ICT.H409

Optics in Information Processing III

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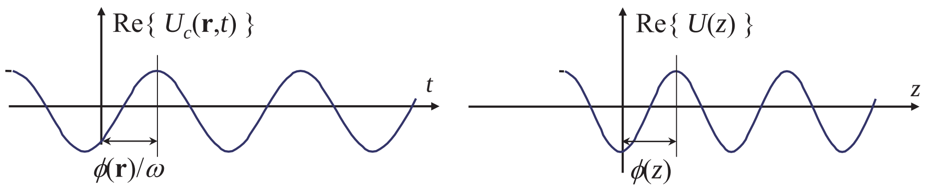
Wave optics



$$U_c(\boldsymbol{r},t) = A(\boldsymbol{r}) \exp\{-j\,(\omega t - \phi(\boldsymbol{r}))\}$$
$$= U(\boldsymbol{r}) \exp(-j\omega t)$$

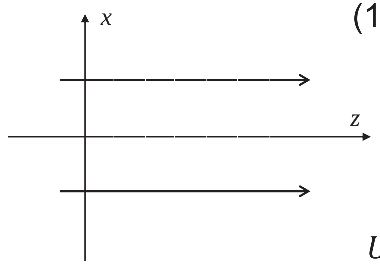
$U(\boldsymbol{r}) = A(\boldsymbol{r}) \exp\{j\,\phi(\boldsymbol{r})\}$: Complex amplitude

$A(\boldsymbol{r})$: Amplitude

$\phi(\boldsymbol{r})$: Phase

$I(\boldsymbol{r}) = <|U_c(\boldsymbol{r},t)|2> = |U(\boldsymbol{r})|^2 = A(\boldsymbol{r})^2$: Intensity

Plane wave traveling in z-direction



(1) In wave optics, parallel beam is a plane wave.
The electromagnetic wave traveling in parallel to the z-axis can be written as

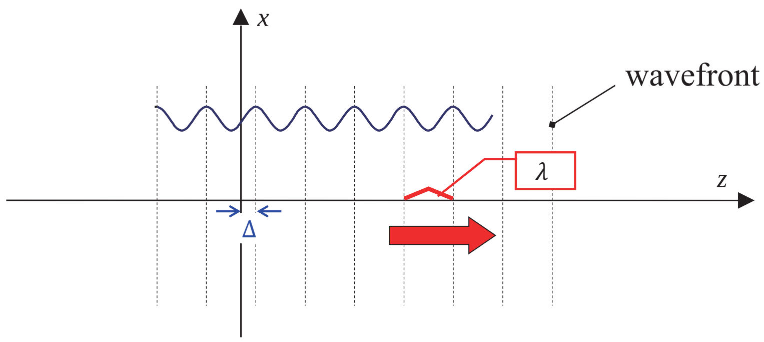
$$U_c(x,y,z,t) = A(x,y,z)\exp(jkz - j\omega t)$$

By separating the time dependent component, we have "complex amplitude":

$$U(x,y,z) = A\exp(jkz)$$

where $k = 2\pi/\lambda$ and λ is the wavelength.

Derive the equation of the surfaces with constant phase, and sketch them on the x-z plane.



The surface with constant phase satisfies

$z - \Delta = m\lambda$ (If we ignore initial phase, $z = m\lambda$)

where Δ is a constant and m is an integer.

