12.Resonator

Operation principles and designs are addressed in several types of resonators. Also, the class includes an important parameter, Q factor.

12.1 Transmission line resonator

12.2 Ring resonator

12.3 Application to a wavelength meter

12.4 Q factor

Resonant condition : transmission line

Current flowing in a series connection of Z_1 and Z_2:

$$I = \frac{E}{Z_1 + Z_2}$$

$Z_1 + Z_2 = 0$ ⇒ $I = \infty$ (resonance)

Circuit can be divided into two at an arbitrary position.

Example : transmission line terminated with short circuit:

input impedance (length l) $Z_{in} = jR_c \tan \beta l$

$$R_c(\tan \beta l_1 + \tan \beta l_2) = 0$$

$$\tan \beta(l_1 + l_2) = \frac{\tan \beta l_1 + \tan \beta l_2}{1 - \tan \beta l_1 \tan \beta l_2} = 0$$

$$\beta(l_1 + l_2) = m\pi$$

$$\therefore l_1 + l_2 = m\frac{\lambda}{2} \qquad (m = 1,2,\ldots)$$

Resonance : voltage and current distribution

$$V(y) = V_i e^{j\beta y} + V_r e^{-j\beta y}$$

$$I(y) = \frac{1}{R_c}\left(V_i e^{j\beta y} - V_r e^{-j\beta y}\right)$$

V=0 at y=0 ⇒ $V_i = -V_r$

$$\therefore V(y) = V_i\left(e^{j\beta y} - e^{-j\beta y}\right) = 2jV_i \sin\beta y$$

$$I(y) = \frac{V_i}{R_c} 2\cos\beta y$$

⇓

$$\therefore v(y,t) = \sqrt{2}\,\mathrm{Re}\left[V(y)e^{j\omega t}\right] = 2\sqrt{2}V_i \sin\beta y \cos(\omega t + \frac{\pi}{2})$$

$$i(y,t) = 2\sqrt{2}\frac{V_i}{R_c}\cos\beta y \cos\omega t$$

(a) The phase of fields (voltage and current) is not a function of position (constant in the cavity).
(b) 90-deg phase difference exists between a magnetic and an electric field.

Q factor

Quality factor of resonator:

$$Q = 2\pi f_0 \frac{L}{R} = \frac{\frac{1}{2} 2\pi f_0 L |I_0|^2}{\frac{1}{2} R |I_0|^2} = \omega_0 \frac{\text{Maximum energy stored in an inductance}}{\text{Energy dissipated in one cycle}}$$

if conductor loss is dominant:

$$Q = \omega_0 \frac{\frac{\mu}{2} \iiint |H|^2 dv}{\frac{R_s}{2} \iint |\mathbf{i}|^2 dS}$$

$R_s = \frac{1}{\sigma\delta}$: a surface resistance. (δ :skin depth)

$|\mathbf{i}| = |\mathbf{H}_t|$ ($\mathbf{H}_t$: a magnetic field component tangential to a conductor surface)

Transmission line : $Q = \frac{\omega_0 L}{R + G\frac{L}{C}} = \frac{\omega_0}{\frac{R}{L} + \frac{G}{C}} = \frac{\omega_0}{\frac{2\alpha}{\sqrt{LC}}} = \frac{\beta_0}{2\alpha}$

External Q

total Quality factor :

$$Q_t = \frac{\omega_0 L}{R + R_{ext}}$$

$$\frac{1}{Q_t} = \frac{R}{\omega_0 L} + \frac{R_{ext}}{\omega_0 L} = \frac{1}{Q_0} + \frac{1}{Q_{ext}}$$

$$Q_{ext} \equiv \frac{\omega_0 L}{R_{ext}} > 0 : \textit{external}\ \mathrm{Q}$$

$$\frac{1}{Q_t} > \frac{1}{Q_0}$$

Q_0:unloaded Q

13.Multi/Demultiplexer

13.1 Equivalent circuit of a resonator

13.2 Band-pass filter

13.3 Band-stop (band-reject) filter

13.4 Demultiplexer (mainly used in lightwave circuit)

- Arrayed waveguide grating (AWG)
- Mach-Zehnder interferometer

Equivalent circuit of resonator

short-open transmission lie

$$Y_{in} = \frac{1}{jR_c}\cot \beta l \quad \Rightarrow \quad \text{resonance at } \beta l = \frac{\pi}{2}$$

