

3. Plane Wave in Free Space

Starting from Maxwell's equation, we obtain set of differential equations similar to the transmission line equations.

$$\begin{array}{ll} \text{rot}\mathbf{E} = -j\omega\mathbf{B} & \text{rot}\mathbf{E} = -j\omega\mu\mathbf{H} \\ \text{rot}\mathbf{H} = j\omega\mathbf{D} + \mathbf{J}_c + \mathbf{J}_s & \text{rot}\mathbf{H} = j\omega\varepsilon\mathbf{E} \\ \mathbf{D} = \varepsilon\mathbf{E}, \mathbf{B} = \mu\mathbf{H}, \mathbf{J}_c = \sigma\mathbf{E} & \rho = \mathbf{J}_s = 0 \end{array} \quad \Rightarrow$$

Assumption: time dependence $\exp(j\omega t)$
propagation along z direction
no variation in transverse directions (x, y)
no E_y (E_x only)

Then, Maxwell's equations are reduced to

$$\begin{array}{ll} 0 = -j\omega\mu H_x & -\frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x \\ \frac{\partial E_x}{\partial z} = -j\omega\mu H_y & \frac{\partial H_x}{\partial z} = j\omega\varepsilon E_y = 0 \\ 0 = -j\omega\mu H_z & 0 \end{array} \quad \Rightarrow \quad H_x = H_z = 0$$

Plane Wave in Free Space

plane wave

$$\frac{dE_x}{dz} = -j\omega\mu H_y = -(j\omega\mu' + \omega\mu'')H_y$$

$$\frac{dH_y}{dz} = -j\omega\varepsilon E_x = -(j\omega\varepsilon' + \omega\varepsilon'')E_x$$

transmission line

$$\frac{dV(z)}{dz} = -Z_d I(z) \quad (Z_d = R + j\omega L)$$

$$\frac{dI(z)}{dz} = -Y_d V(z) \quad (Y_d = G + j\omega C)$$

from the similarity to the transmission line equation

$$E_x = E_{x1}e^{-\gamma z} + E_{x2}e^{+\gamma z}$$

$$H_y = \frac{1}{Z_c}(E_{x1}e^{-\gamma z} - E_{x2}e^{+\gamma z}) \quad \text{with} \quad \gamma = j\omega\sqrt{\varepsilon\mu}, \quad Z_c = \sqrt{\frac{\mu}{\varepsilon}}$$

+z propagating wave:

$$E_x^+ = E_{x1}e^{j(\omega t - \beta z)}$$

$$H_y^+ = \frac{1}{Z_c}E_{x1}e^{j(\omega t - \beta z)}$$

$$\text{wave impedance: } \frac{E_x^+}{H_y^+} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$$

$$\text{phase velocity: } v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\varepsilon\mu}}$$

4. Transmission Line Composed of Two Conductors

4.1 Electro-static model

The electromagnetic (EM) wave of **TEM mode** can be determined by an electro-static model.

$$\mathbf{E} = \mathbf{E}_t(x, y)e^{-\gamma z}$$

$$\mathbf{H} = \mathbf{H}_t(x, y)e^{-\gamma z}$$

Use $\nabla = \nabla_t + \mathbf{k} \frac{\partial}{\partial z} = \nabla_t - \gamma \mathbf{k}$ in $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$

then, $(\nabla_t - \gamma \mathbf{k}) \times \mathbf{E}_t e^{-\gamma z} = -j\omega\mu\mathbf{H}_t e^{-\gamma z}$