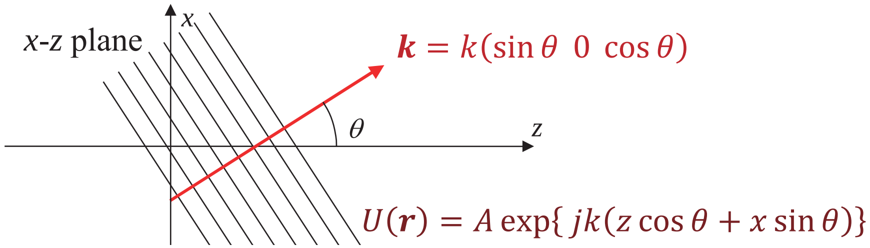
or, $k(z-\Delta) = 2\pi m$ ➡ Wavefront

$k = \dfrac{2\pi}{\lambda}$: wave number

$$U(\boldsymbol{r}) = A\exp\{j\,k\,(z-\Delta)\}$$
$$= A\exp\{j\,(k\,z-\psi)\}$$

$$\psi = \frac{2\pi\Delta}{\lambda}$$

Plane wave traveling along the wave vector $\boldsymbol{k}$



$U(\boldsymbol{r}) \; = \; A(\boldsymbol{r})\exp\{ j\,\phi(\boldsymbol{r}) \}$

The surface with constant phase: $\phi(\boldsymbol{r}) = const. + 2\pi m$, where

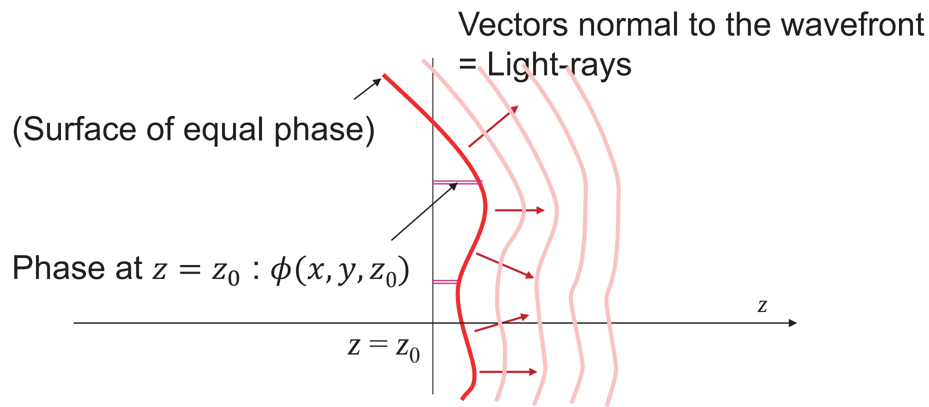
$\phi(\boldsymbol{r}) \; = \; \boldsymbol{k}\cdot\boldsymbol{r} \; = \; k_x x \; + \; k_y y \; + \; k_z z$

$\boldsymbol{k} \; = \; (\, k_x \;\; k_y \;\; k_z \,)$: wave vector, $\cdot$: inner product

Complex amplitude of plane wave:

$$U(\boldsymbol{r}) = A(\boldsymbol{r})\exp(\; j\,\boldsymbol{k}\cdot\boldsymbol{r}\,)$$
$$= A(\boldsymbol{r})\exp\{\; j\,(\,k_x x \; + \; k_y y \; + \; k_z z\,)\,\}$$

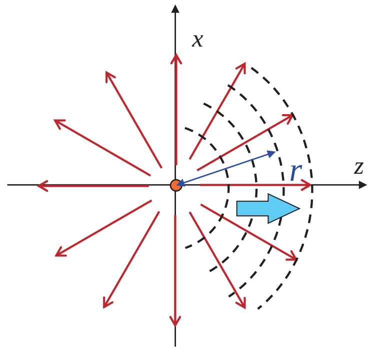
Wavefront



Complex amplitude at $z = z_0$:

$$U(x, y) \; = \; A(x, y)\exp\{\; j\,\phi(x, y)\,\}$$

Spherical wave



Exercise 3: (2)

The surface with constant phase satisfies

$$r - \Delta = m\lambda$$

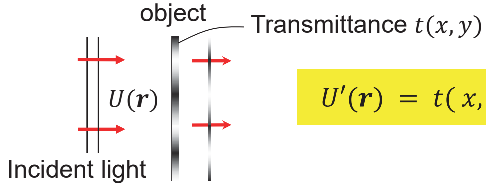
Δ : constant, m : integer.

Complex amplitude of spherical wave:

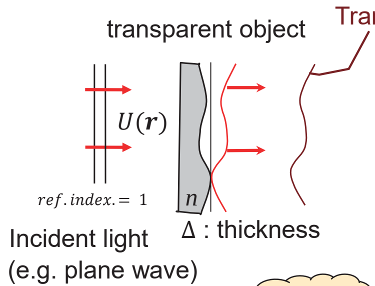
$$U(r) = \frac{U_0\exp(jkr)}{r}$$

$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

Modulation of wavefront



$U'(\boldsymbol{r}) \; = \; t(\,x, y\,)\,U(\boldsymbol{r})$ Amplitude modulation

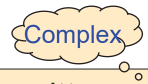


$(n-1)\Delta$: Optical path length difference

$$\phi(\,x, y\,) \; = \; \frac{2\pi\Delta(\,x, y\,)}{\lambda}$$ Phase modulation

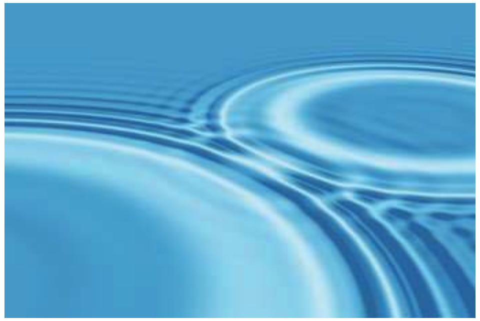
$$U'(\boldsymbol{r}) \; = \; \exp\{\,j\phi(\,x, y\,)\}\; U(\boldsymbol{r})$$

$$t(x, y) \; = \; \exp\{\,j\phi(\,x, y\,)\}$$



Amplitude transmittance $t(x, y) \; \rightarrow U'(x, y) \; = \; t(x, y)\,U(x, y)$

Interference



Interference

- Interference of two wavefronts

$$U_{1c}(\boldsymbol{r},t) = A_1(\boldsymbol{r})\exp\{-j(\omega_1 t - \phi_1(\boldsymbol{r}))\}$$
$$U_{2c}(\boldsymbol{r},t) = A_2(\boldsymbol{r})\exp\{-j(\omega_2 t - \phi_2(\boldsymbol{r}))\}$$

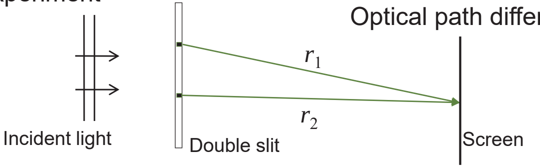
Intensity

$$I(\boldsymbol{r}) = <|U_{1c}(\boldsymbol{r},t) + U_{2c}(\boldsymbol{r},t)|^2>$$
$$= A_1(\boldsymbol{r})^2 + A_2(\boldsymbol{r})^2$$
$$+2A_1(\boldsymbol{r})A_2(\boldsymbol{r}) < \cos\{(\omega_1-\omega_2)t - (\phi_1(\boldsymbol{r}) - \phi_2(\boldsymbol{r}))\} >$$