$$Y_{in} = \frac{1}{jR_c}\cot \beta l = -j\frac{1}{R_c}\cot(\omega\sqrt{\varepsilon\mu}l) = -j\frac{1}{R_c}\cot(\omega_r\sqrt{\varepsilon\mu}l + \Delta\omega\sqrt{\varepsilon\mu}l)$$

$$= -j\frac{1}{R_c}\cot(\frac{\pi}{2} + \Delta\omega\sqrt{\varepsilon\mu}l) \approx j\frac{1}{R_c}\Delta\omega\sqrt{\varepsilon\mu}l = j\frac{1}{R_c}\frac{\pi}{2}\frac{\Delta\omega}{\omega_r}$$

$L_e \ // \ C_e$ $\Rightarrow$

$$Y_e = j\omega C_e + \frac{1}{j\omega L_e} = j(\omega_r + \Delta\omega)C_e + \frac{1}{j(\omega_r + \Delta\omega)L_e}$$

$$\approx j(\omega_r + \Delta\omega)C_e + \frac{1}{j\omega_r L_e}\left(1 - \frac{\Delta\omega}{\omega_r}\right) = j\Delta\omega C_e + j\frac{1}{\omega_r L_e}\frac{\Delta\omega}{\omega_r}$$

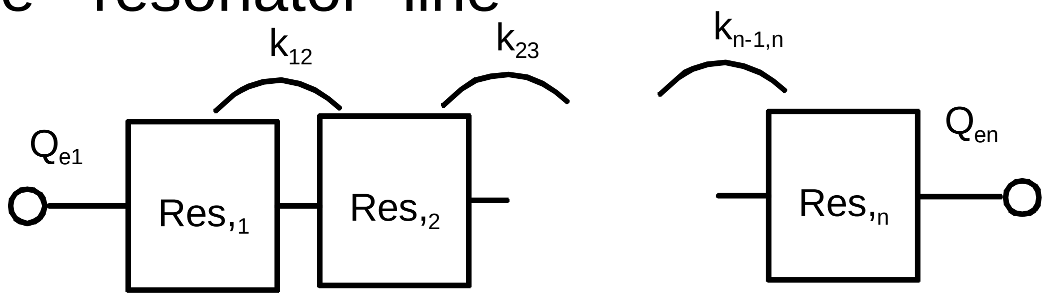
$$= j\Delta\omega C_e\left(1 + \frac{1}{\omega_r{}^2 L_e C_e}\right) = j2\Delta\omega C_e$$

