

8. Coupled Mode Theory

8.1 Coupled mode equation

unperturbed (uncoupled) eigen mode

$$\begin{aligned}\nabla \times \tilde{\mathbf{H}}_i &= j\omega\epsilon_0 N_i^2 \tilde{\mathbf{E}}_i \\ \nabla \times \tilde{\mathbf{E}}_i &= -j\omega\mu_0 \tilde{\mathbf{H}}_i \\ \begin{cases} \tilde{\mathbf{E}}_i = \mathbf{E}_i e^{j(\omega t - \beta_i z)} \\ \tilde{\mathbf{H}}_i = \mathbf{H}_i e^{j(\omega t - \beta_i z)} \end{cases} & (i=1,2)\end{aligned}$$

Perturbed (coupled) mode is expressed by superposition of unperturbed modes.

$$\begin{cases} \tilde{\mathbf{E}} = A(z)\tilde{\mathbf{E}}_1 + B(z)\tilde{\mathbf{E}}_2 \\ \tilde{\mathbf{H}} = A(z)\tilde{\mathbf{H}}_1 + B(z)\tilde{\mathbf{H}}_2 \end{cases}$$

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu_0 \tilde{\mathbf{H}} \quad \frac{dA}{dz} \mathbf{u}_z \times \tilde{\mathbf{E}}_1 + \frac{dB}{dz} \mathbf{u}_z \times \tilde{\mathbf{E}}_2 = 0 \quad (\text{a})$$

$$\nabla \times \tilde{\mathbf{H}} = j\omega\epsilon_0 N^2 \tilde{\mathbf{E}} \quad \frac{dA}{dz} \mathbf{u}_z \times \tilde{\mathbf{H}}_1 + \frac{dB}{dz} \mathbf{u}_z \times \tilde{\mathbf{H}}_2 - j\omega\epsilon_0 (N^2 - N_1^2) A \tilde{\mathbf{E}}_1 - j\omega\epsilon_0 (N^2 - N_2^2) B \tilde{\mathbf{E}}_2 = 0 \quad (\text{b})$$

Coupled mode equation

$$\begin{aligned} \iint (\tilde{E}_1^* \cdot (b) - \tilde{H}_1^* \cdot (a)) dS = 0 \\ \iint (\tilde{E}_2^* \cdot (b) - \tilde{H}_2^* \cdot (a)) dS = 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \frac{dA}{dz} + C_{12} \frac{dB}{dz} e^{-j(\beta_2 - \beta_1)z} + j\kappa_{11}A + j\kappa_{12}Be^{-j(\beta_2 - \beta_1)z} = 0 \\ \frac{dB}{dz} + C_{21} \frac{dA}{dz} e^{j(\beta_2 - \beta_1)z} + j\kappa_{21}Ae^{j(\beta_2 - \beta_1)z} + j\kappa_{22}B = 0 \end{aligned}$$

$$\begin{aligned} C_{ij} &= \frac{\iint \mathbf{u}_z \cdot (\mathbf{E}_i^* \times \mathbf{H}_j + \mathbf{E}_j \times \mathbf{H}_i^*) dS}{\iint \mathbf{u}_z \cdot (\mathbf{E}_i^* \times \mathbf{H}_i + \mathbf{E}_i \times \mathbf{H}_i^*) dS} \\ \kappa_{ij} &= \frac{\omega \epsilon_0 \iint (N^2 - N_j^2) \mathbf{E}_i^* \cdot \mathbf{E}_j dS}{\iint \mathbf{u}_z \cdot (\mathbf{E}_i^* \times \mathbf{H}_i + \mathbf{E}_i \times \mathbf{H}_i^*) dS} \quad (i, j = 1, 2) \end{aligned}$$

normalization condition:

$$\iint \mathbf{u}_z \cdot (\mathbf{E}_i^* \times \mathbf{H}_i + \mathbf{E}_i \times \mathbf{H}_i^*) dS = 4P_i = 1 \quad \Rightarrow \quad \begin{cases} C_{12}^* = C_{21} \\ \kappa_{11} = \kappa_{11}^*, \quad \kappa_{22} = \kappa_{22}^* \end{cases}$$