< > Ensemble average (Temporal average)

Interference term

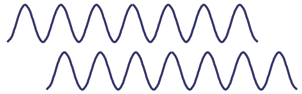
Young's experiment



Optical path difference (OPD) $= n \cdot |r_1 - r_2|$

$$\phi_1(\boldsymbol{r}) = n \cdot r_1 \cdot \frac{2\pi}{\lambda}$$
$$|\phi_1(\boldsymbol{r}) - \phi_2(\boldsymbol{r})| = \frac{2\pi}{\lambda} \cdot \mathrm{OPD}$$

Coherence

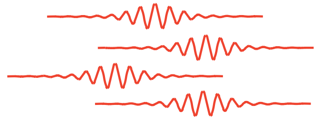


- If $\omega_1 \neq \omega_2$, $< \cos\{(\omega_1 - \omega_2)t + \phi\} > = 0$

  for the observation time $\gg \frac{2\pi}{\omega}$

  $I(\boldsymbol{r}) = A_1(\boldsymbol{r})^2 + A_2(\boldsymbol{r})^2$

  → no interference term: incoherent (temporal)



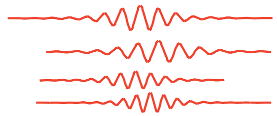
- If $\omega_1 = \omega_2$, → coherent

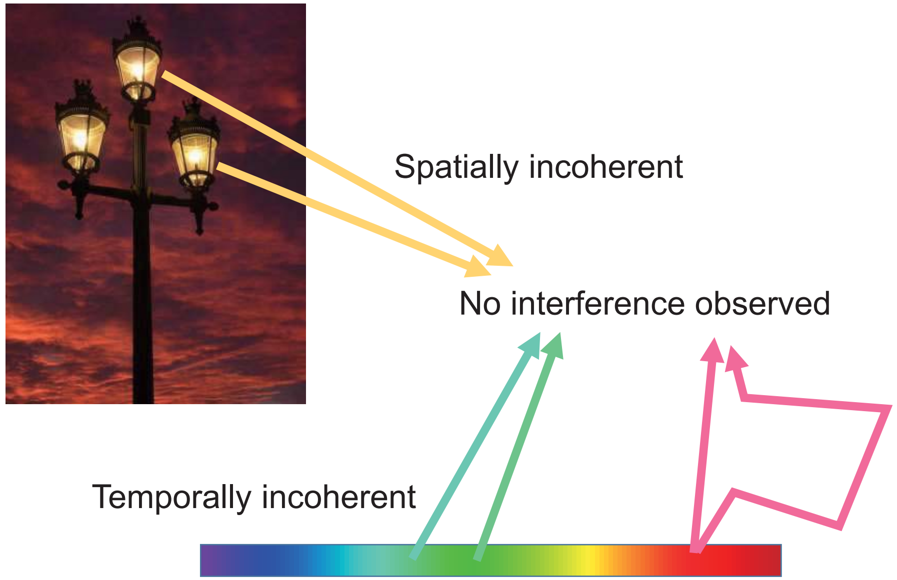
  $I(\boldsymbol{r}) = <|U_{1c}(\boldsymbol{r},t) + U_{2c}(\boldsymbol{r},t)|^2>$

  $= |U_1(\boldsymbol{r}) + U_2(\boldsymbol{r})|^2$

  $= A_1(\boldsymbol{r})^2 + A_2(\boldsymbol{r})^2 + 2A_1(\boldsymbol{r})A_2(\boldsymbol{r})\cos\{\phi_1(\boldsymbol{r}) - \phi_2(\boldsymbol{r})\}$

- (Partially coherent)

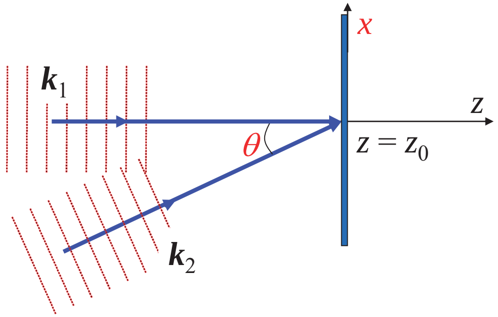




Bandwidth > 0

cf. coherence length < ∞

Example: Interference of two plane waves



$$\phi_1(\boldsymbol{r}) = \boldsymbol{k_1} \cdot \boldsymbol{r} + \varphi_1 = kz_0 + \varphi_1$$

$$\phi_2(\boldsymbol{r}) = \boldsymbol{k_2} \cdot \boldsymbol{r} + \varphi_2 = k \sin\theta \cdot x + k\cos\theta \cdot z_0 + \varphi_2$$

