11.Coupler and divider

11.1 Magic-T

11.2 Directional coupler: metallic hollow waveguide

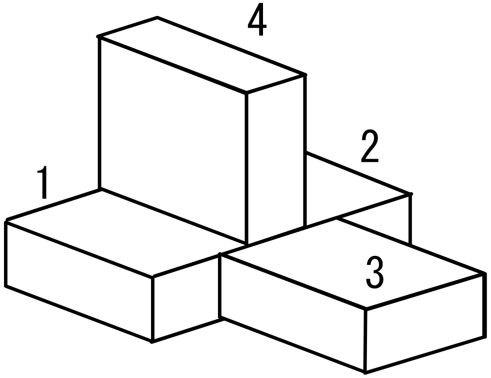
11.3 Design of a strip line hybrid

11.4 Directional coupler: coupled dielectric waveguide

Magic-T

4 port reciprocal circuit:

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{22} & S_{23} & S_{24} \\ S_{13} & S_{23} & S_{33} & S_{34} \\ S_{14} & S_{24} & S_{34} & S_{44} \end{bmatrix}$$



symmetry

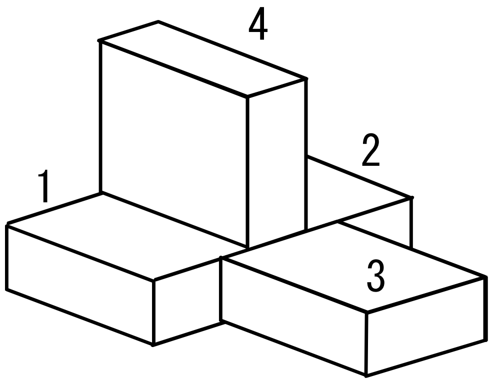
input from port 4: $S_{14} = -S_{24}$

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{11} & S_{13} & S_{24} \\ S_{13} & S_{13} & S_{33} & S_{34} \\ S_{14} & S_{24} & S_{34} & S_{44} \end{bmatrix} \Rightarrow \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{11} & S_{13} & -S_{14} \\ S_{13} & S_{13} & S_{33} & S_{34} \\ S_{14} & -S_{14} & S_{34} & S_{44} \end{bmatrix}$$

Magic-T

port 3 and 4 are isolated. $S_{34} = S_{43} = 0$

port 3 and 4 are matched. $S_{33} = S_{44} = 0$



$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{11} & S_{13} & -S_{14} \\ S_{13} & S_{13} & 0 & 0 \\ S_{14} & -S_{14} & 0 & 0 \end{bmatrix}$$

loss-less: $SS^{t*} = I$

$$\begin{cases} |S_{11}|^2 + |S_{12}|^2 + |S_{13}|^2 + |S_{14}|^2 = 1 \\ 2|S_{13}|^2 = 1 \\ 2|S_{14}|^2 = 1 \end{cases}$$

$$S_{13} = \frac{1}{\sqrt{2}} e^{j\varphi}$$

$$S_{14} = \frac{1}{\sqrt{2}} e^{j\theta}$$

$$S_{11} = S_{12} = 0$$

$$\begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

Directional coupler

reciprocity → symmetry of S

matched at port 1 and 2

port 1-2, port 3-4 are uncoupled.

$$\begin{bmatrix} 0 & 0 & S_{13} & S_{14} \\ 0 & 0 & S_{23} & S_{24} \\ S_{13} & S_{23} & S_{33} & 0 \\ S_{14} & S_{24} & 0 & S_{44} \end{bmatrix}$$

loss-less

$$\begin{cases} |S_{13}|^2 + |S_{14}|^2 = 1 \\ |S_{23}|^2 + |S_{24}|^2 = 1 \\ |S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 = 1 \\ |S_{14}|^2 + |S_{24}|^2 + |S_{44}|^2 = 1 \\ S_{13}S_{23}^* + S_{14}S_{24}^* = 0 \\ S_{13}S_{33}^* = 0 \\ S_{14}S_{44}^* = 0 \\ S_{23}S_{33}^* = 0 \\ S_{24}S_{44}^* = 0 \\ S_{13}S_{14}^* + S_{23}S_{24}^* = 0 \end{cases}$$

