.

Input: $u_{oM}(\tilde{x}_o, \tilde{y}_o) = \frac{1}{M} u_o\left(\frac{\tilde{x}_o}{M}, \frac{\tilde{y}_o}{M}\right)$

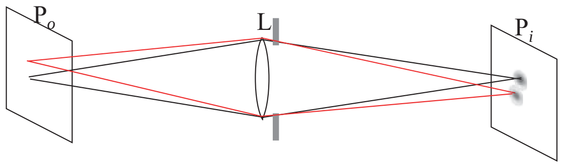
-- Complex amplitude of the magnified object.

Output: $u_i(x_i, y_i)$ -- Complex amplitude of the image.

Impulse response (PSF): $h_C(x_i, y_i) = \mathcal{F}\{P(\lambda d_i x, \lambda d_i y)\}$

-- Fourier transform of the pupil function

Incoherent case



Each point source at P_o plane does not interfere with the light from other point sources at observation plane.

Intensity distribution at P_o plane: $I_o(x_o, y_o) = |u_o(x_o, y_o)|^2$

$I_{oM}(x_o, y_o) = |u_{oM}(x_o, y_o)|^2$

Intensity distribution at P_i plane: $I_i(x_i, y_i) = |u_i(x_i, y_i)|^2$

$$I_i(x_i, y_i) = \int\int_{-\infty}^{\infty} |h(x_i - \tilde{x}_o, y_i - \tilde{y}_o)|^2\, I_{oM}(\tilde{x}_o, \tilde{y}_o)\, d\tilde{x}_o d\tilde{y}_o = |h(x_i, y_i)|^2 * I_{oM}(x_i, y_i)$$

$$I_i(x_i, y_i) = h_I(x_i, y_i) * I_{oM}(x_i, y_i)$$

$$h_I(x_i, y_i) = |h(x_i - \tilde{x}_o, y_i - \tilde{y}_o)|^2 = |\mathcal{F}\{P(\lambda d_i x, \lambda d_i y)\}|^2$$

Incoherent imaging case

The point spread function (impulse response) of an imaging lens system is the Fourier energy spectrum of a pupil function.

Input: $I_{oM}(\tilde{x}_o, \tilde{y}_o) = \frac{1}{M^2} I_o\left(\frac{\tilde{x}_o}{M}, \frac{\tilde{y}_o}{M}\right)$

-- Intensity of the magnified object.

Output: $I_i(x_i, y_i)$ -- Intensity of the image.

Impulse response (PSF): $h_I(x_i, y_i) = |\mathcal{F}\{P(\lambda d_i x, \lambda d_i y)\}|^2$

-- Fourier power spectrum of the pupil function

Frequency response
(Transfer function)

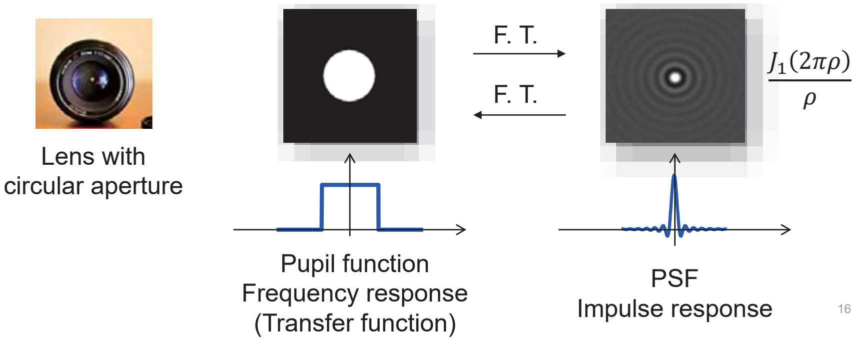
Fourier transform of the impulse response

Coherent imaging case

$$H_C(u, v) = \mathcal{F}\{h_C(x_i, y_i)\} = \mathcal{F}\{\mathcal{F}\{P(\lambda d_i x, \lambda d_i y)\}\} = P(-\lambda d_i x, -\lambda d_i y)$$

$\mathcal{F}\{\text{ circ}(r)\} = \text{jinc}(2\pi\rho) = \frac{J_1(2\pi\rho)}{\rho}$ (Fourier-Bessel transform)

J_1 : Bessel function of the first kind, order 1.



