10. Eigen Vector (Eigen Excitation) and Eigen Value

10.1 Eigen vector

The input impedance measured at every port , eg., becomes identical in eigen excitation. This impedance is an eigen value of the impedance matrix of the circuit.

10.2 Eigen values and eigen vectors in circuit matrices

When eigen values are obtained for one of circuit matrices, those of remaining matrices are determined. Also, eigen vectors are identical for all matrices in a given circuit.

10.3 Method for determination of eigen values and eigen vectors

The eigen values and eigen vectors of rotationally symmetric circuit is calculated as an example.

Eigen vector and eigen value : Definition

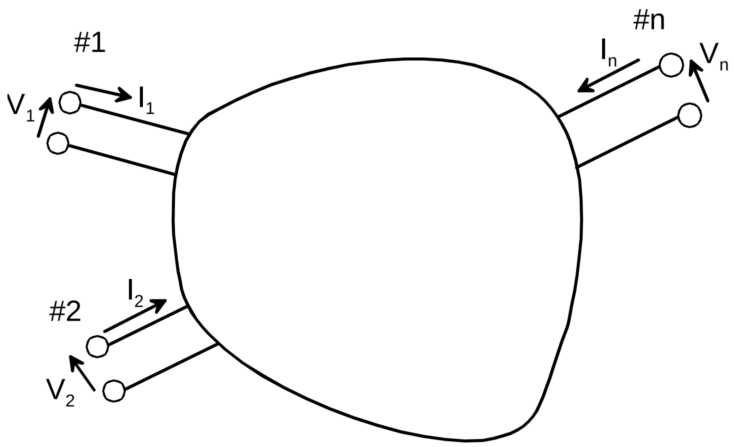
Consider the case that the following relation holds. That is, the input impedance measured at every port is identical.

$$\frac{V_1^i}{I_1^i} = \frac{V_2^i}{I_2^i} = \cdots = \frac{V_n^i}{I_n^i} = z_i$$

$$[V^i] = z_i [I^i] = [Z][I^i]$$

$$([Z] - z_i[1])[I] = 0$$

$$\det([Z] - z_i[1]) = 0$$



n solutions for z_i. ➡ eigen values

Eigen vector and eigen value : Example

Symmetric 2 port circuit

$$[Z]=\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{11} \end{bmatrix}$$

$$\det([Z]-z[1])=0$$

$$(Z_{11}-z)^2-Z_{12}{}^2=0$$

$$\therefore z_i=Z_{11}\pm Z_{12}$$

(a) $z_1=Z_{11}+Z_{12}$

$$\begin{bmatrix} -Z_{12} & Z_{12} \\ Z_{12} & -Z_{12} \end{bmatrix}\begin{bmatrix} I_1^1 \\ I_2^1 \end{bmatrix}=0$$

$\therefore I_1^1=I_2^1$ (even excitation)

(b) $z_2=Z_{11}-Z_{12}$

$$\begin{bmatrix} Z_{12} & Z_{12} \\ Z_{12} & Z_{12} \end{bmatrix}\begin{bmatrix} I_1^2 \\ I_2^2 \end{bmatrix}=0$$

$\therefore I_1^2=-I_2^2$ (odd excitation)

Relation between circuit matrices

Rational function of a square matrix [M]

$$f([M]) = c_0([M]-c_1[1])([M]-c_2[1])\cdots([M]-c_{-1}[1])^{-1}([M]-c_{-2}[1])^{-1}\cdots$$

$$[M]\mathbf{E}^i = \lambda_i \mathbf{E}^i$$

Ei : the eigen vector of [***M***].

$$c_k[1]\mathbf{E}^i = c_k\mathbf{E}^i$$

$$([M]-c_k[1])\mathbf{E}^i = (\lambda_i - c_k)\mathbf{E}^i$$

$$\mathbf{E}^i = (\lambda_i - c_k)([M]-c_k[1])^{-1}\mathbf{E}^i \Rightarrow ([M]-c_k[1])^{-1}\mathbf{E}^i = (\lambda_i - c_k)^{-1}\mathbf{E}^i$$

$$f([M])\mathbf{E}^i = c_0([M]-c_1[1])([M]-c_2[1])\cdots([M]-c_{-1}[1])^{-1}([M]-c_{-2}[1])^{-1}\cdots\mathbf{E}^i$$

$$= c_0([M]-c_1[1])([M]-c_2[1])\cdots([M]-c_{-1}[1])^{-1}(\lambda_i - c_{-2})^{-1}\cdots\mathbf{E}^i$$

$$= c_0(\lambda_i - c_1)(\lambda_i - c_2)\cdots(\lambda_i - c_{-1})^{-1}(\lambda_i - c_{-2})^{-1}\cdots\mathbf{E}^i$$

the eigen vector of rational matrix function f([M])= the eigen vector of matrix [M].

