

7. Optical Fiber

7.1 Electromagnetic wave guided in an optical fiber

Cylindrical coordinate system:

$$\mathbf{E} = \mathbf{E}(r, \theta)e^{j(\omega t - \beta z)}, \mathbf{H} = \mathbf{H}(r, \theta)e^{j(\omega t - \beta z)}$$

$$\begin{aligned} \frac{1}{r} \frac{\partial E_z}{\partial \theta} + j\beta E_\theta &= -j\omega\mu_0 H_r \\ -j\beta E_r - \frac{\partial E_z}{\partial r} &= -j\omega\mu_0 H_\theta \\ \frac{1}{r} \frac{\partial(rE_\theta)}{\partial r} - \frac{1}{r} \frac{\partial E_r}{\partial \theta} &= -j\omega\mu_0 H_z \\ \frac{1}{r} \frac{\partial H_z}{\partial \theta} + j\beta H_\theta &= j\omega\epsilon_0 n^2 E_r \\ -j\beta H_r - \frac{\partial H_z}{\partial r} &= j\omega\epsilon_0 n^2 E_\theta \\ \frac{1}{r} \frac{\partial(rH_\theta)}{\partial r} - \frac{1}{r} \frac{\partial H_r}{\partial \theta} &= j\omega\epsilon_0 n^2 E_z \end{aligned} \quad \begin{aligned} &\longrightarrow \frac{1}{r} \frac{\partial E_z}{\partial \theta} + \frac{\beta}{\omega\epsilon_0 n^2} \left(-j\beta H_r - \frac{\partial H_z}{\partial r} \right) = -j\omega\mu_0 H_r \\ &H_r = \frac{j}{\beta_t^2} \left(\frac{\omega\epsilon_0 n^2}{r} \frac{\partial E_z}{\partial \theta} - \beta \frac{\partial H_z}{\partial r} \right) \\ &\beta_t^2 = n^2 k_0^2 - \beta^2 \\ &E_\theta = -\frac{j}{\beta_t^2} \left(\frac{\beta}{r} \frac{\partial E_z}{\partial \theta} - \omega\mu_0 \frac{\partial H_z}{\partial r} \right) \end{aligned}$$

EMW in optical fiber

$$E_r = -\frac{j}{\beta_t^2} \left(\beta \frac{\partial E_z}{\partial r} + \omega \mu_0 \frac{1}{r} \frac{\partial H_z}{\partial \theta} \right)$$

$$H_\theta = -\frac{j}{\beta_t^2} \left(\omega \varepsilon_0 n^2 \frac{\partial E_z}{\partial r} + \frac{\beta}{r} \frac{\partial H_z}{\partial \theta} \right)$$

wave equations:
$$\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r} \frac{\partial H_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H_z}{\partial \theta^2} + \beta_t^2 H_z = 0$$

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r} \frac{\partial E_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \theta^2} + \beta_t^2 E_z = 0$$

variable separation $H_z, E_z = R(r)\Theta(\theta)$

$$\frac{r^2}{R} \left(\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) + \beta_t^2 r^2 + \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = 0$$

$$\left\{ \begin{array}{l} \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = -n^2 \\ \frac{r^2}{R} \left(\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) + \beta_t^2 r^2 = n^2 \end{array} \right.$$

EMW in optical fiber

Bessel differential equations:

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left(\beta_t^2 - \frac{n^2}{r^2} \right) R = 0$$