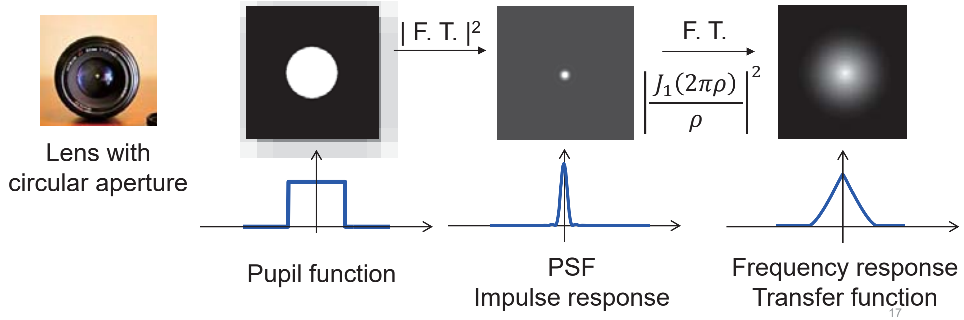
Frequency response
(Transfer function)

Fourier transform of the impulse response

Incoherent imaging case

$$H_I(u, v) = \mathcal{F}\{h_I(x_i, y_i)\} = \mathcal{F}\{|\mathcal{F}\{P(\lambda d_i x, \lambda d_i y)\}|^2\}$$

$$= \int\int_{-\infty}^{\infty} P(\tilde{u}, \tilde{v})\, P^*(\tilde{u} + u, \tilde{v} + v)\, d\tilde{u}\, d\tilde{v} \qquad \text{(Autocorrelation)}$$



Optical transfer function (OTF)

Frequency response of an incoherent imaging system after normalization is called OTF. OTF is commonly used for the characterization of an optical imaging system.

$$OTF(u, v) = \frac{H_I(u, v)}{H_I(0,0)}$$

Modulation Transfer Function

$$\mathrm{MTF} = \frac{\text{Contrast of output image } (u,v)}{\text{Contrast of input image } (u,v)}$$

$$\mathrm{MTF} = |\,\mathrm{OTF}\,|$$

$$\mathrm{OTF}(u,v) = \mathrm{MTF}(u,v)\ \exp\{\,j\phi(u,v)\}$$

$\phi(u,v)$: Phase transfer function (PTF)

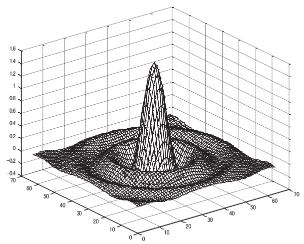
Impulse response of a circular aperture

$$h_I(x_i, y_i) = |\mathcal{F}\{P(\lambda d_i x, \lambda d_i y)\}|^2$$

$$= \left|P_f\left(\frac{x_i}{\lambda d_i}, \frac{y_i}{\lambda d_i}\right)\right|^2 = \left|\frac{\pi D^2}{2}\cdot\frac{J_1\left(\frac{\pi D r_i}{\lambda d_i}\right)}{\frac{\pi D r_i}{\lambda d_i}}\right|^2$$

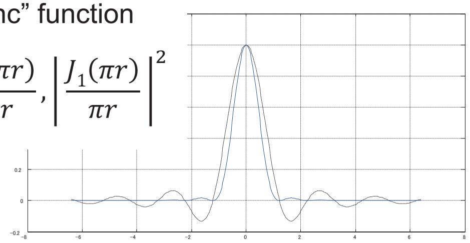
J_1 : Bessel function of the first kind, order 1.