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$$

$$\nabla_t \times \mathbf{E}_t = 0$$

$$-\gamma \mathbf{k} \times \mathbf{E}_t = -j\omega\mu\mathbf{H}_t$$

$$\nabla_t \times \mathbf{H}_t = 0$$

$$-\gamma \mathbf{k} \times \mathbf{H}_t = j\omega\epsilon\mathbf{E}_t$$

$$-\gamma \mathbf{k} \times (\gamma \mathbf{k} \times \mathbf{E}_t) = -\omega^2 \epsilon\mu\mathbf{E}_t$$

$$\therefore \gamma^2 = -\omega^2 \epsilon\mu$$

$$\mathbf{E}_t = -\nabla_t \phi_e$$

$$\mathbf{H}_t = -\nabla_t \phi_m$$

Transmission line with two conductors

$$\begin{array}{l} \mathbf{E}_t = -\nabla_t \phi_e \\ \text{div} \mathbf{E} = 0 \end{array} \longrightarrow \boxed{\nabla_t^2 \phi_e = 0}$$

$$-\gamma \mathbf{k} \times \mathbf{E}_t = -j\omega\mu \mathbf{H}_t \longrightarrow \mathbf{H}_t = \frac{\gamma \mathbf{k} \times \mathbf{E}_t}{j\omega\mu}$$

$$\begin{aligned} \text{div} \mathbf{H} = \text{div} \mathbf{H}_t &= \nabla \cdot \left(\frac{\gamma}{j\omega\mu} \mathbf{k} \times \mathbf{E}_t \right) \\ &= \frac{\gamma}{j\omega\mu} (\mathbf{E}_t \cdot \nabla \times \mathbf{k} - \mathbf{k} \cdot \nabla \times \mathbf{E}_t) = \frac{-\gamma}{j\omega\mu} \mathbf{k} \cdot \nabla \times \mathbf{E}_t = 0 \end{aligned}$$

$$\mathbf{E} = \mathbf{E}_t(x, y)e^{-\gamma z}$$

$$\mathbf{H} = \mathbf{H}_t(x, y)e^{-\gamma z}$$

$$\therefore \gamma^2 = -\omega^2 \epsilon \mu$$

Coaxial line and strip line

4.2 Coaxial line

charge density per unit length= $q[\text{Cm}^{-1}]$

→voltage between inner and outer conductors= V

→capacitance $C[\text{Fm}^{-1}]$

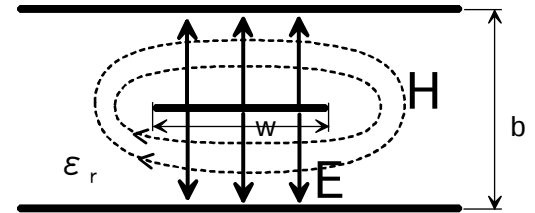
→characteristic impedance $Z_c = \frac{V}{I} = \frac{\frac{q}{C}}{\frac{q}{\sqrt{\epsilon\mu}}} = \frac{\sqrt{\epsilon\mu}}{C}$

4.3 Strip line

Dominant guided mode is a TEM mode.

$$C = 2 \times \epsilon \frac{w}{b} = \frac{4\epsilon w}{b}$$

$$Z_c = \frac{b}{4w} \sqrt{\frac{\mu}{\epsilon}}$$



Micro-strip line

4.4 Micro-strip line

a quasi TEM mode

approximate representation of characteristic impedance

0-th order approximation

$$C = \varepsilon \frac{w}{h} \rightarrow Z_c = \sqrt{\frac{\mu}{\varepsilon}} \frac{h}{w}$$

1-st order approximation (Schneider's expression)

$$\varepsilon_{eff} = \frac{\varepsilon_r + 1}{2} + \frac{\varepsilon_r - 1}{2} \left(1 + \frac{10h}{w} \right)^{-\frac{1}{2}}$$

$$\beta = \sqrt{\varepsilon_{eff}} \beta_0 \quad (\beta_0 = \omega \sqrt{\varepsilon_0 \mu_0}), \quad Z_c = \frac{1}{\sqrt{\varepsilon_{eff}}} Z_{c0}$$

$$Z_{c0} = 60 \ln \left[f \frac{h}{w_0} + \sqrt{1 + \left(\frac{2h}{w_0} \right)^2} \right], \quad f = 6 + 0.283 \exp \left[- \left(30.7 \frac{h}{w_0} \right)^{0.753} \right]$$

$$w_0 = w + \Delta w = w + \frac{t}{\pi} \ln \left(4e / \sqrt{\left(\frac{t}{h} \right)^2 + \frac{1}{\pi^2 (w/t + 1.1)^2}} \right)$$