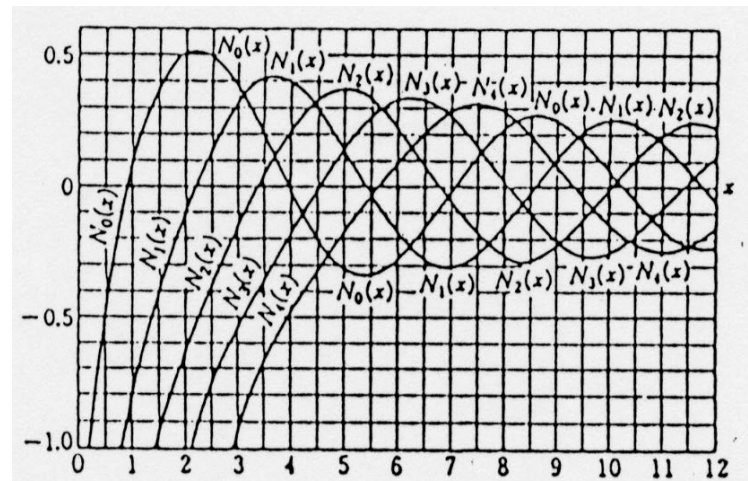
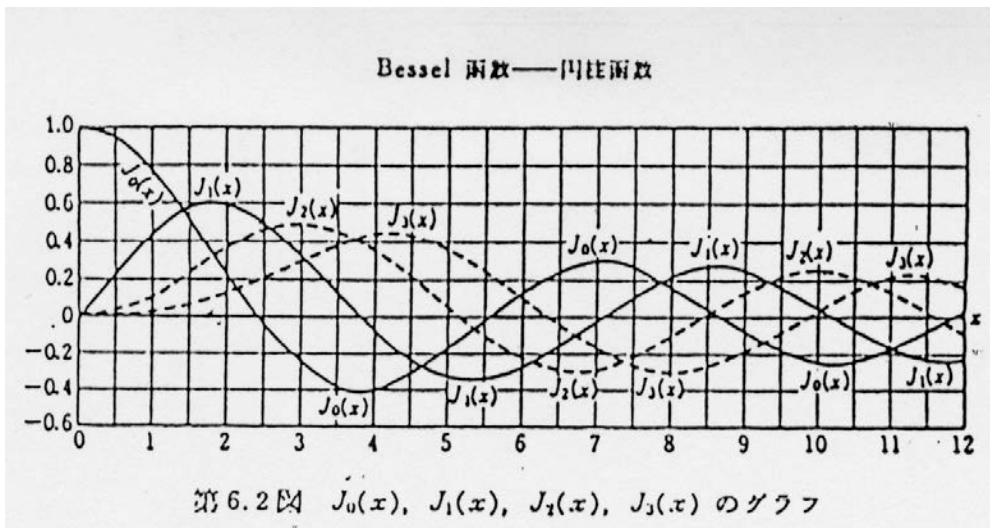
$$R(r) = \begin{cases} AJ_n(\beta_t r) + BN_n(\beta_t r) & : \beta_t^2 > 0 \\ CK_n(|\beta_t| r) + DI_n(|\beta_t| r) & : \beta_t^2 < 0 \end{cases}$$

core region: $\beta_t^2 = n_1^2 k_0^2 - \beta^2 > 0$
 $N_n(\beta_t r) \rightarrow \infty \quad (r \rightarrow 0) \quad \therefore B = 0$

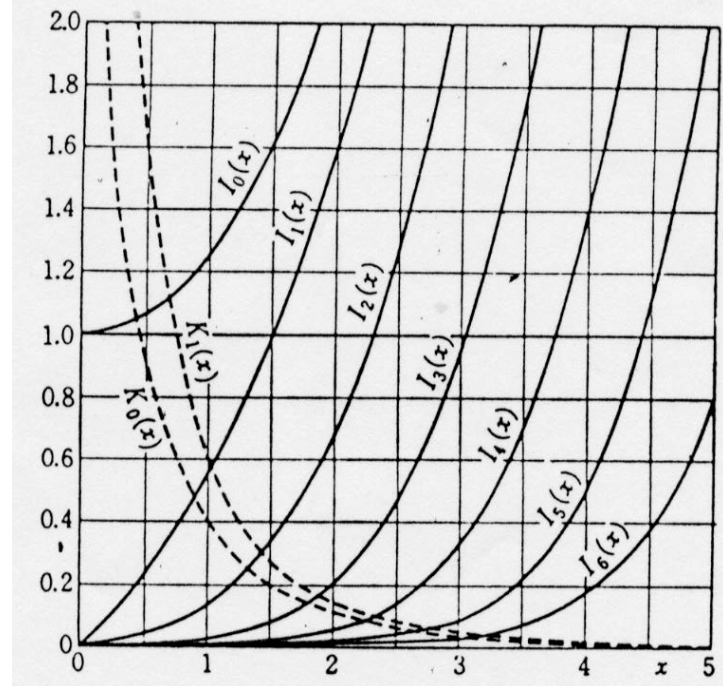
clad region: $\beta_t^2 = n_2^2 k_0^2 - \beta^2 < 0$
 $I_n(\beta_t r) \rightarrow \infty \quad (r \rightarrow \infty) \quad \therefore D = 0$

$$\Theta = \begin{cases} \cos n\theta \\ \sin n\theta \end{cases}$$

Bessel functions



第 6.3 図 $N_0(x)$, $N_1(x)$, $N_2(x)$, $N_3(x)$, $N_1'(x)$ のグラフ



EMW in optical fiber

$$E_z = \begin{cases} AJ_n\left(\frac{u}{a}r\right)\sin n\theta & : r \leq a \\ CK_n\left(\frac{w}{a}r\right)\sin n\theta & : r \geq a \end{cases}$$