Coupled mode equation

power carried by electromagnetic wave

$$\begin{aligned}
 P &= \frac{1}{2} \iint (\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*) \cdot \mathbf{u}_z dS \\
 &= \frac{1}{4} (|A|^2 + |B|^2 + AB^* C_{12}^* e^{j2\delta z} + A^* B C_{12} e^{-j2\delta z}) \\
 \delta &= \frac{\beta_2 - \beta_1}{2}
 \end{aligned}$$

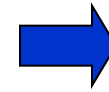
power conservation condition : $\frac{dP}{dz} = 0$

$$\begin{aligned}
 &\frac{dA}{dz} (A^* + B^* C_{12}^* e^{j2\delta z}) + \frac{dA^*}{dz} (A + B C_{12} e^{-j2\delta z}) \\
 &+ \frac{dB}{dz} (B^* + A^* C_{12} e^{-j2\delta z}) + \frac{dB^*}{dz} (B + A C_{12}^* e^{j2\delta z}) \\
 &+ j2\delta (AB^* C_{12}^* e^{j2\delta z} - A^* B C_{12} e^{-j2\delta z}) = 0 \\
 &(-\kappa_{21} + \kappa_{12}^* + 2\delta C_{12}^*) AB^* e^{j2\delta z} + (-\kappa_{12} + \kappa_{21}^* - 2\delta C_{12}) A^* B e^{-j2\delta z} = 0 \\
 &-\kappa_{21} + \kappa_{12}^* + 2\delta C_{12}^* = 0 \\
 &\therefore \kappa_{21} = \kappa_{12}^* + 2\delta C_{12}^*
 \end{aligned}$$

Coupled mode equation

in case of symmetric coupled waveguide: $\delta = 0$

in case of weakly coupled waveguides : $C_{12} \approx 0$



$$\kappa_{21} = \kappa_{12}^*$$

$$\kappa_a = \frac{\kappa_{12} - \kappa_{22}C_{12}}{1 - |C_{12}|^2}, \quad \kappa_b = \frac{\kappa_{21} - \kappa_{11}C_{21}}{1 - |C_{12}|^2}$$

$$\alpha_a = \frac{-\kappa_{11} + \kappa_{21}C_{12}}{1 - |C_{12}|^2}, \quad \alpha_b = \frac{-\kappa_{22} + \kappa_{12}C_{21}}{1 - |C_{12}|^2}$$

$$\frac{dA}{dz} = -j\kappa_a B(z)e^{-j2\delta z} + j\alpha_a A(z)$$

$$\frac{dB}{dz} = -j\kappa_b A(z)e^{j2\delta z} + j\alpha_b B(z)$$

$$C_{12} = 0 \quad \kappa_{11} = \kappa_{22} = 0 \quad \Rightarrow \quad \kappa_a = \kappa_{12} \quad \kappa_b = \kappa_{21} \quad \alpha_a = \alpha_b = 0$$

$$\frac{dA}{dz} = -j\kappa_{12} B(z)e^{-j(\beta_2 - \beta_1)z}$$

$$\frac{dB}{dz} = -j\kappa_{21} A(z)e^{j(\beta_2 - \beta_1)z} \quad (\text{coupled mode equation})$$

8.2 Coupling coefficients

Consider a TE mode propagating in a slab waveguide extending y-z plane.