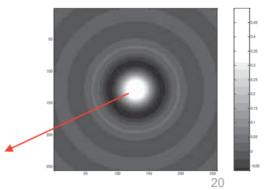
$$h_I(x_i, y_i) = 0 \quad \text{when} \quad \frac{D r_i}{\lambda d_i} = 1.220$$



"Jinc" function

$$\frac{J_1(\pi r)}{\pi r}, \left|\frac{J_1(\pi r)}{\pi r}\right|^2$$

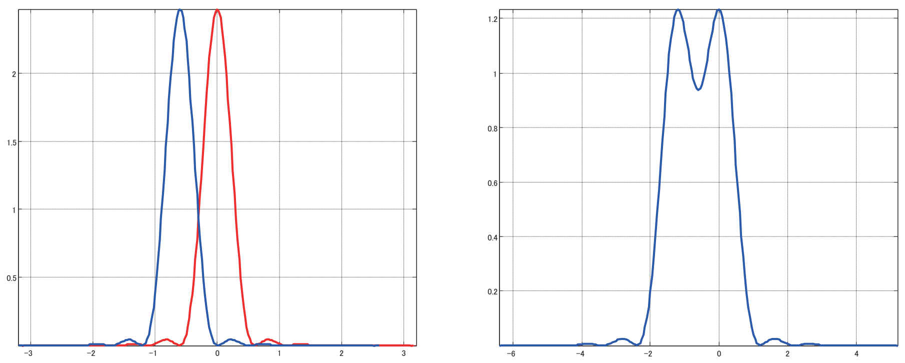




Airy Disk

Resolution of a lens system

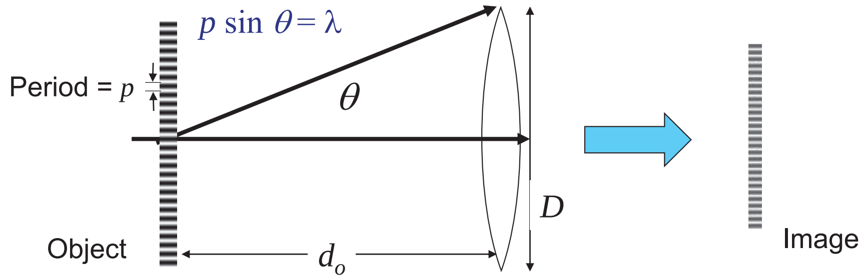
- Rayleigh criterion -



Rayleigh limit $L = 1.22\dfrac{\lambda d_i}{D}$

(Diffraction limit)

Estimating the resolution of a lens system



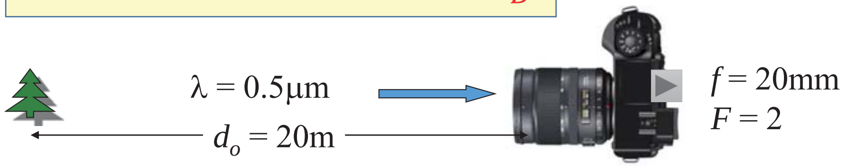
$$\sin\theta \cong \theta \cong \frac{D}{2do} \Rightarrow U_{max} = \frac{1}{p_{min}} = \frac{\sin\theta}{\lambda} \cong \frac{D}{2\lambda d_o} \quad \text{or} \quad p_{min} = \frac{2\lambda do}{D}$$

Rayleigh limit (image plane)

$$L = 1.22\frac{\lambda d_i}{D} \longrightarrow \text{Object plane} \quad \frac{L}{M} = 1.22\frac{\lambda d_i}{DM} = 1.22\frac{\lambda d_o}{D}$$

Example

F-number of a lens system $F = \frac{f}{D}$



Lens diameter $D = \frac{f}{F} = 10mm$

$$L_o = 1.22\frac{\lambda d_o}{D} = 1.22\,\frac{0.5 \times 10^{-6} \times 20}{10 \times 10^{-3}} = 1.22 \times 10^{-3}\ [m]$$

Object larger than ≅ 1.22mm can be resolved.