$$H_z = \begin{cases} BJ_n\left(\frac{u}{a}r\right)\cos n\theta & : r \leq a \\ DK_n\left(\frac{w}{a}r\right)\cos n\theta & : r \geq a \end{cases}$$

$$u = \beta_{t1}a = \sqrt{n_1^2k_0^2 - \beta^2}a$$

$$w = \beta_{t2}a = \sqrt{\beta^2 - n_2^2k_0^2}a$$

$n = 0$: TE or TM $\rightarrow$ axial symmetry

$n \neq 0$: hybrid $\rightarrow E_z \neq 0$ and $H_z \neq 0$

7.2 Eigen value equation

boundary conditions: $E_z, H_z, E_\theta, H_\theta$ are to be continuous at $r=a$.

$$AJ_n(u) = CK_n(w)$$

$$BJ_n(u) = DK_n(w)$$

$$-A \frac{j\beta}{(u/a)^2} \frac{n}{a} J_n(u) + B \frac{j\omega\mu_0}{u/a} J_n'(u) = C \frac{j\beta}{(w/a)^2} \frac{n}{a} K_n(w) - D \frac{j\omega\mu_0}{w/a} K_n'(w)$$

$$-A \frac{j\omega\varepsilon_0 n_1^2}{u/a} J_n'(u) + B \frac{j\beta}{(u/a)^2} \frac{n}{a} J_n(u) = C \frac{j\omega\varepsilon_0 n_2^2}{w/a} K_n'(w) - D \frac{j\beta}{(w/a)^2} \frac{n}{a} K_n(w)$$

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0 \quad - \quad \det[\mathbf{M}] = 0 \quad \downarrow$$

$$\left[\frac{J_n'(u)}{uJ_n(u)} + \frac{K_n'(w)}{wK_n(w)} \right] \left[\frac{n_1^2}{n_2^2} \frac{J_n'(u)}{uJ_n(u)} + \frac{K_n'(w)}{wK_n(w)} \right] = n^2 \left(\frac{1}{u^2} + \frac{1}{w^2} \right) \left(\frac{n_1^2}{n_2^2} \frac{1}{u^2} + \frac{1}{w^2} \right)$$

eigen mode in an optical fiber: TE and TM mode

(1) $n = 0$ $\Theta(\theta) = \text{const}$

$r \geq a$

$$E_z = CK_0\left(\frac{w}{a}r\right)$$

$$E_r = C \frac{j\beta}{w/a} K_0'\left(\frac{w}{a}r\right)$$

$$E_\theta = -D \frac{j\omega\mu_0}{w/a} K_0'\left(\frac{w}{a}r\right)$$

$$H_z = DK_0\left(\frac{w}{a}r\right)$$

$$H_r = D \frac{j\beta}{w/a} K_0'\left(\frac{w}{a}r\right)$$

$$H_\theta = C \frac{j\omega\varepsilon_0 n_2^2}{w/a} K_0'\left(\frac{w}{a}r\right)$$

$a \geq r$

$$E_z = AJ_0\left(\frac{u}{a}r\right)$$

$$E_r = -A \frac{j\beta}{u/a} J_0'\left(\frac{u}{a}r\right)$$

$$E_\theta = B \frac{j\omega\mu_0}{u/a} J_0'\left(\frac{u}{a}r\right)$$

$$H_z = BJ_0\left(\frac{u}{a}r\right)$$

$$H_r = -B \frac{j\beta}{u/a} J_0'\left(\frac{u}{a}r\right)$$

$$H_\theta = -A \frac{j\omega\varepsilon_0 n_1^2}{u/a} J_0'\left(\frac{u}{a}r\right)$$

TM_{0l}

$$\frac{n_1^2}{n_2^2} \frac{J_0'(u)}{uJ_0(u)} + \frac{K_0'(w)}{wK_0(w)} = 0$$

TE_{0l}

$$\frac{J_0'(u)}{uJ_0(u)} + \frac{K_0'(w)}{wK_0(w)} = 0$$

eigen mode in an optical fiber: hybrid mode

(2) $n \neq 0$: hybrid $\rightarrow E_z \neq 0$ and $H_z \neq 0$

$n_1 \approx n_2$ weakly guided approximation

$$\left[\frac{J_n'(u)}{uJ_n(u)} + \frac{K_n'(w)}{wK_n(w)} \right]^2 = n^2 \left(\frac{1}{u^2} + \frac{1}{w^2} \right)^2$$

$$\therefore \frac{J_n'(u)}{uJ_n(u)} + \frac{K_n'(w)}{wK_n(w)} = \pm n \left(\frac{1}{u^2} + \frac{1}{w^2} \right)$$