$$|S_{13}|^2 + |S_{14}|^2 + |S_{23}|^2 + |S_{24}|^2 + |S_{33}|^2 + |S_{44}|^2 = 2$$

$$S_{33} = S_{44} = 0$$

$$|S_{13}|^2 + |S_{23}|^2 = 1$$

$$|S_{14}| = |S_{23}|$$

$$|S_{13}| = |S_{24}|$$

Directional coupler

$$\begin{cases} |S_{13}|^2 + |S_{14}|^2 = 1 \\ |S_{23}|^2 + |S_{24}|^2 = 1 \\ |S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 = 1 \\ |S_{14}|^2 + |S_{24}|^2 + |S_{44}|^2 = 1 \\ S_{13}S_{23}^* + S_{14}S_{24}^* = 0 \\ S_{13}S_{33}^* = 0 \\ S_{14}S_{44}^* = 0 \\ S_{23}S_{33}^* = 0 \\ S_{24}S_{44}^* = 0 \\ S_{13}S_{14}^* + S_{23}S_{24}^* = 0 \end{cases}$$

$$|S_{13}| = |S_{24}|$$

$$S_{13} = S_{24} = \alpha$$

$$S_{23}^* = -S_{14}$$

$$S_{14} = j\beta$$

$$S_{23} = (-S_{14})^* = j\beta$$

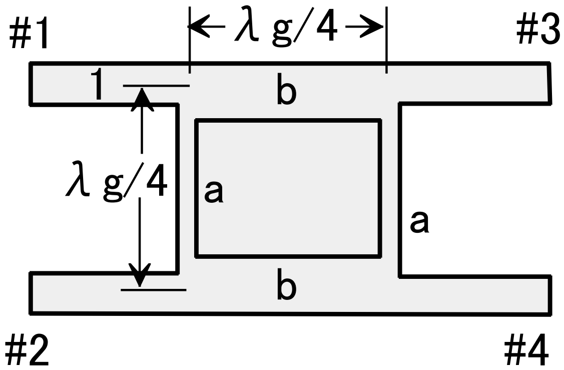
$$\begin{bmatrix} 0 & 0 & S_{13} & S_{14} \\ 0 & 0 & S_{23} & S_{24} \\ S_{13} & S_{23} & S_{33} & 0 \\ S_{14} & S_{24} & 0 & S_{44} \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 0 & 0 & \alpha & j\beta \\ 0 & 0 & j\beta & \alpha \\ \alpha & j\beta & 0 & 0 \\ j\beta & \alpha & 0 & 0 \end{bmatrix}$$

$\alpha = \beta = \dfrac{1}{\sqrt{2}}$: 3dB coupler

Design of a strip line directional coupler

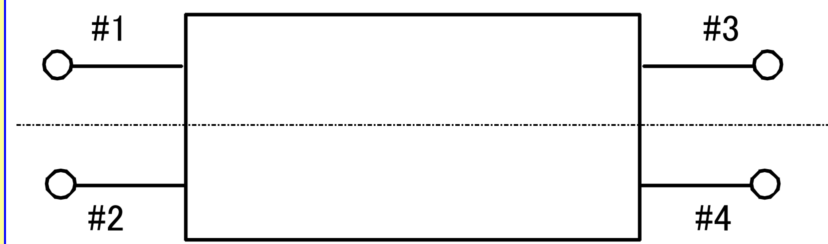


even odd

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)$$

uniaxial symmetry



$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{21} & S_{31} & S_{41} \\ S_{21} & S_{11} & S_{41} & S_{31} \\ S_{31} & S_{41} & S_{33} & S_{43} \\ S_{41} & S_{31} & S_{43} & S_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