$$\mathbf{E} = (0, E_y, 0)$$

$$\mathbf{H} = (H_x, 0, H_z)$$

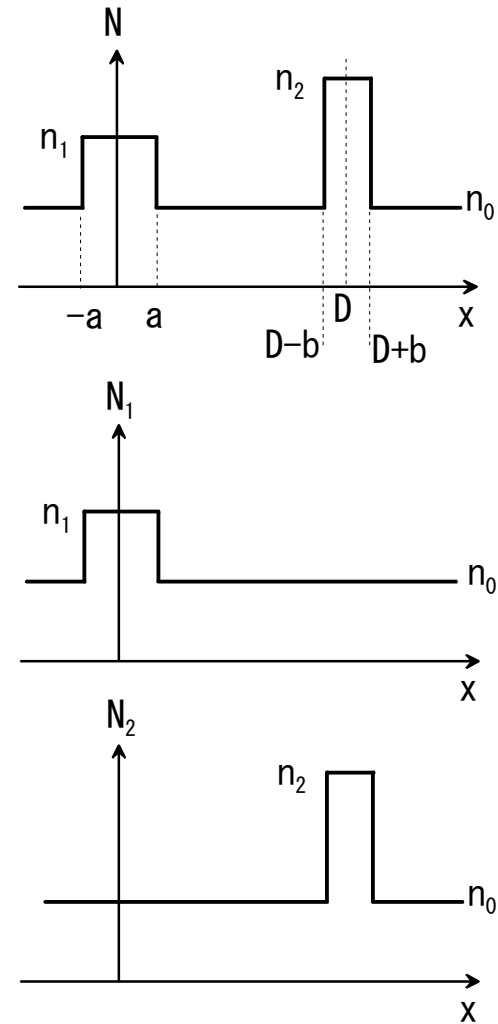
$$H_x = -\frac{\beta}{\omega\mu_0} E_y$$

$$(\mathbf{E}_1^* \times \mathbf{H}_1 + \mathbf{E}_1 \times \mathbf{H}_1^*) \cdot \mathbf{u}_z$$

$$= 2\text{Re}[(\mathbf{E}_1^* \times \mathbf{H}_1) \cdot \mathbf{u}_z] = \frac{2\beta}{\omega\mu_0} |E_{1y}|^2$$

normalization condition:

$$\iint (\mathbf{E}_1^* \times \mathbf{H}_1 + \mathbf{H}_1 \times \mathbf{E}_1^*) \cdot \mathbf{u}_z = 4P_1 = 1$$



Coupling coefficients

coupling coefficient:

$$C_{12} = \int (\mathbf{E}_1^* \times \mathbf{H}_2 + \mathbf{E}_2 \times \mathbf{H}_1^*) \cdot \mathbf{u}_z dx$$

$$= \int (-E_{1y}^* H_{2x} - E_{2y} H_{1x}^*) dx$$

$$= \frac{\beta_1 + \beta_2}{\omega \mu_0} \int E_{1y}^* E_{2y} dx$$

$$\cong \frac{2\omega \varepsilon_0 n}{k_0} \int E_{1y}^* E_{2y} dx$$

$$\kappa_{11} = \omega \varepsilon_0 \int (N^2 - N_1^2) E_1^* \cdot E_1 dx = \omega \varepsilon_0 \int_{D-b}^{D+b} (n_2^2 - n_0^2) |E_{1y}|^2 dx$$

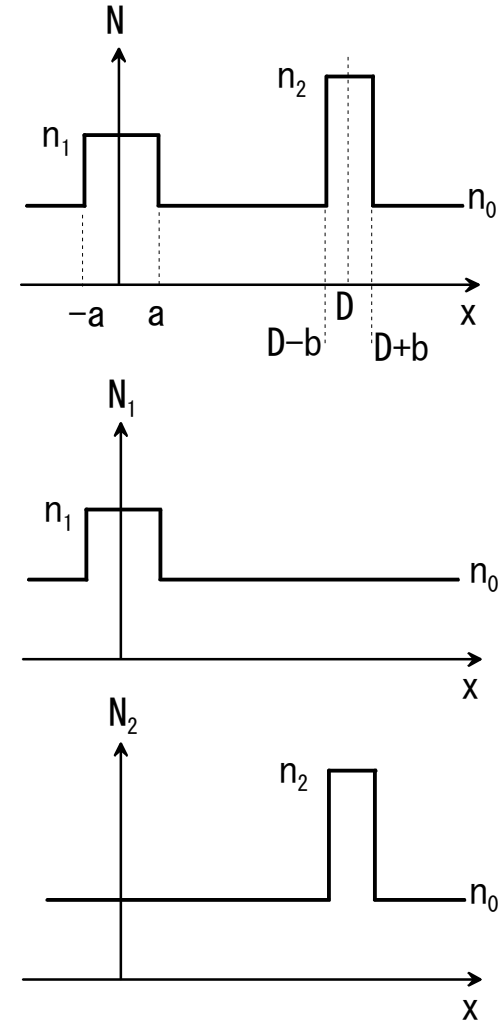
$$\kappa_{12} = \omega \varepsilon_0 \int (N^2 - N_2^2) E_1^* \cdot E_2 dx = \omega \varepsilon_0 \int_{-a}^a (n_1^2 - n_0^2) E_{1y}^* E_{2y} dx$$