With the help of Bessel function's equalities, all the modes can be included in the following characteristic equation.

$$\frac{uJ_{m-1}(u)}{J_m(u)} = -\frac{wK_{m-1}(w)}{K_m(w)}$$

$$m = \begin{cases} 1 & : TE, TM \text{ mode} \\ n+1 & : EH \text{ mode} \\ n-1 & : HE \text{ mode} \end{cases} \quad LP_{ml}$$

7.3 Guided mode

Normalized frequency:

$$V^2 = u^2 + w^2 = k_0^2(n_1^2 - n_2^2)a^2 \approx k_0^2 n_1^2 a^2 2 \frac{n_1 - n_2}{n_1} = k_0^2 n_1^2 a^2 2\Delta \quad V = k_0 n_1 a \sqrt{2\Delta}$$

Characteristic equation:

$$\frac{u J_{m-1}(u)}{J_m(u)} + \frac{w K_{m-1}(w)}{K_m(w)} = 0$$

$$V^2 = u^2 + w^2$$

Cut-off : $w=0$

$$J_{m-1}(u) = 0 \text{ the } l\text{-th zero of order } m-1 \text{ Bessel function } u = j_{m-1,l}$$

$$\downarrow$$

$$V_c = j_{m-1,l} \quad (m \geq 1, l \geq 1) \longrightarrow LP_{ml} \text{ mode} (EH_{m-1,l} \quad HE_{m+1,l})$$

$$LP_{1l} \text{ mode} (TM_{0l} \quad TE_{0l} \quad HE_{2l})$$

$$LP_{01} (HE_{11} \text{ } n=1, m=0) \text{ mode} \longrightarrow V_c = 0$$

$$LP_{11} \text{ mode} \longrightarrow J_0(V_c) = 0 \rightarrow V_c = 2.405$$

single mode

$$V \leq 2.405$$

7.4 Group velocity and wavelength dispersion

group velocity:

$$v_g = \frac{1}{\frac{\partial \beta}{\partial \omega}} = v_p - \lambda \frac{dv_p}{d\lambda}$$

Group Velocity Dispersion (GVD)

$$\frac{dv_g}{d\lambda} = \frac{d}{d\lambda} \left(\frac{1}{t_g} \right) = -\frac{1}{t_g^2} \frac{dt_g}{d\lambda} = -v_g^2 \frac{dt_g}{d\lambda}$$