5.1 Rectangular waveguide

EM field of TE and TM mode

orthogonal relation among eigen modes

$$\begin{array}{ll}
 \gamma E_y = -j\omega\mu H_x & (5.1.a) \\
 -\gamma E_x = -j\omega\mu H_y & (5.1.b) \\
 \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z & (5.1.c) \\
 \frac{\partial H_z}{\partial y} + \gamma H_y = j\omega\epsilon E_x & (5.1.d) \\
 -\gamma H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y & (5.1.e) \\
 \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 & (5.1.f)
 \end{array}$$

$$\begin{array}{ll}
 E_x = \frac{-j\omega\mu}{\gamma^2 + \omega^2\epsilon\mu} \frac{\partial H_z}{\partial y} = \frac{-j\omega\mu}{\beta_c^2} \frac{\partial H_z}{\partial y} & H_x = \frac{-\gamma}{\beta_c^2} \frac{\partial H_z}{\partial x} \\
 E_y = \frac{j\omega\mu}{\beta_c^2} \frac{\partial H_z}{\partial x} & H_y = \frac{-\gamma}{\beta_c^2} \frac{\partial H_z}{\partial y}
 \end{array}$$

Rectangular waveguide

$$\frac{j\omega\mu}{\beta_c^2} \frac{\partial^2 H_z}{\partial x^2} + \frac{j\omega\mu}{\beta_c^2} \frac{\partial^2 H_z}{\partial y^2} = -j\omega\mu H_z$$

$$\therefore \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = -\beta_c^2 H_z$$

$$\frac{1}{X(x)} \frac{d^2 X}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y}{dy^2} = -\beta_c^2$$

$$\frac{1}{X(x)} \frac{d^2 X}{dx^2} = -\beta_x^2$$

$$\frac{1}{Y(y)} \frac{d^2 Y}{dy^2} = -\beta_y^2$$

$$\beta_x^2 + \beta_y^2 = \beta_c^2$$

$$X(x) = A \sin \beta_x x + B \cos \beta_x x$$

$$Y(y) = C \sin \beta_y y + D \cos \beta_y y$$

Rectangular waveguide

$$H_z = X(x)Y(y)$$

$$X(x) = A \sin \beta_x x + B \cos \beta_x x$$

$$Y(y) = C \sin \beta_y y + D \cos \beta_y y$$

boundary condition at $x=0, a$

$$E_y = \frac{j\omega\mu}{\beta_c^2} \frac{\partial H_z}{\partial x} = \frac{j\omega\mu}{\beta_c^2} \beta_x (A \cos \beta_x x - B \sin \beta_x x) Y(y)$$

$$A = 0 \quad (E_y = 0 \quad \text{at} \quad x = 0)$$

$$\sin \beta_x a = 0 \quad (E_y = 0 \quad \text{at} \quad x = a)$$

$$\therefore \beta_x = \frac{m\pi}{a}$$

boundary condition for E_x at $y=0, b$

$$C = 0$$

$$\sin \beta_y b = 0 \quad \therefore \beta_y = \frac{n\pi}{b}$$

Rectangular waveguide

Field components are determined as

$$H_z = H_0 \cos \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \quad (\text{Either } n \text{ or } m \text{ can be } 0)$$

$$E_x = \frac{j\omega\mu}{\beta_c^2} H_0 \frac{n\pi}{b} \cos \frac{m\pi}{a} x \sin \frac{n\pi}{b} y$$

$$E_y = \frac{-j\omega\mu}{\beta_c^2} H_0 \frac{m\pi}{a} \sin \frac{m\pi}{a} x \cos \frac{n\pi}{b} y$$

$$H_x = \frac{\gamma}{\beta_c^2} H_0 \frac{m\pi}{a} \sin \frac{m\pi}{a} x \cos \frac{n\pi}{b} y$$

$$H_y = \frac{\gamma}{\beta_c^2} H_0 \frac{n\pi}{b} \cos \frac{m\pi}{a} x \sin \frac{n\pi}{b} y$$

$$\beta_c^2 = \left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 = \gamma^2 + \omega^2 \epsilon \mu$$

$$\therefore \gamma = \pm \sqrt{\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 - \omega^2 \epsilon \mu}$$