even excitation

$$a_1 = a_2 = \frac{1}{\sqrt{2}} \qquad a_3 = a_4 = 0$$

$$b_1 = b_2 = \frac{1}{\sqrt{2}}(S_{11} + S_{21}),$$

$$b_3 = b_4 = \frac{1}{\sqrt{2}}(S_{31} + S_{41})$$

$$\Gamma_e = \frac{b_1}{a_1} = \frac{b_2}{a_2} = S_{11} + S_{21}$$

$$T_e = \frac{b_3}{a_1} = \frac{b_4}{a_2} = S_{31} + S_{41}$$

odd excitation

$$a_1 = -a_2 = \frac{1}{\sqrt{2}} \qquad a_3 = a_4 = 0$$

$$b_1 = -b_2 = \frac{1}{\sqrt{2}}(S_{11} - S_{21}),$$

$$b_3 = -b_4 = \frac{1}{\sqrt{2}}(S_{31} - S_{41})$$

$$\Gamma_o = \frac{b_1}{a_1} = \frac{b_2}{a_2} = S_{11} - S_{21}$$

$$T_o = \frac{b_3}{a_1} = \frac{b_4}{a_2} = S_{31} - S_{41}$$

uniaxial symmetric structure : S_{ij} and Γ, T

A uniaxially symmetric four-port circuit can be treated as two types of two port circuit.

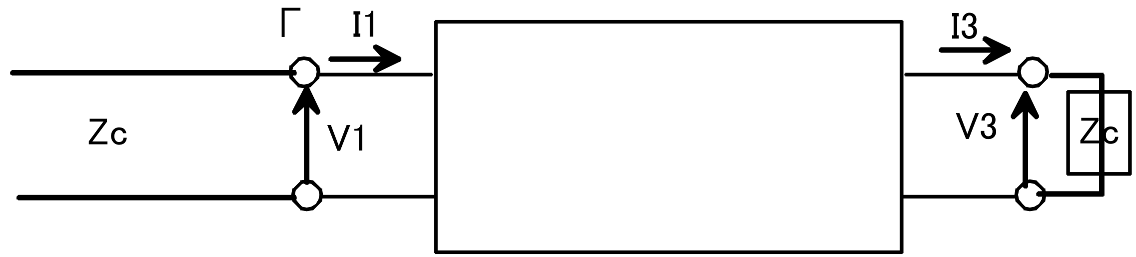
$$S_{11} = \frac{1}{2}(\Gamma_e + \Gamma_o)$$

$$S_{21} = \frac{1}{2}(\Gamma_e - \Gamma_o)$$

$$S_{31} = \frac{1}{2}(T_e + T_o)$$

$$S_{41} = \frac{1}{2}(T_e - T_o)$$

How to define Γ and T ?



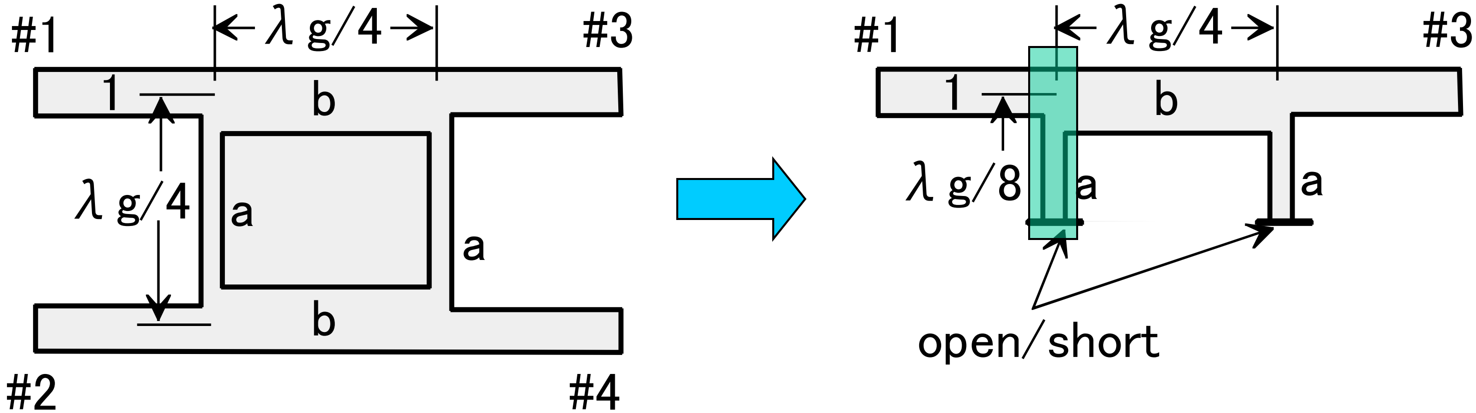
$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V_3 \\ I_3 \end{pmatrix}$$