material dispersion

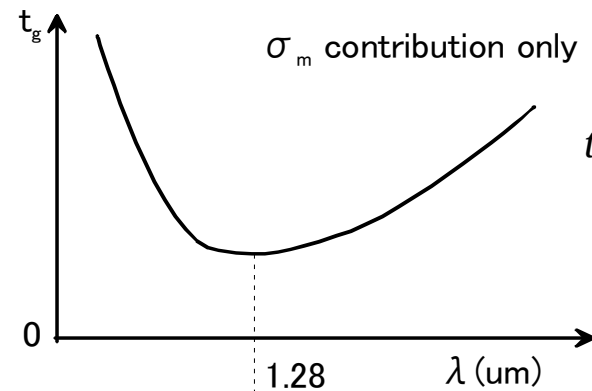
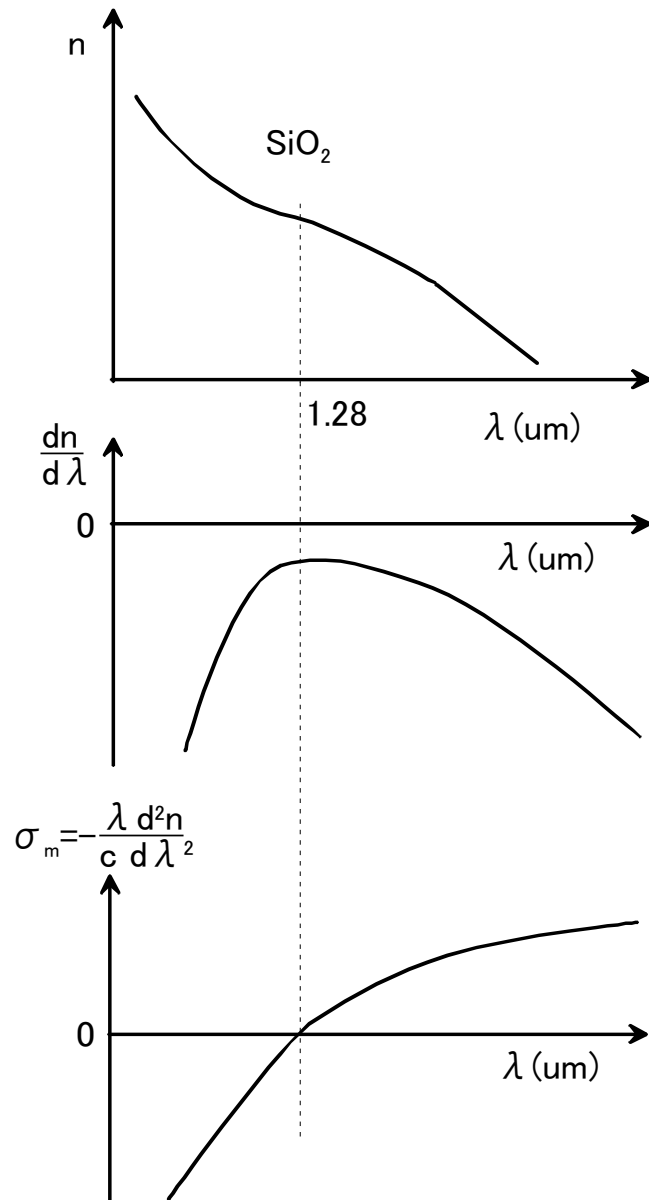
$$v_g = \frac{c}{n} - \lambda \frac{d}{d\lambda} \left(\frac{c}{n} \right) = \frac{c}{n} \left(1 + \frac{\lambda}{n} \frac{dn}{d\lambda} \right)$$

$$t_g = v_g^{-1} \approx \frac{n}{c} \left(1 - \frac{\lambda}{n} \frac{dn}{d\lambda} \right)$$

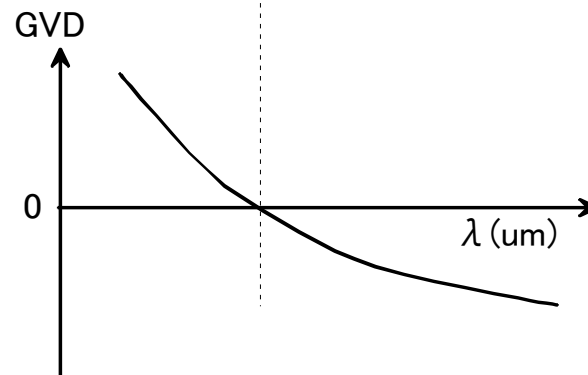
$$\therefore \sigma_m = \frac{dt_g}{d\lambda} = -\frac{\lambda}{c} \frac{d^2 n}{d\lambda^2}$$

$$\text{GVD} \quad \frac{dv_g}{d\lambda} = \left(\frac{c}{n} \right)^2 \left(1 + \frac{\lambda}{n} \frac{dn}{d\lambda} \right)^2 \frac{\lambda}{c} \frac{d^2 n}{d\lambda^2} \approx \frac{c\lambda}{n^2} \frac{d^2 n}{d\lambda^2} = -\left(\frac{c}{n} \right)^2 \sigma_m$$

material dispersion



$$t_g = v_g^{-1} \approx \frac{n}{c} \left(1 - \frac{\lambda}{n} \frac{dn}{d\lambda} \right)$$