Rectangular waveguide

TM mode

$$E_z = E_0 \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \quad (m, n \geq 1)$$

$$E_x = \frac{-\gamma}{\beta_c^2} E_0 \frac{m\pi}{a} \cos \frac{m\pi}{a} x \sin \frac{n\pi}{b} y$$

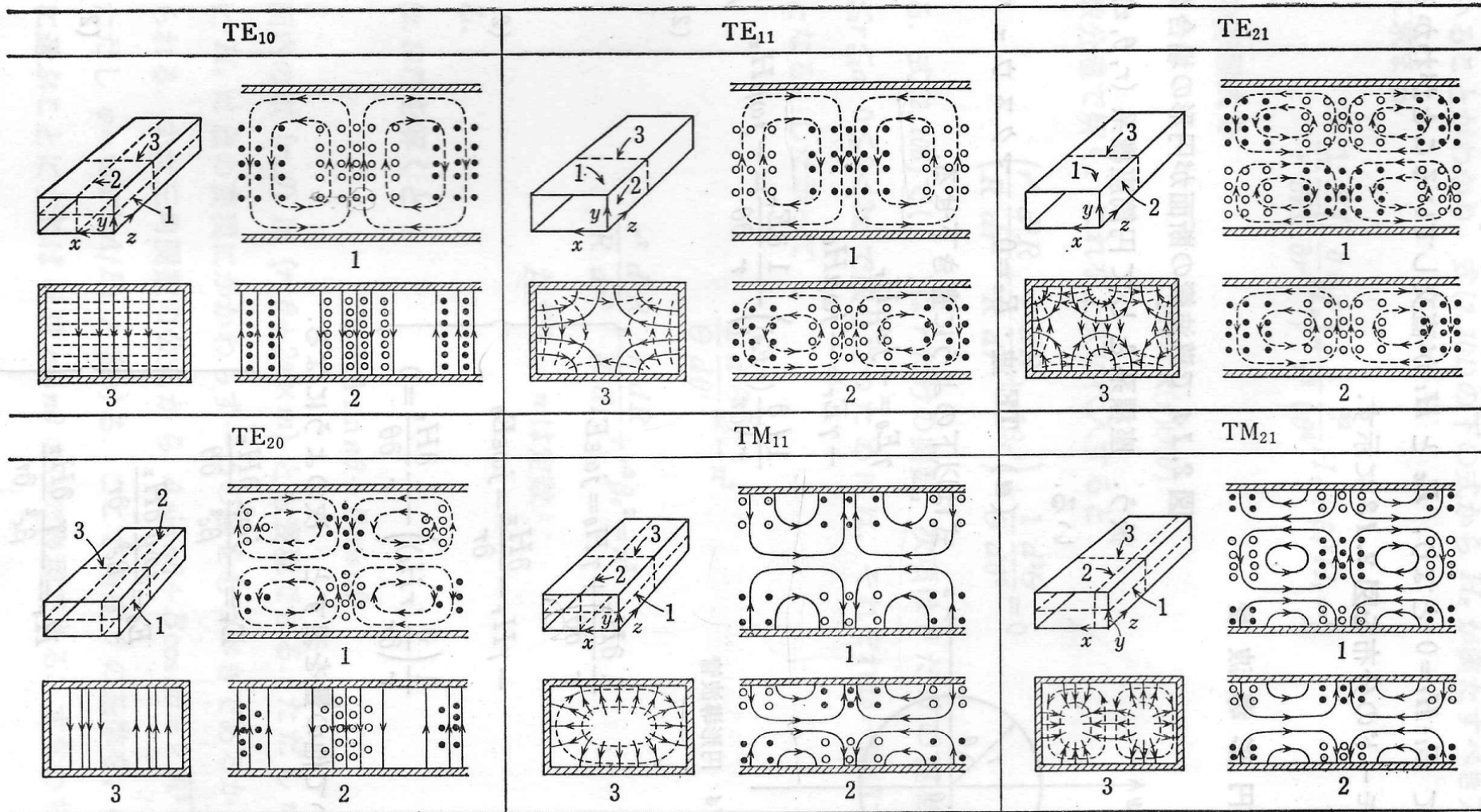
$$E_y = \frac{-\gamma}{\beta_c^2} E_0 \frac{n\pi}{b} \sin \frac{m\pi}{a} x \cos \frac{n\pi}{b} y$$