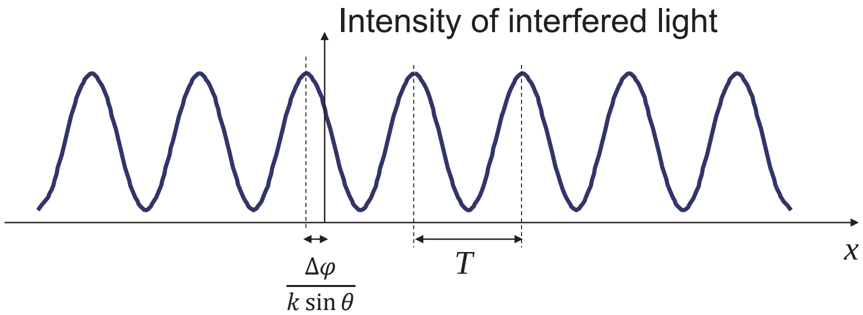
Constant

$$\phi_1(\boldsymbol{r}) - \phi_2(\boldsymbol{r}) = k \sin\theta \cdot x + \Delta\varphi$$

$$I(\boldsymbol{r}) = A_1(\boldsymbol{r})^2 + A_2(\boldsymbol{r})^2 + 2A_1(\boldsymbol{r})A_2(\boldsymbol{r})\cos\{\phi_1(\boldsymbol{r}) - \phi_2(\boldsymbol{r})\}$$
$$= I_0(\boldsymbol{r}) + I_i(\boldsymbol{r})\cos(k\sin\theta \cdot x + \Delta\varphi)$$

Interference fringe

$$I(\boldsymbol{r}) = I_0(\boldsymbol{r}) + I_i(\boldsymbol{r})\cos(k\sin\theta \cdot x + \Delta\varphi)$$

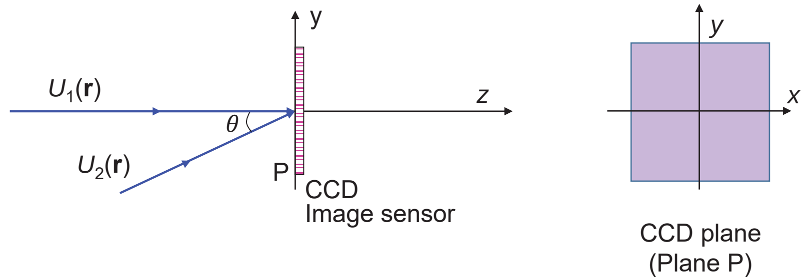


What is the period T of the interference fringe?

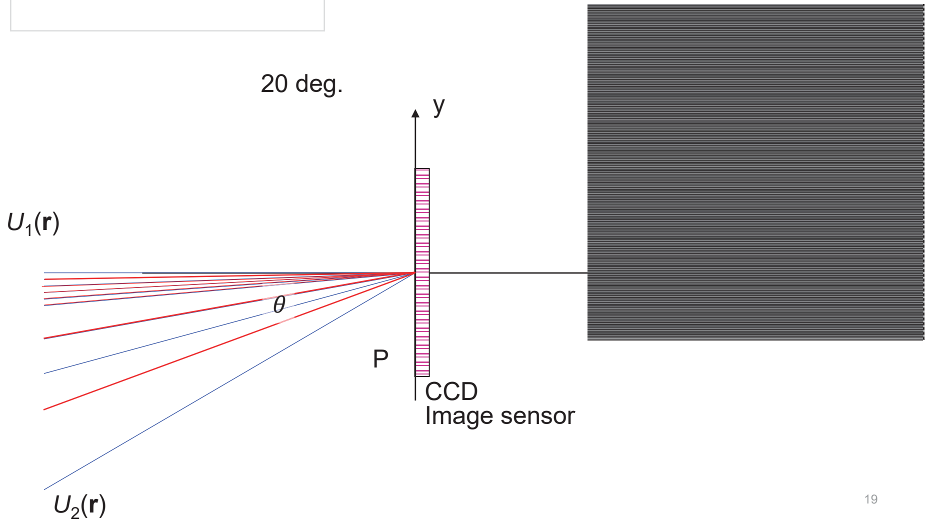
Let us consider to capture the interference pattern of two plane waves U_1 and U_2 using a CCD image sensor. U_1 is a plane wave traveling in parallel to z-axis, and the incident angle of U_2 onto the CCD plane P is θ. The wavelength of the light is assumed to be λ.

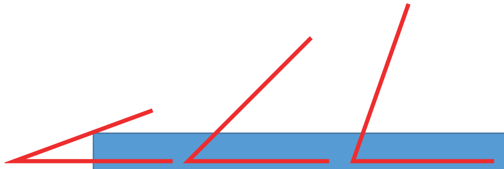
(1) Derive the light intensity pattern (interference pattern) on the plane P, and draw it schematically.

(2) Draw the 2D Fourier transform of the interference pattern of (1), schematically.

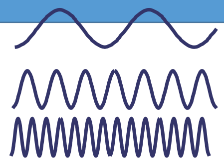


(3) Derive the relationship between the spatial frequency of the interference fringe, u_i [cycles/m], and the incident angle of U_2, θ.





Angle corresponds to Frequency



OCT (Optical Coherence Tomography)

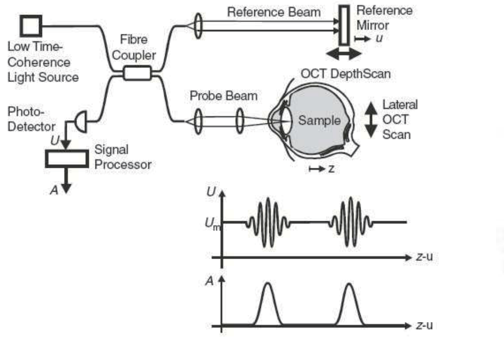


Figure 11. Time-domain reflectometer LCI in fibre optics technology. $U - U_m = U_G(\tau) =$ LCI signal. A is the real envelope. Signals generated at only two light re-emitting sites (anterior corneal surface and Bruch's membrane in the fundus of the eye) are indicated.