field intensity : core region=1, clad region= $\eta (<< 1)$

$$\kappa_{11} \propto \omega \varepsilon_0 (n_2^2 - n_0^2) \eta^2$$

$$\kappa_{12} \propto \omega \varepsilon_0 (n_1^2 - n_0^2) \eta$$

$$C_{12} \propto (\omega \varepsilon_0 n / k_0) \eta \quad n/k_0 \ll 1$$



8.3 Co-directional mode coupling

$$C_{12} = 0 \quad \kappa_{21} = \kappa_{12}^*$$

coupled mode equation:

$$\frac{dA}{dz} = -j\kappa_{12}B(z)e^{-j(\beta_2 - \beta_1)z}$$

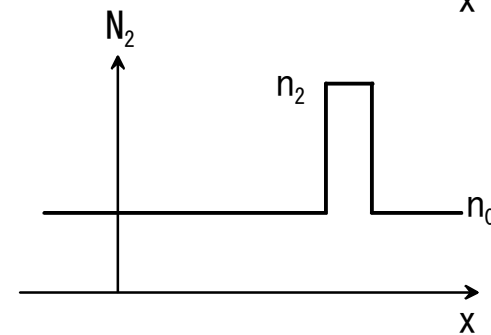
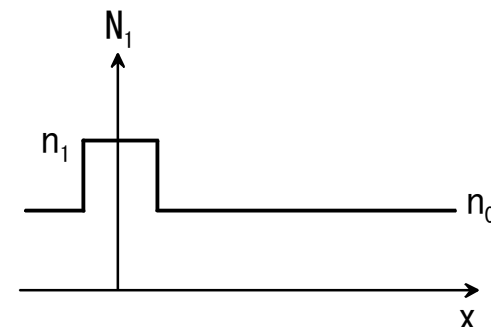
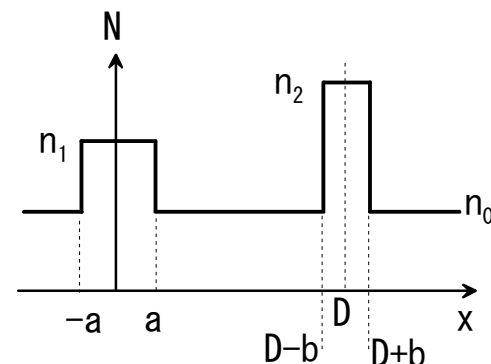
$$\frac{dB}{dz} = -j\kappa_{21}A(z)e^{j(\beta_2 - \beta_1)z}$$

$$\begin{cases} A(z) = (a_1 e^{jqz} + a_2 e^{-jqz}) e^{-j\delta z} \\ B(z) = (b_1 e^{jqz} + b_2 e^{-jqz}) e^{j\delta z} \end{cases}$$