$$Z_1 = \frac{V_1}{I_1} = \frac{AV_3 + BI_3}{CV_3 + DI_3} = \frac{AZ_c + B}{CZ_c + D}$$

$$\Gamma = \frac{Z_1 - Z_c}{Z_1 + Z_c} = \frac{A + \frac{B}{Z_c} - CZ_c - D}{A + \frac{B}{Z_c} + CZ_c + D}$$

$$T = \frac{b_3}{a_1} = \frac{V_3 - Z_c(-I_3)}{V_1 + Z_c I_1} = \frac{2}{A + \frac{B}{Z_c} + CZ_c + D}$$

Design of a strip line directional coupler : F-matrix



1) open (or short) stub:

$$V_1 = V_2$$

$$I_1 = \frac{V_2}{z} + I_2$$

$$F = \begin{pmatrix} 1 & 0 \\ \frac{1}{z} & 1 \end{pmatrix}$$

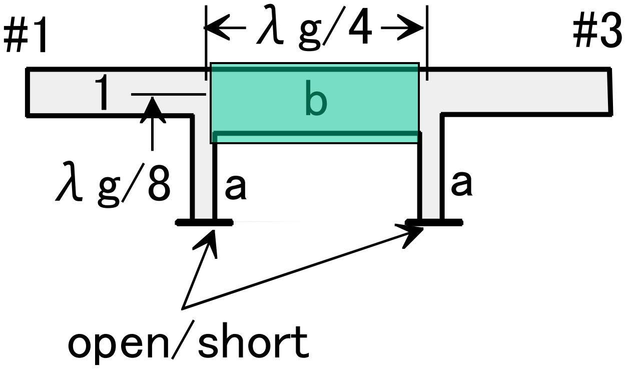
$$Z(l) = Z_c \frac{Z_L + jZ_c \tan \beta l}{Z_c + jZ_L \tan \beta l}$$

$$open: z = -jz_c = -j\frac{1}{a} \qquad (\because \beta l = \frac{\pi}{4})$$

$$short: z = jz_c = j\frac{1}{a}$$

Design of a strip line directional coupler : F-matrix

2) transmission line of length $\lambda_g/4$ ($\beta l = \dfrac{\pi}{2}$)



$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} \cos\beta l & j\dfrac{1}{b}\sin\beta l \\ jb\sin\beta l & \cos\beta l \end{pmatrix} \begin{pmatrix} V_2 \\ I_2 \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & j\dfrac{1}{b} \\ jb & 0 \end{pmatrix}$$

G and T for eigen excitaion

even excitation:

$$[F_e] = \begin{pmatrix} 1 & 0 \\ ja & 1 \end{pmatrix} \begin{pmatrix} 0 & j\frac{1}{b} \\ jb & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ ja & 1 \end{pmatrix} = \begin{pmatrix} -\frac{a}{b} & j\frac{1}{b} \\ j\left(b - \frac{a^2}{b}\right) & -\frac{a}{b} \end{pmatrix}$$

$$\Gamma_e = \frac{j\left(\frac{1}{b} - b + \frac{a^2}{b}\right)}{-\frac{2a}{b} + j\left(\frac{1}{b} + b - \frac{a^2}{b}\right)}, \qquad T_e = \frac{2}{-\frac{2a}{b} + j\left(\frac{1}{b} + b - \frac{a^2}{b}\right)}$$

odd excitation:

$$[F_o] = \begin{pmatrix} 1 & 0 \\ -ja & 1 \end{pmatrix} \begin{pmatrix} 0 & j\frac{1}{b} \\ jb & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -ja & 1 \end{pmatrix} = \begin{pmatrix} \frac{a}{b} & j\frac{1}{b} \\ j\left(b - \frac{a^2}{b}\right) & \frac{a}{b} \end{pmatrix}$$

$$\Gamma_o = \frac{j\left(\frac{1}{b} - b + \frac{a^2}{b}\right)}{\frac{2a}{b} + j\left(\frac{1}{b} + b - \frac{a^2}{b}\right)}, \qquad T_o = \frac{2}{\frac{2a}{b} + j\left(\frac{1}{b} + b - \frac{a^2}{b}\right)}$$

3dB condition

$$S_{11} = \frac{1}{2}(\Gamma_e + \Gamma_o) = 0, \quad S_{21} = \frac{1}{2}(\Gamma_e - \Gamma_o) = 0$$

⇓

$$\Gamma_e = \Gamma_o = 0$$

$$\therefore 1 + a^2 = b^2$$

$$T_e = \frac{2}{-\frac{2a}{b} + j\left(\frac{1}{b} + b - \frac{a^2}{b}\right)} \qquad T_o = \frac{2}{\frac{2a}{b} + j\left(\frac{1}{b} + b - \frac{a^2}{b}\right)}$$

$$S_{31} = \frac{1}{2}(T_e + T_o), \quad S_{41} = \frac{1}{2}(T_e - T_o)$$

⇓

$$S_{31} = -j\frac{1}{b}, S_{41} = -\frac{a}{b}$$

3-dB $|S_{31}| = |S_{41}| = \frac{1}{\sqrt{2}}$ ⇒ $a = 1, \quad b = \sqrt{2}$

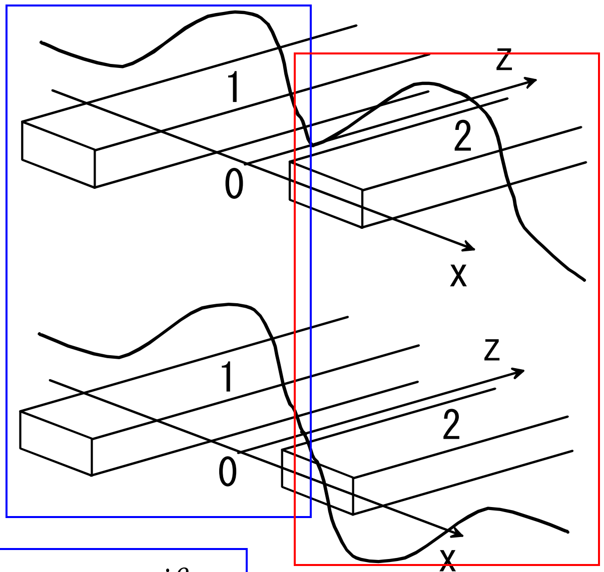
$$\begin{bmatrix} 0 & 0 & j\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & j\frac{1}{\sqrt{2}} \\ j\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & j\frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

Directional coupler : coupled dielectric waveguide

eigen excitation

even excitation:

odd excitation:



waveguide #1

$$E_1(x,z) = E_e f_e(-x) e^{-j\beta_e z} + E_o f_o(-x) e^{-j\beta_o z}$$

waveguide #2

$$E_2(x,z) = E_e f_e(x) e^{-j\beta_e z} - E_o f_o(x) e^{-j\beta_o z}$$

$$f_e(x) \approx f_o(x) \approx f(x) = f(-x)$$

Coupled dielectric waveguide : input from port 1

$$\begin{pmatrix} E_1(x,z) \\ E_2(x,z) \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} E_1(x,0) \\ E_2(x,0) \end{pmatrix}$$

input from port 1

$$\begin{pmatrix} E_1(x,z) \\ E_2(x,z) \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} f(-x) \\ 0 \end{pmatrix}$$