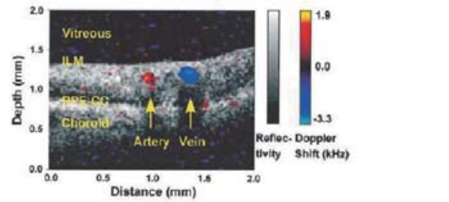
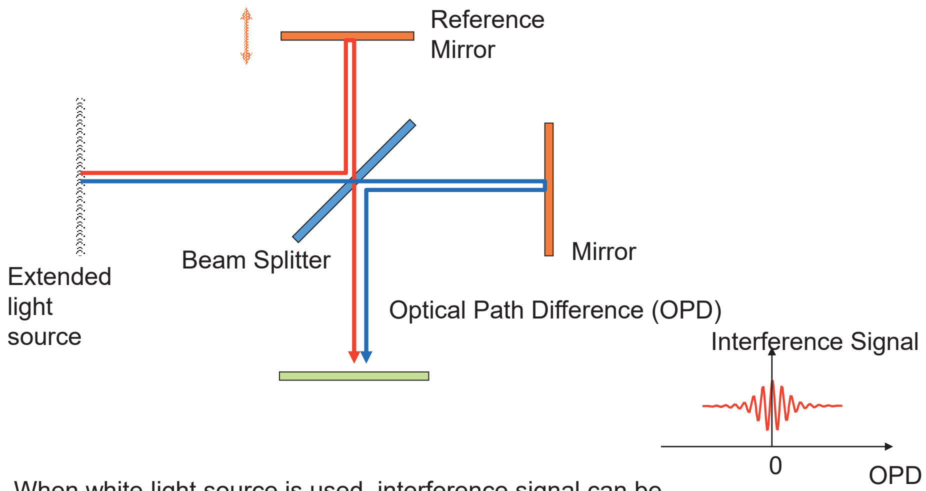


Figure 20. *In vivo* DOCT image of human retina superior to the optic nerve head. Velocity data were thresholded, colour coded and superimposed on the conventional OCT image encoded in grey scale. Reprinted from Yazdanfar *et al* (2000) by permission of the Optical Society of America.

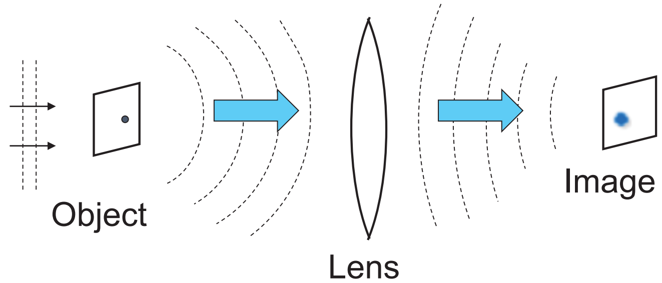
A. F. Fercher, W. Drexler, C. K. Hitzenberger, and T Lasser, "Optical coherence tomography—principles and applications," Reports on Progress in Physics, 239–303, (2003)

Michelson Interferometer

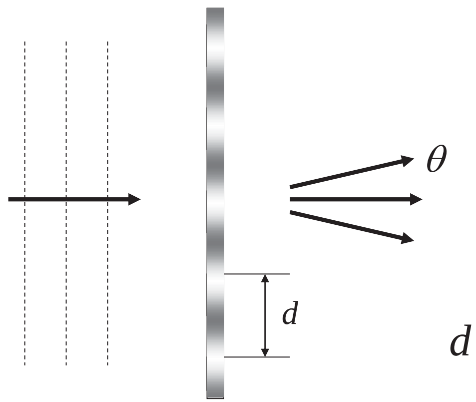


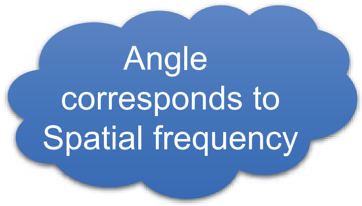
When white light source is used, interference signal can be observed only when OPD ≈ 0