the eigen value of rational matrix function f([M])

$$c_0(\lambda_i - c_1)(\lambda_i - c_2)\cdots(\lambda_i - c_{-1})^{-1}(\lambda_i - c_{-2})^{-1}\cdots$$

Relation between circuit matrices

$$[Z][I^i] = z_i[I^i]$$

$$[Y][V^i] = y_i[V^i]$$

Here, the following relation holds.

$$[Y] = f([Z]) = [Z]^{-1} = ([Z] - c_{-1}[1])^{-1} \quad c_{-1} = 0$$

eigen vectors of impedance matrix = eigen vectors of admittance matrix

eigen values of admittance matrix is given by the following simple eq. :

$$y_i = (z_i - c_{-1})^{-1} = z_i{}^{-1}$$

Scattering matrix and impedance matrix

$$[S] = (\frac{1}{R_0}[Z] + 1)^{-1}(\frac{1}{R_0}[Z] - 1) = ([Z] + R_0)^{-1}([Z] - R_0)$$

$$S_i = (z_i + R_0)^{-1}(z_i - R_0) = \frac{z_i - R_0}{z_i + R_0}$$

How to find eigen vectors and eigen values ?

$$[P]\mathbf{u}^i = p_i \mathbf{u}^i$$

$$[Q][P]\mathbf{u}^i = p_i [Q]\mathbf{u}^i$$

Commutative matrices [***P***] and [***Q***]: $[P][Q] = [Q][P]$

$$[Q][P]\mathbf{u}^i = [P][Q]\mathbf{u}^i = [P]([Q]\mathbf{u}^i) = p_i([Q]\mathbf{u}^i)$$

$[Q]\mathbf{u}^i$ is the eigen vector of [***P***], and has the eiven value p_i.
Therefore, $[Q]\mathbf{u}^i$ is to be proportional to $\mathbf{u}^i$.

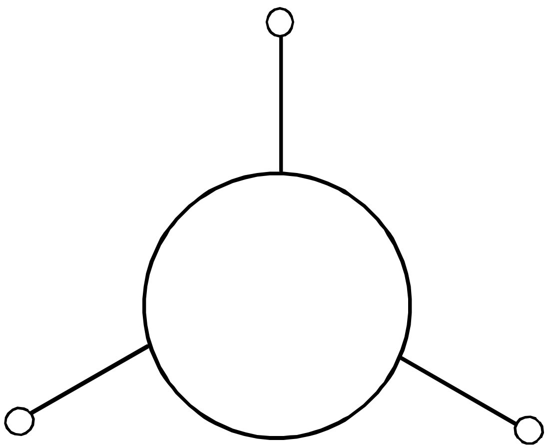
$$\therefore [Q]\mathbf{u}^i = q_i \mathbf{u}^i$$

$\mathbf{u}^i$ is also the eigen vector of matrix [***Q***].

Eigen vectors $\mathbf{u}^i$ are obtainable from either [***P***] or [***Q***], if [***P***] and [***Q***] are commutative.

How to find eigen vectors and eigen values ?

rotationally symmetric 3 port circuit



$$[R]=\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

$$[R][Z]\mathbf{i}=[R]\mathbf{v}=\mathbf{v}'=[Z]\mathbf{i}'=[Z][R]\mathbf{i}$$

$$\therefore [R][Z]=[Z][R]$$

[***R***] and [***Z***] are commutative.

The eigen vectors of $[Z]$ become identical to that of $[R]$.

$$[R]\mathbf{u}^i=r^i\mathbf{u}^i$$

$$\det([R]-r[1])=0$$

$$\det\begin{pmatrix}-r & 0 & 1\\ 1 & -r & 0\\ 0 & 1 & -r\end{pmatrix}=0, \quad r^3-1=0 \qquad \therefore r=1, e^{j\frac{2\pi}{3}}, e^{-j\frac{2\pi}{3}}$$

How to find eigen vectors and eigen values ?

$(i)\, r = 1$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \Rightarrow \mathbf{u}^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$(ii)\, r = e^{j\frac{2\pi}{3}}$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = e^{j\frac{2\pi}{3}}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \Rightarrow \mathbf{u}^2 = \begin{pmatrix} 1 \\ e^{-j\frac{2\pi}{3}} \\ e^{j\frac{2\pi}{3}} \end{pmatrix}$$

$(iii)\, r = e^{-j\frac{2\pi}{3}}$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = e^{-j\frac{2\pi}{3}}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \Rightarrow \mathbf{u}^3 = \begin{pmatrix} 1 \\ e^{j\frac{2\pi}{3}} \\ e^{-j\frac{2\pi}{3}} \end{pmatrix}$$