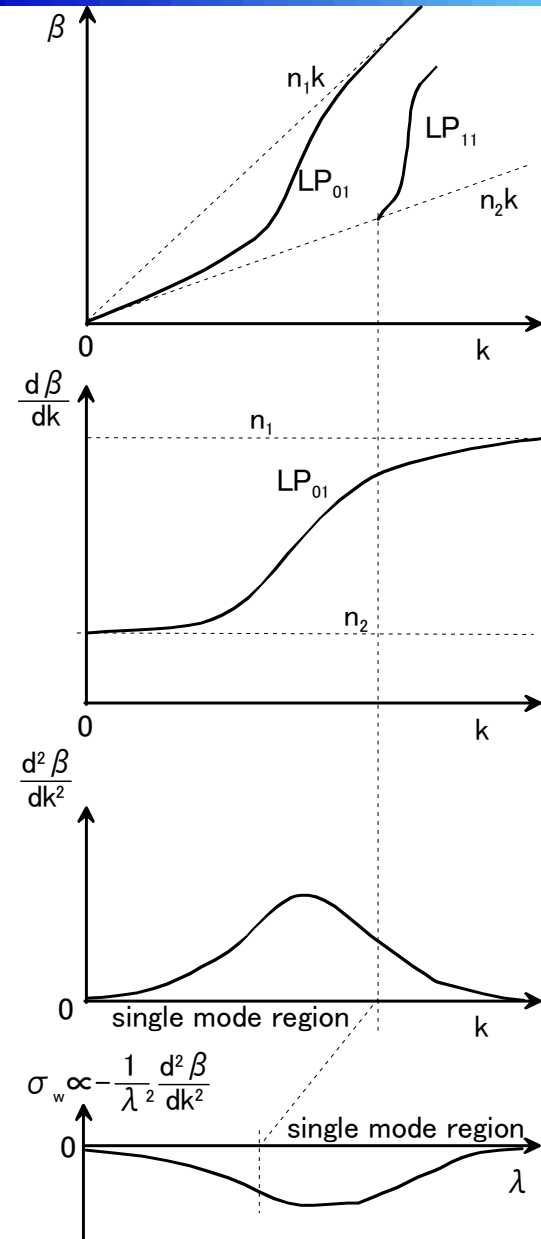
$$\frac{dv_g}{d\lambda} = -\left(\frac{c}{n} \right)^2 \sigma_m$$

waveguide dispersion

β changes as wavelength changes due to confinement variation into waveguide core.

$$t_g = v_g^{-1} = \frac{d\beta}{d\omega} = \frac{1}{c} \frac{d\beta}{dk}$$

$$\therefore \sigma_w = \frac{dt_g}{d\lambda} = \frac{1}{c} \frac{d}{d\lambda} \left(\frac{d\beta}{dk} \right) = -\frac{2\pi}{c\lambda^2} \frac{d^2\beta}{dk^2}$$



total dispersion

total dispersion: $\sigma = \sigma_m + \sigma_w$

(1) normal dispersion ($GVD > 0$, $\sigma < 0$):
 v_g increases as a wavelength becomes longer.

(2) anomalous dispersion ($GVD < 0$, $\sigma > 0$):
 v_g decreases as a wavelength becomes longer.

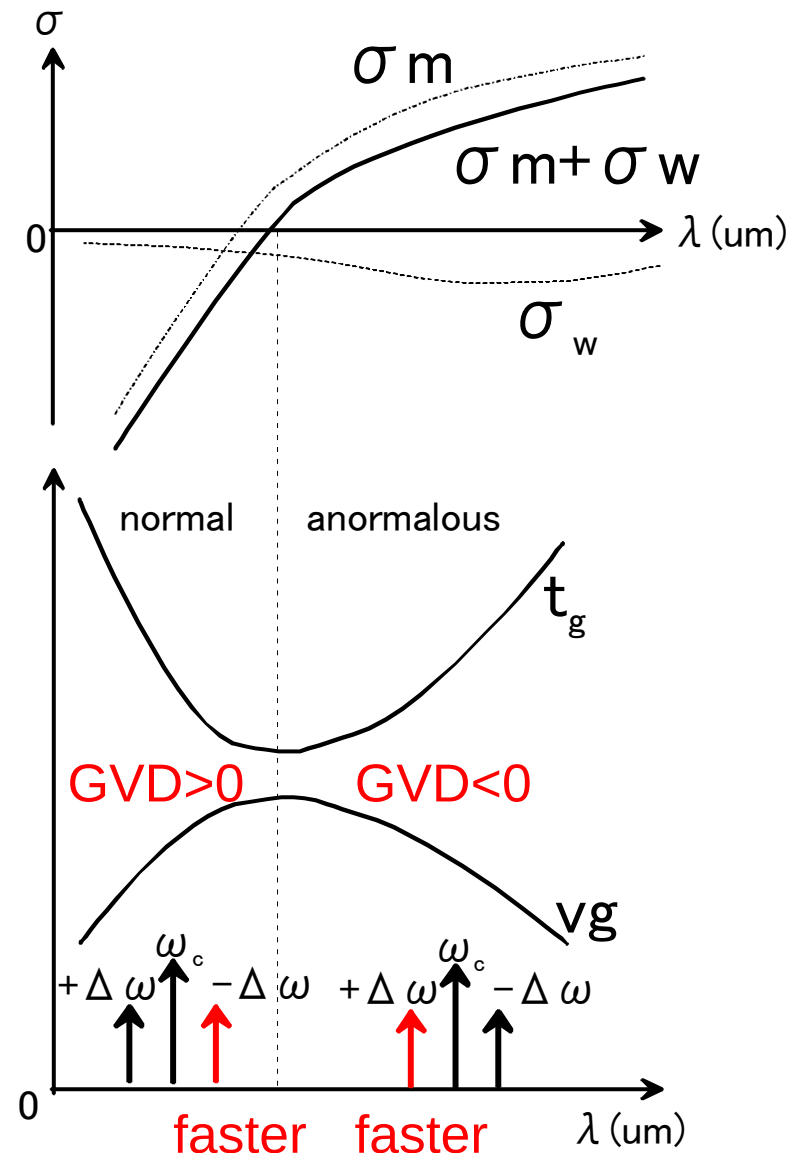
pulse broadening: $\Delta T = L|\sigma|\delta\lambda$
 W: bit rate $\rightarrow$ limit $\Delta T < T = \frac{1}{W}$