Diffraction and wave propagation



Diffraction Grating





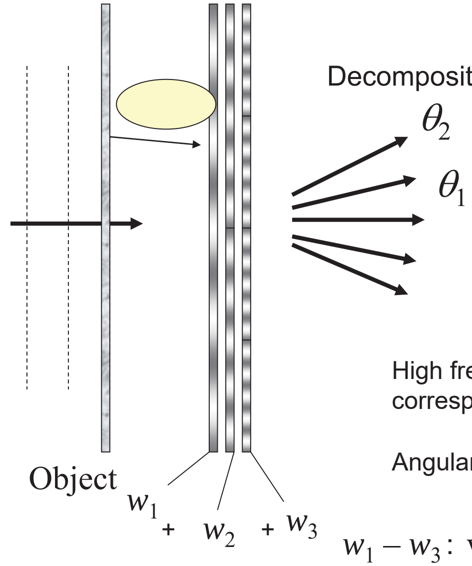
Angular spectrum

$$d \sin\theta = m\lambda$$

If sinusoidal grating,

$$d \sin\theta = 0、\pm\lambda$$

Superposition of sinusoidal gratings



Decomposition to sinusoidal gratings

θ_2

θ_1

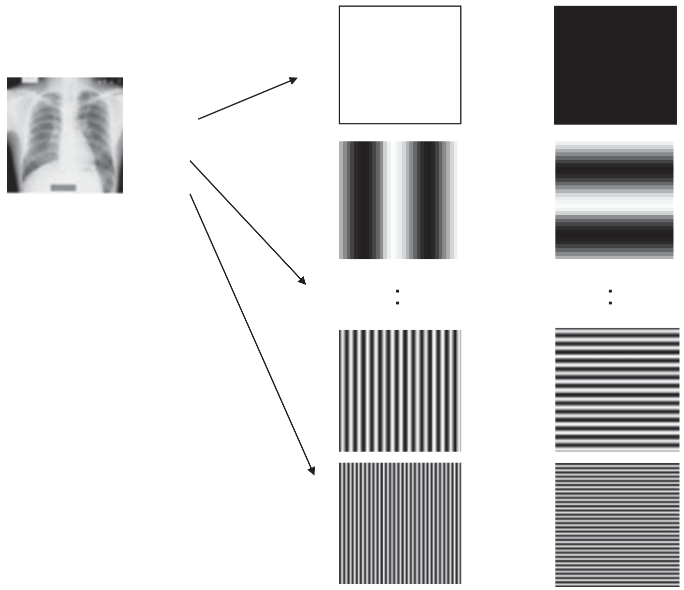
High frequency components in an image correspond to large diffraction angle.

Angular spectrum

$w_1 - w_3$: weights

2-D Fourier transform

$$G(u,v) = \iint g(x,y)\exp\{-j2\pi(xu + yv)\}dxdy$$



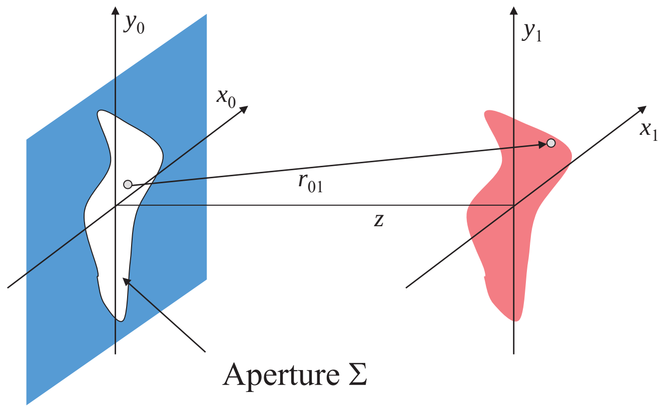
$\times G(u_0, v_0)$

$\times G(u_k, v_k)$

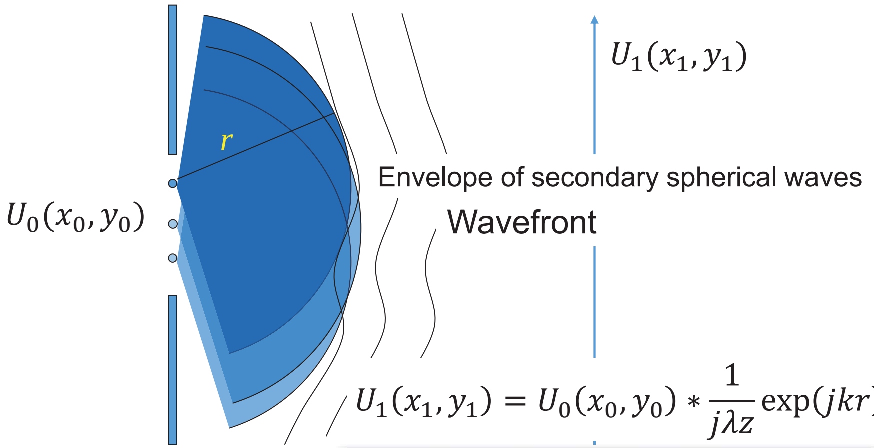
$\times G(u_{k+1}, v_{k+1})$

Scalar diffraction theory

$$U_1(x_1, y_1) = \frac{1}{j\lambda z}\iint_{\Sigma} U_0(x_0, y_0)\exp(jkr_{01})dx_0 dy_0$$



Huygens-Fresnel Principle



$$U_1(x_1, y_1) = U_0(x_0, y_0) * \frac{1}{j\lambda z}\exp(jkr)$$

Convolution with spherical wave

Rewriting the Fresnel diffraction equation

$$U_1(x_1, y_1) = \frac{\exp(jkz)}{j\lambda z}\int\int_\Sigma U_0(x_0, y_0)\exp\left\{j\frac{k}{2z}\left[(x_0 - x_1)^2 + (y_0 - y_1)^2\right]\right\}dx_0 dy_0$$

$$= \frac{\exp(jkz)}{j\lambda z}\int\int_\Sigma U_0(x_0, y_0)\exp\left\{j\frac{k}{2z}(x_0^2 + y_0^2)\right\}\exp\left\{j\frac{k}{2z}(x_1^2 + y_1^2)\right\}$$
$$\exp\left\{j\frac{k}{z}(x_0 x_1 + y_0 y_1)\right\}dx_0 dy_0$$

$$= C\int\int_\Sigma U_0(x_0, y_0)\exp\left\{j\frac{k}{2z}(x_0^2 + y_0^2)\right\}\exp\left\{j\frac{k}{z}(x_0 x_1 + y_0 y_1)\right\}dx_0 dy_0$$