$Y_{in} = Y_e$ $\Rightarrow$

$$C_e = \frac{\pi}{4\omega_r R_c}, \qquad L_e = \frac{4R_c}{\pi\omega_r}$$

Band-pass filter

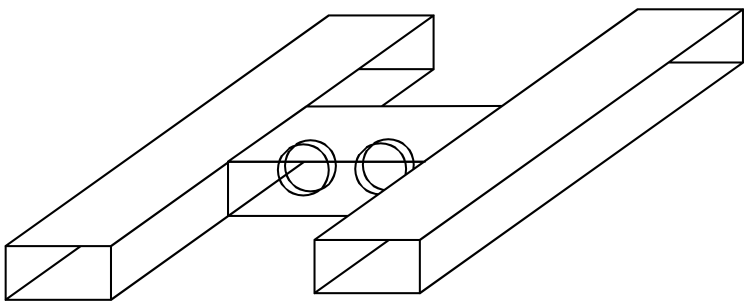
Line - resonator -line

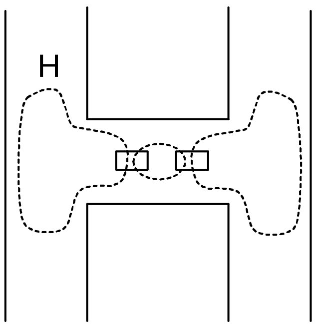


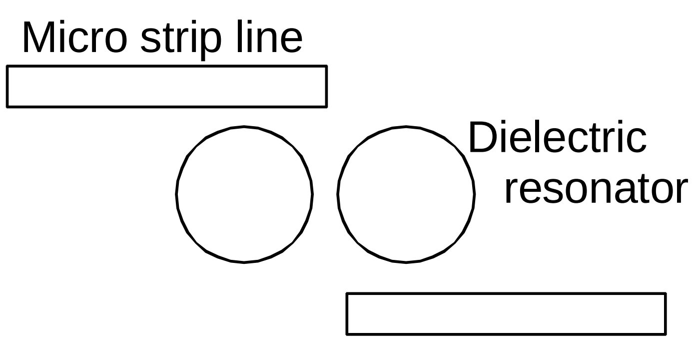
electrical length between resonators

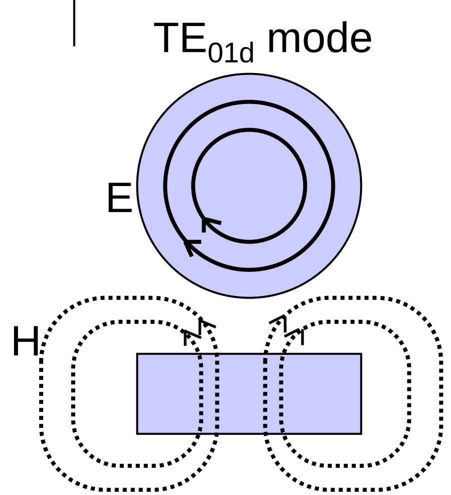
d=0 : double size (2R, 2L), invariant Q

d=$\pi/4$: transformer improved Q



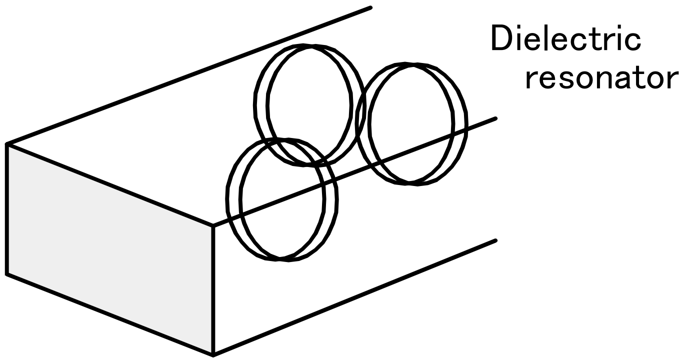


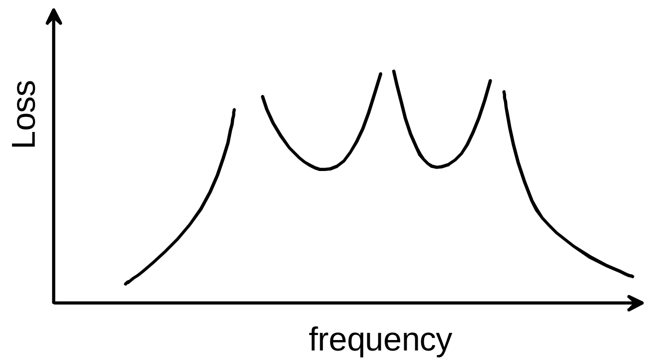




Band-stop filter

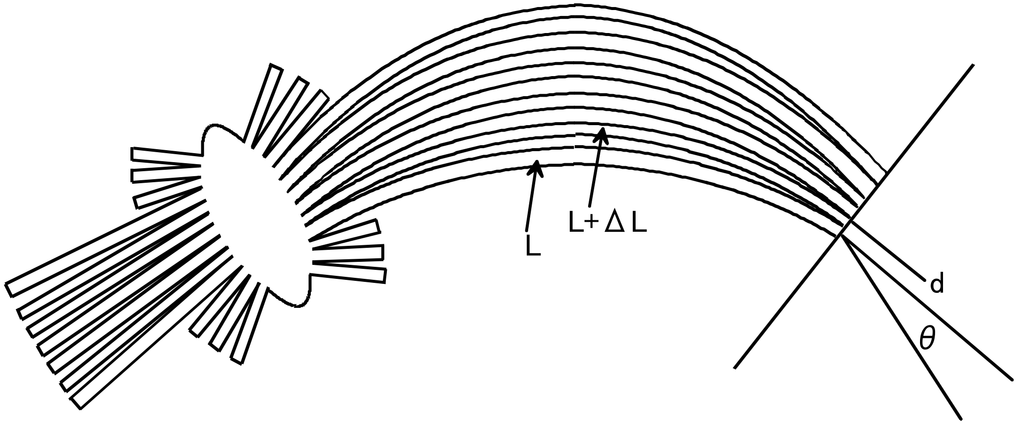
at resonance condition : Resonator prohibits transmission.