$$H_x = \frac{j\omega\epsilon}{\beta_c^2} E_0 \frac{n\pi}{b} \sin \frac{m\pi}{a} x \cos \frac{n\pi}{b} y$$

$$H_y = \frac{-j\omega\epsilon}{\beta_c^2} E_0 \frac{m\pi}{a} \cos \frac{m\pi}{a} x \sin \frac{n\pi}{b} y$$

$$\gamma = \pm \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \epsilon \mu}$$

Field distribution



Reference: Y. Naito
Micro and Millimeter Wave Engineering

Rectangular waveguide: orthogonality

TE mode (between transverse field component)

$$E_t^e = \frac{j\omega\mu}{\beta_c^2} \left(-\frac{\partial H_z}{\partial y} \mathbf{i} + \frac{\partial H_z}{\partial x} \mathbf{j} \right)$$

$$H_t^e = \frac{-\gamma}{\beta_c^2} \left(\frac{\partial H_z}{\partial x} \mathbf{i} + \frac{\partial H_z}{\partial y} \mathbf{j} \right)$$

$$\therefore E_t^e \cdot H_t^e = \frac{-j\omega\mu\gamma}{\beta_c^4} \left(-\frac{\partial H_z}{\partial x} \frac{\partial H_z}{\partial y} + \frac{\partial H_z}{\partial x} \frac{\partial H_z}{\partial y} \right) = 0$$

TM mode (between transverse field component)

$$E_t^m = \frac{-\gamma}{\beta_c^2} \left(\frac{\partial E_z}{\partial x} \mathbf{i} + \frac{\partial E_z}{\partial y} \mathbf{j} \right)$$

$$H_t^m = \frac{j\omega\epsilon}{\beta_c^2} \left(\frac{\partial E_z}{\partial y} \mathbf{i} - \frac{\partial E_z}{\partial x} \mathbf{j} \right)$$

$$\therefore E_t^m \cdot H_t^m = \frac{-j\omega\epsilon\gamma}{\beta_c^4} \left(\frac{\partial E_z}{\partial x} \frac{\partial E_z}{\partial y} - \frac{\partial E_z}{\partial x} \frac{\partial E_z}{\partial y} \right) = 0$$

5.2 Circular waveguide

$$\nabla \times H = i_r \left(\frac{1}{r} \frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z} \right) + i_\theta \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right) + i_z \frac{1}{r} \left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta} \right)$$

TE mode $H_z = H_0 J_n(\beta_c r) \cos n\theta$

$$E_r = \frac{-j\omega\mu}{\beta_c^2} \frac{1}{r} \frac{\partial H_z}{\partial \theta}$$

$$E_\theta = \frac{j\omega\mu}{\beta_c^2} \frac{\partial H_z}{\partial r} \longrightarrow E_\theta = \frac{j\omega\mu}{\beta_c^2} \beta_c H_0 J_n'(\beta_c r) \cos n\theta$$