$$\begin{cases} A(0) = a_1 + a_2 \\ B(0) = b_1 + b_2 \end{cases}$$

$$\begin{bmatrix} A(z) \\ B(z) \end{bmatrix} = \begin{bmatrix} (\cos qz + j \frac{\delta}{q} \sin qz) e^{-j\delta z} & -j \frac{\kappa_{12}}{q} \sin qz e^{-j\delta z} \\ -j \frac{\kappa_{12}^*}{q} \sin qz e^{j\delta z} & (\cos qz - j \frac{\delta}{q} \sin qz) e^{j\delta z} \end{bmatrix} \begin{bmatrix} A(0) \\ B(0) \end{bmatrix}$$

$$q = \sqrt{|\kappa_{12}|^2 + \delta^2}$$



Co-directional coupling

input from guide-1: $A(0) = A_0$ $B(0) = 0$

$$\begin{cases} A(z) = A_0 \left(\cos qz + j \frac{\delta}{q} \sin qz \right) e^{-j\delta z} \\ B(z) = A_0 \left(-j \frac{\kappa_{12}^*}{q} \right) \sin qz e^{j\delta z} \end{cases}$$

$$\frac{P_a}{P_0} = \frac{|A(z)|^2}{|A_0|^2} = \cos^2 qz + \frac{\delta^2}{q^2} \sin^2 qz = 1 - \frac{|\kappa_{12}|^2}{q^2} \sin^2 qz$$

$$\frac{P_b}{P_0} = \frac{|\kappa_{12}|^2}{q^2} \sin^2 qz = \frac{1}{1 + \frac{\delta^2}{|\kappa_{12}|^2}} \sin^2 qz$$

coupling length: $L_c = \frac{\pi}{2q} = \frac{\pi}{2\sqrt{|\kappa_{12}|^2 + \delta^2}} \quad \Rightarrow \quad \frac{P_b}{P_0} = \frac{1}{1 + \frac{\delta^2}{|\kappa_{12}|^2}}$

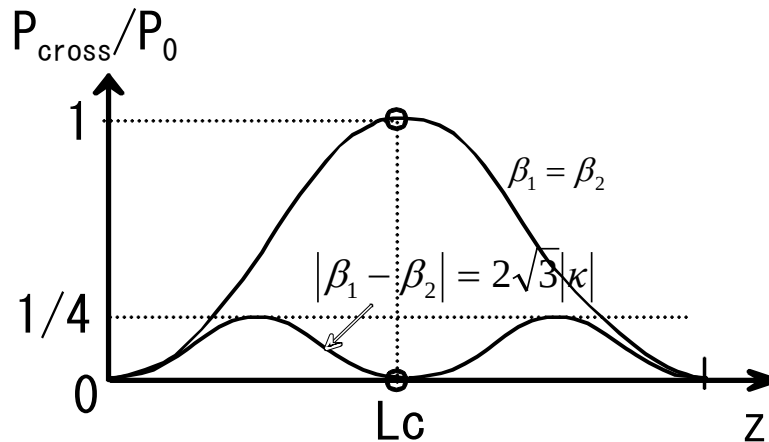
phase matched ($\delta = 0$) $\Rightarrow L_c = \frac{\pi}{2|\kappa_{12}|}$, $\frac{P_b}{P_0} = 1$

Co-directional coupling

coupled power: $\frac{P_{cross}}{P_{in}} = \frac{1}{1 + \frac{(\beta_2 - \beta_1)^2}{4|\kappa_{12}|^2}} \sin^2 \sqrt{|\kappa_{12}|^2 + \frac{(\beta_2 - \beta_1)^2}{4}} z$

symmetric coupler $\beta_1 = \beta_2$: $\frac{P_{cross}}{P_{in}} = \sin^2 |\kappa_{12}| z$

asymmetric coupler: $|\beta_1 - \beta_2| = 2\sqrt{3}|\kappa| \Rightarrow \frac{P_{cross}}{P_{in}} = \frac{1}{4} \sin^2 2|\kappa| z$



cross state: $\beta_1 = \beta_2$

bar state: $|\beta_1 - \beta_2| = 2\sqrt{3}|\kappa|$