Fourier Transform of $U_0(x_0, y_0)\exp\left\{j\frac{k}{2z}(x_0^2 + y_0^2)\right\}$

× Phase Term

Fresnel approximation

If $|x_0 - x_1| << z$ and $|y_0 - y_1| << z$

$$r_{01} = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}$$
$$= z\sqrt{1 + (\frac{x_0 - x_1}{z})^2 + (\frac{y_0 - y_1}{z})^2}$$
$$\cong z\left[1 + \frac{1}{2}(\frac{x_0 - x_1}{z})^2 + \frac{1}{2}(\frac{y_0 - y_1}{z})^2\right]$$

⇨ Paraxial approximation

Spherical wave is approximated by quadratic (parabolic) wave:

Spherical wave located at a point light source located at (x_0, y_0) is given by

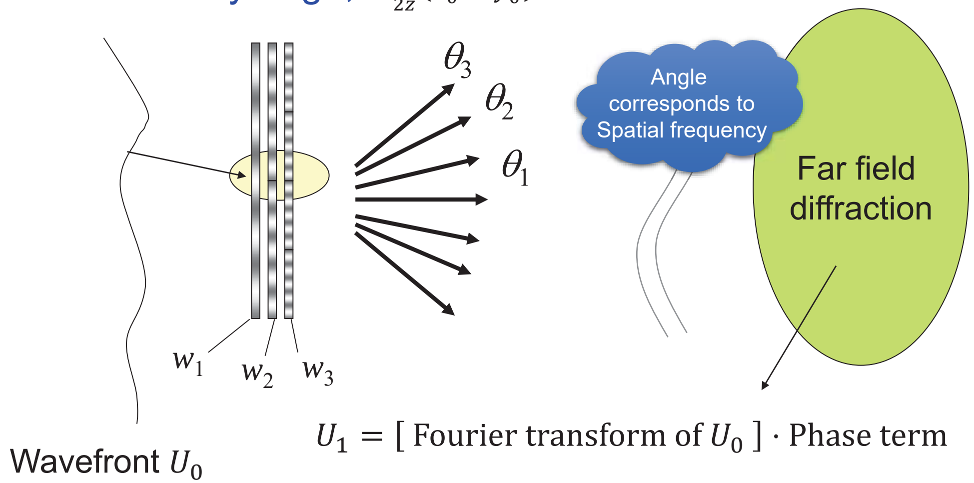
$$U_1(x_1, y_1) = \frac{\exp(jkr)}{r} \cong \frac{\exp(jkz)}{z}\exp\left\{j\frac{k}{2z}\left[(x_0 - x_1)^2 + (y_0 - y_1)^2\right]\right\}$$

Fresnel diffraction

$$U_1(x_1, y_1) = \frac{\exp(jkz)}{j\lambda z}\int\int_\Sigma U_0(x_1, y_1)\exp\left\{j\frac{k}{2z}\left[(x_0 - x_1)^2 + (y_0 - y_1)^2\right]\right\}dx_0 dy_0$$

Fraunhofer Diffraction

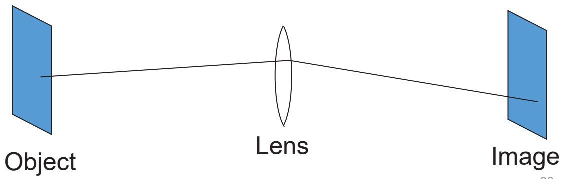
If z is very large, $\frac{k}{2z}(x_0^2 + y_0^2)$ is almost constant



U_1 = [Fourier transform of U_0] · Phase term

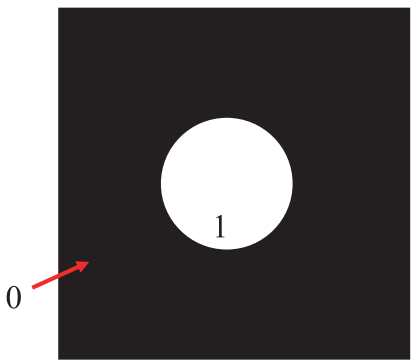
Light intensity = |Fourier transform of $U_0|^2$

Image formation by a lens system (1)

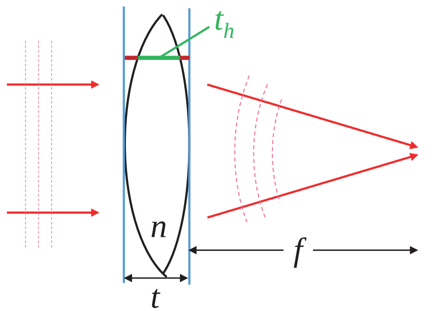


Lens aperture = Pupil function

Amplitude $P(x_l, y_l)$



Phase



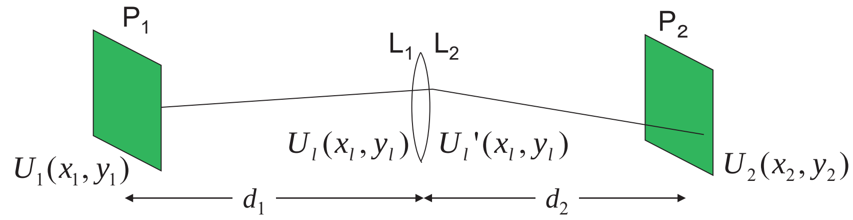
Optical path length:

$$\Delta = n \cdot t_h(x_l, y_l) + (t - t_h(x_l, y_l))$$

Transformation of wavefront Spherical wave → Plane wave
Plane wave → Spherical wave