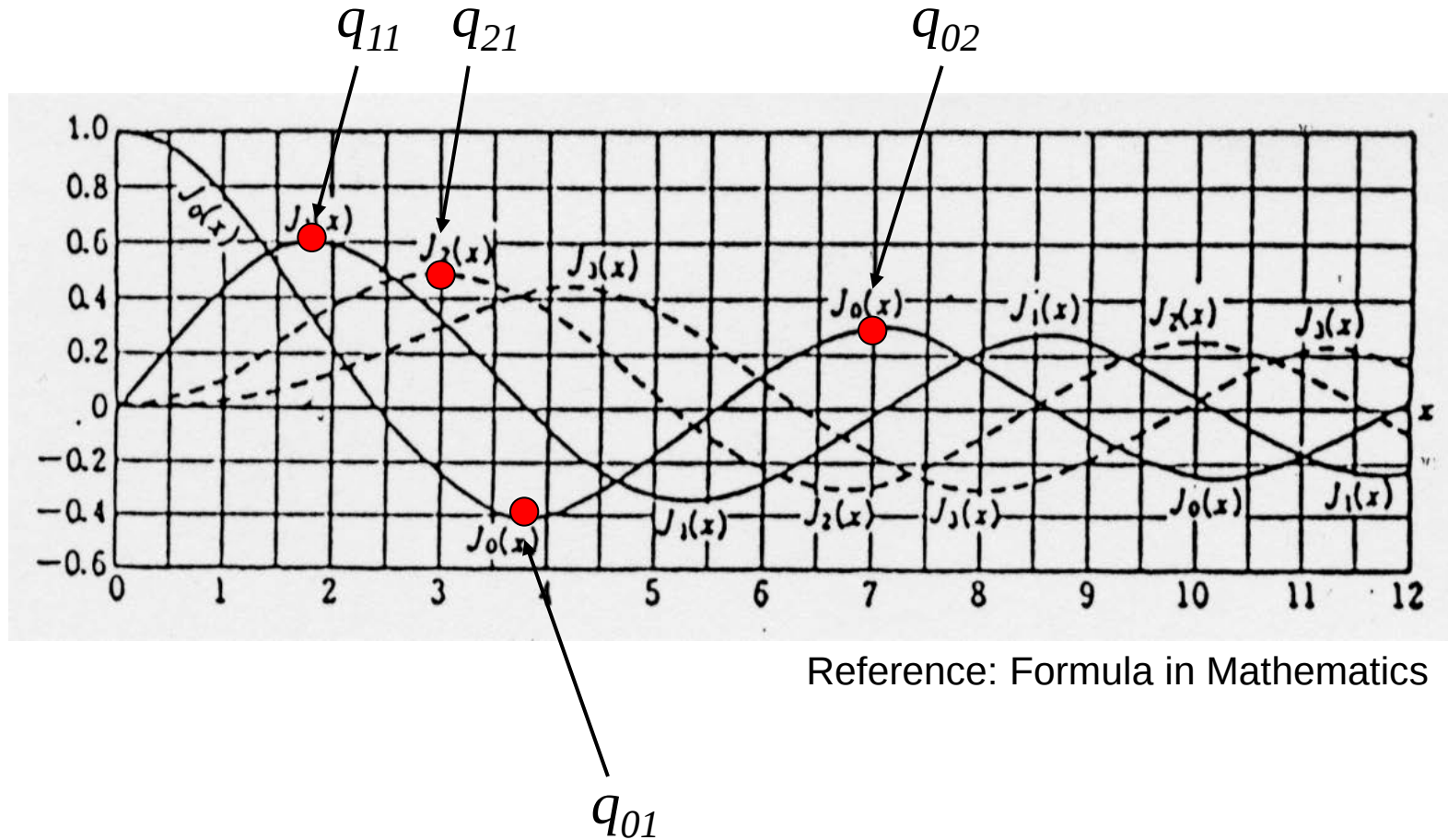
$$H_r = \frac{-\gamma}{\beta_c^2} \frac{\partial H_z}{\partial r}$$

$$H_\theta = \frac{-\gamma}{\beta_c^2} \frac{1}{r} \frac{\partial H_z}{\partial \theta}$$

$$(\beta_c^2 = \gamma^2 + \omega^2 \epsilon \mu)$$

$$J_n'(\beta_c a) = 0 \rightarrow q_{nm} = \beta_c a \quad \therefore \beta_c = \frac{q_{nm}}{a}, \quad \gamma = \pm \sqrt{\left(\frac{q_{nm}}{a} \right)^2 - \omega^2 \epsilon \mu}$$

Bessel function $J_n(x)$



Circular waveguide : TM mode

TM mode

$$E_z = E_0 J_n(\beta_c r) \cos n\theta$$

$$E_r = \frac{-\gamma}{\beta_c^2} \frac{\partial E_z}{\partial r}$$

$$E_\theta = \frac{-\gamma}{\beta_c^2} \frac{1}{r} \frac{\partial E_z}{\partial \theta}$$

$$H_r = \frac{j\omega\epsilon}{\beta_c^2} \frac{1}{r} \frac{\partial E_z}{\partial \theta}$$