Demultiplexer : AWG

Arrayed waveguide grating (AWG)



path difference ΔL ⇒ $(n_s d \sin\theta + n_c \Delta L)k_0$

in-phase condition

$$(n_s d \sin\theta + n_c \Delta L)k_0 = 2m\pi$$

$$n_s d \sin\theta + n_c \Delta L = m\lambda$$

$$\therefore \sin\theta = \frac{m\lambda - n_c \Delta L}{n_s d}$$

Lightwave of wavelength λ focuses at angle θ.

AWG wavelength sensitivity

$$n_s d \sin\theta_1 + n_c \Delta L = m\lambda_1$$

$$n_s d \sin\theta_2 + n_c \Delta L = m\lambda_2$$

$$\begin{aligned} m(\lambda_1 - \lambda_2) = m\Delta\lambda &= n_s d(\sin\theta_1 - \sin\theta_2) \\ &= n_s d(\sin\theta_1 - \sin\theta_1 \cos\Delta\theta - \cos\theta_1 \sin\Delta\theta) \\ &\approx -n_s d\Delta\theta\cos\theta \end{aligned}$$

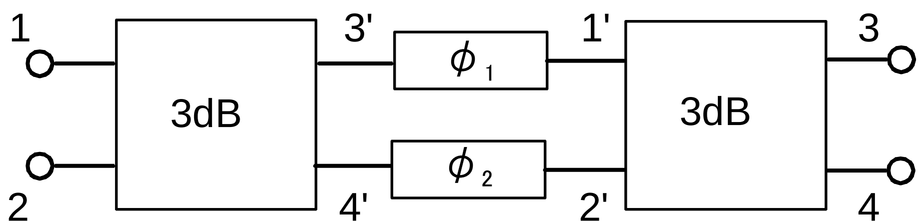
$$\therefore \Delta\lambda = \frac{-n_s d\cos\theta}{m}\Delta\theta$$

angular displacement -(lens with focal distance *f*) --> spatial displacement

$$\Delta x = f\Delta\theta = \frac{-mf}{n_s d\cos\theta}\Delta\lambda$$

$$\therefore \frac{\Delta x}{\Delta\lambda} = \frac{-mf}{n_s d\cos\theta}$$

Mach-Zehnder interferometer



3dB directional coupler

$$\begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & -j\frac{1}{\sqrt{2}} \\ 0 & 0 & -j\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -j\frac{1}{\sqrt{2}} & 0 & 0 \\ -j\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

Output

$$\Rightarrow \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}(e^{j\varphi_1} - e^{j\varphi_2}) \\ \frac{-j}{2}(e^{j\varphi_1} + e^{j\varphi_2}) \end{bmatrix}$$

output from cross port

$$\varphi_2 - \varphi_1 = 2n\pi \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ -je^{j\varphi_1} \end{bmatrix}$$

output from bar port

$$\varphi_2 - \varphi_1 = (2n-1)\pi \Rightarrow \begin{bmatrix} 0 \\ 0 \\ e^{j\varphi_1} \\ 0 \end{bmatrix}$$

Mach-Zehnder interferometer

When the phase difference is

(a) controlled by an external field:

$$\varphi_2 - \varphi_1 = \begin{cases} 2n\pi \\ (2n-1)\pi \end{cases} \Rightarrow \text{switch / modulator}$$

(b) dependent on frequency :

$$\varphi_2(f) - \varphi_1(f) = \begin{cases} 2n\pi \\ (2n-1)\pi \end{cases} \Rightarrow \text{demultiplexer}$$

(c) dependent on propagation direction :

$$\varphi_2 - \varphi_1\big|_{\pm} = \begin{cases} 2n\pi \\ (2n-1)\pi \end{cases} \Rightarrow \text{circulator}$$