$$WL|\sigma|\delta\lambda < 1$$

modulation $\rightarrow \delta\lambda = \frac{\lambda^2}{c} W$

$$\therefore W^2 L |\sigma| < \frac{c}{\lambda^2}$$

$\sigma = 16 \text{ ps/km/nm}$ ($\lambda = 1550 \text{ nm}$)
 $W = 2.5 \text{ Gbps} \rightarrow L < 1248 \text{ km}$
 $W = 10 \text{ Gbps} \rightarrow L < 78 \text{ km}$



7.5.1 Nonlinear effect (SPM) and soliton

optical Kerr effect: $n = n_0 + n_2(\omega)|E|^2$

$$\Delta\phi = k_0 n_2(\omega)|E|^2 L \quad \Delta\omega = -\frac{d\Delta\phi}{dt} = -\frac{2\pi}{\lambda} n_2 L \frac{d|E|^2}{dt}$$

rising edge: $\Delta\omega < 0$

falling edge: $\Delta\omega > 0$

(1) normal dispersion region (GVD>0):

t_g is larger for shorter wavelength
(at higher frequency)

rising edge : faster

trailing edge : slowly

pulse broadening

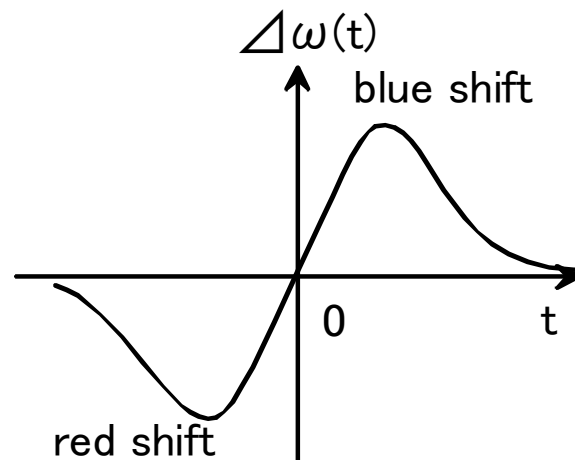
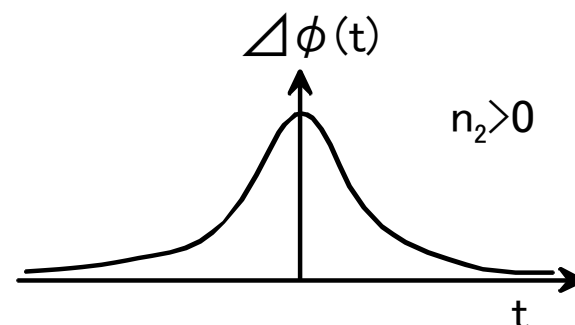
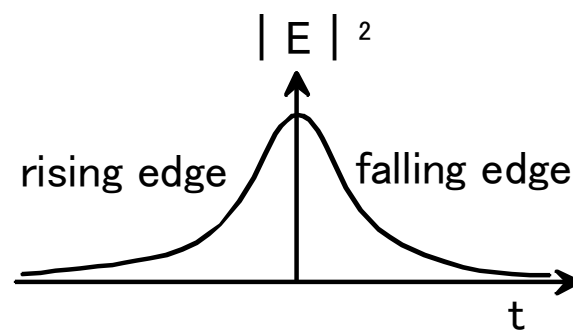
(2) anomalous region (GVD<0):

t_g is larger for longer wavelength
(at lower frequency)