Phase modulation: $\phi_L(x_l, y_l) = \dfrac{k}{2f}(x_l^2 + y_l^2)$

Imaging through a lens system



$P_1 \rightarrow L_1$ Fresnel Diffraction

$$U_l(x_l, y_l) = \frac{\exp(jkd_1)}{j\lambda d_1} \int\int_{-\infty}^{\infty} U_1(x_1, y_1) \exp\{j \frac{k}{2d_1}[(x_l - x_1)^2 + (y_l - y_1)^2]\} dx_1 dy_1$$

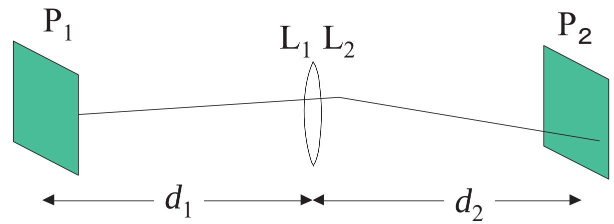
$L_1 \rightarrow L_2$ Phase modulation by lens

$$U_l'(x_l, y_l) = U_l(x_l, y_l) P(x_l, y_l) \exp\{-j \frac{k}{2f}(x_l^2 + y_l^2)\}$$

$L_2 \rightarrow P_2$ Fresnel Diffraction

$$U_2(x_2, y_2) = \frac{\exp(jkd_2)}{j\lambda d_2} \int\int_{-\infty}^{\infty} U_l'(x_l, y_l) \exp\{j \frac{k}{2d_2}[(x_2 - x_l)^2 + (y_2 - y_l)^2]\} dx_l dy_l$$

Wavefront at P_1 and P_2 planes



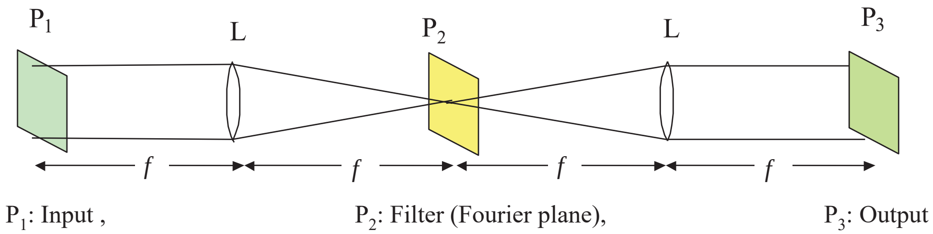
$$U_2(x_2, y_2) = A \int\int_{-\infty}^{\infty} h(x_2, y_2; x_1, y_1) U_1(x_1, y_1) dx_1 dy_1$$

$$h(x_2, y_2; x_1, y_1) = \frac{1}{\lambda^2 d_1 d_2} \exp\left[j \frac{k}{2d_2}(x_2^2 + y_2^2)\right] \exp\left[j \frac{k}{2d_1}(x_1^2 + y_1^2)\right]$$

$$\int\int_{-\infty}^{\infty} P(x, y) \exp\left[j \frac{k}{2}(\frac{1}{d_1} + \frac{1}{d_2} - \frac{1}{f})(x^2 + y^2)\right]$$

$$\exp\left[-jk\{(\frac{x_1}{d_1} + \frac{x_2}{d_2})x + (\frac{y_1}{d_1} + \frac{y_2}{d_2})y\}\right] dxdy$$

Optical Fourier transform and Coherent optical filtering



When $d_1 = f$ and $d_2 = f$,

$$U_2(x_2, y_2) = A\int\int_{-\infty}^{\infty} U_1(x_1, y_1)\exp\{j\frac{2\pi}{\lambda f}[x_1x_2 + y_1y_2]\}dx_1dy_1$$

$$u = x_2 / \lambda f ,\ v = y_2 / \lambda f$$

$$U_2(u, v) = C\ \boldsymbol{F}\{U_1(x_1, y_1)\}$$

Optical Fourier Transform

Phase contrast imaging

- Phase shift of zero frequency component

$$f(x, y) = \exp\{j\phi(x, y)\} \approx 1 + j\phi(x, y) \qquad (\ \phi(x,y) << 1\ )$$

$$g(x, y) = \exp(j\frac{\pi}{2}) + j\phi(x, y) = j + j\phi(x, y)$$

$$I(x, y) = | g(x, y) |^2 \approx 1 + 2\phi(x, y)$$

Summary

- Complex amplitude of light wave
  - Plane wave, spherical wave
  - Wavefront
  - Complex amplitude modulation
- Interference
  - Superposition of two waves
  - Coherence
  - “Angle corresponds to spatial frequency”
- Diffraction and wave propagation
  - Convolution
  - Fresnel approximation
  - Fraunhofer approximation
  - Angular spectrum
- Image formation by a lens system