$$f_e(x) \approx f_o(x) \approx f(x) = f(-x) \qquad E_e = E_o = 1/2$$

$$E_1(x,z) = T_{11} f(-x) = E_e f_e(-x) e^{-j\beta_e z} + E_o f_o(-x) e^{-j\beta_o z}$$

$$= \frac{f(-x)}{2}\left(e^{-j\beta_e z} + e^{-j\beta_o z}\right) = \cos\left(\frac{\beta_e - \beta_o}{2} z\right) e^{-j\frac{\beta_e+\beta_o}{2}z} f(-x) \qquad \therefore T_{11} = \cos\left(\frac{\beta_e - \beta_o}{2} z\right) e^{-j\frac{\beta_e+\beta_o}{2}z}$$

$$E_2(x,z) = T_{21} f(-x) = E_e f_e(x) e^{-j\beta_e z} - E_o f_o(x) e^{-j\beta_o z}$$

$$= -j\sin\left(\frac{\beta_e - \beta_o}{2} z\right) e^{-j\frac{\beta_e+\beta_o}{2}z} f(-x) \qquad \therefore T_{21} = -j\sin\left(\frac{\beta_e - \beta_o}{2} z\right) e^{-j\frac{\beta_e+\beta_o}{2}z}$$

Coupled dielectric waveguide : input from port 2

input from port 2

$$\begin{pmatrix} E_1(x,z) \\ E_2(x,z) \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} 0 \\ f(x) \end{pmatrix} \qquad E_e = -E_o = 1/2$$

$$E_1(x,z) = T_{12} f(x) = E_e f_e(-x) e^{-j\beta_e z} + E_o f_o(-x) e^{-j\beta_o z}$$

$$= \frac{f(-x)}{2}\left(e^{-j\beta_e z} - e^{-j\beta_o z}\right) = -j \sin\left(\frac{\beta_e - \beta_o}{2} z\right) e^{-j\frac{\beta_e+\beta_o}{2}z} f(x) \quad \therefore T_{12} = -j\sin\left(\frac{\beta_e - \beta_o}{2} z\right) e^{-j\frac{\beta_e+\beta_o}{2}z}$$

$$E_2(x,z) = T_{22} f(x) = E_e f_e(x) e^{-j\beta_e z} - E_o f_o(x) e^{-j\beta_o z}$$

$$= \cos\left(\frac{\beta_e - \beta_o}{2} z\right) e^{-j\frac{\beta_e+\beta_o}{2}z} f(x) \qquad \therefore T_{22} = \cos\left(\frac{\beta_e - \beta_o}{2} z\right) e^{-j\frac{\beta_e+\beta_o}{2}z}$$

$$\begin{pmatrix} T_{11} & T_{21} \\ T_{12} & T_{22} \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\beta_e - \beta_o}{2} z\right) & -j\sin\left(\frac{\beta_e - \beta_o}{2} z\right) \\ -j\sin\left(\frac{\beta_e - \beta_o}{2} z\right) & \cos\left(\frac{\beta_e - \beta_o}{2} z\right) \end{pmatrix} e^{-j\frac{\beta_e+\beta_o}{2}z}$$

Directional coupler : coupled dielectric waveguide

coupling coefficient : $\kappa = \dfrac{\beta_e - \beta_o}{2}$

$$[T] = \begin{pmatrix} \cos\kappa z & -j\sin\kappa z \\ -j\sin\kappa z & \cos\kappa z \end{pmatrix} e^{-j\frac{\beta_e+\beta_o}{2}z}$$

100% coupling : $\kappa z = \pi / 2$

$$[T] = \begin{pmatrix} 0 & -j \\ -j & 0 \end{pmatrix} e^{-j\frac{\beta_e+\beta_o}{2}z}$$

50% coupling : $\kappa z = \pi / 4$

$$[T] = \begin{pmatrix} \frac{1}{\sqrt{2}} & -j\frac{1}{\sqrt{2}} \\ -j\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} e^{-j\frac{\beta_e+\beta_o}{2}z}$$