rising edge : slowly

trailing edge : faster

pulse compression

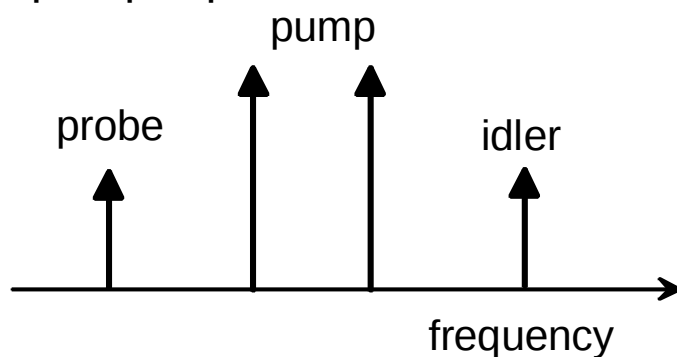


7.5.2 Nonlinear effect (FWM: Four Wave Mixing)

caused by an optical Kerr effect (third-order nonlinearity)

FWM: Four Wave Mixing

2 pump + probe $\rightarrow$ idler



bad news

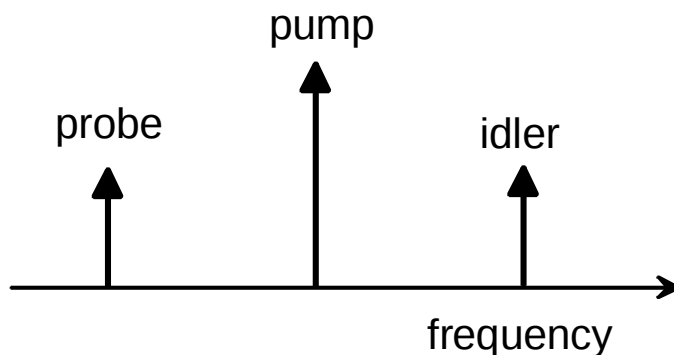
FWM produces a spurious light that could have the same wavelength as a signal to be transmitted in WDM.

good news

FWM can be used as a wavelength converter.

DFWM: Degenerate Four Wave Mixing

2 pump (with same λ) + probe $\rightarrow$ idler



7.5.3 Nonlinear effect (Raman effect)

Frequency shifted light is generated due to Raman effect:

$$\lambda(\omega) \rightarrow \lambda + \Delta\lambda \quad (\omega - \Delta\omega)$$

$-\Delta\omega$: molecular vibration, optical phonon energy

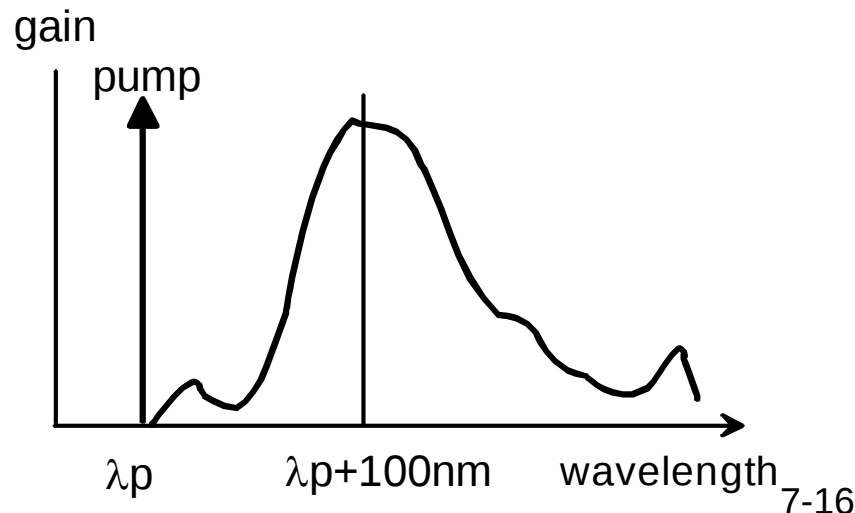
Stimulated Raman effect in optical fibers:

optical gain at a wavelength region approximately 100nm-longer
than a pump wavelength

utilized as an optical amplifier

can be tuned, in principle, at an arbitrary wavelength

fiber itself is a gain medium



7.5.4 Nonlinear effect (Brillouin scattering)

Wavelength-shifted light is generated:

shifted by the frequency corresponding to acoustic phonon energy

acoustic phonon → compressional wave

→ periodic variation in refractive index

→ red-shifted scattered light
(propagating in backward)

coherent and intense incident light

→ Stimulated Brillouin scattering (SBS)

power transfer from incident light
to scattered light

→ suppresses transmission power