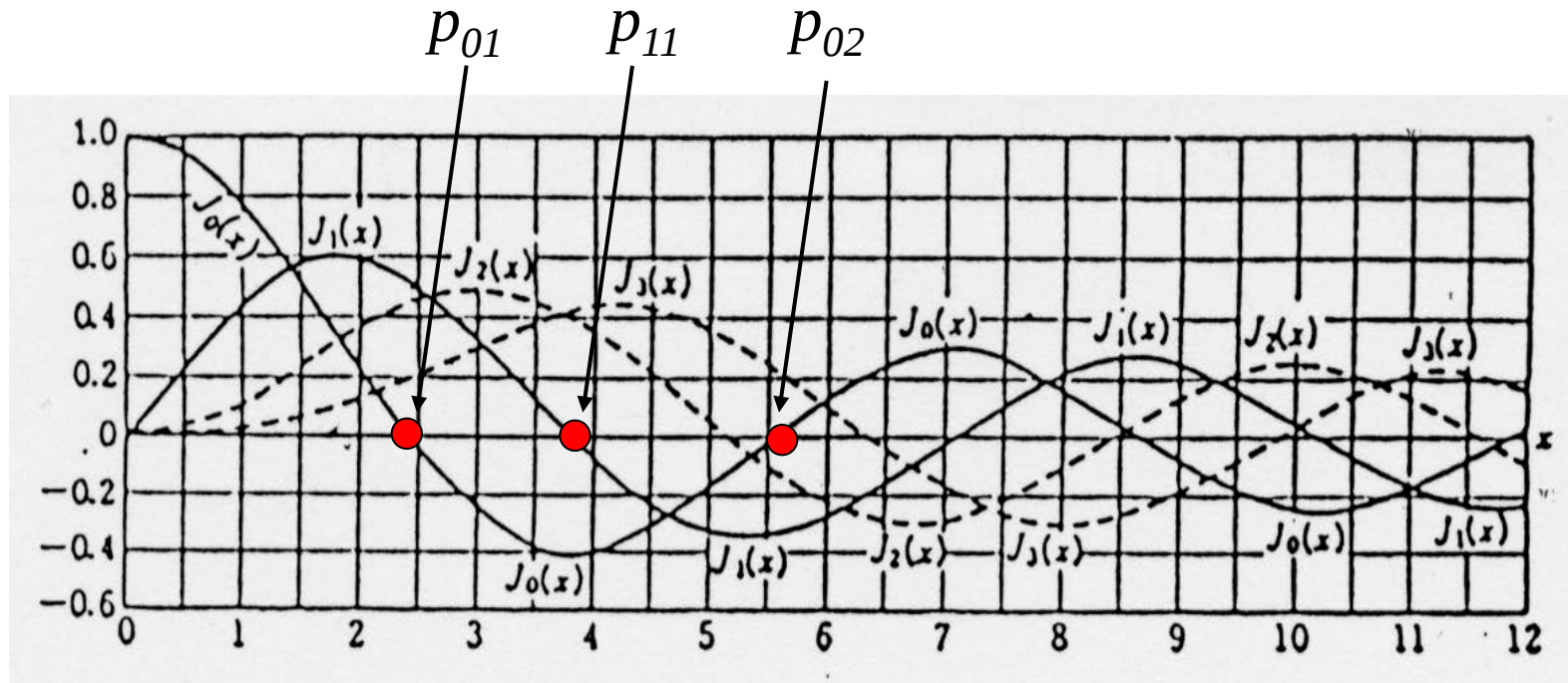
$$H_\theta = \frac{-j\omega\epsilon}{\beta_c^2} \frac{\partial E_z}{\partial r}$$

$$\longrightarrow E_\theta = \frac{\gamma}{\beta_c^2} \frac{n}{r} E_0 J_n(\beta_c r) \sin n\theta$$

$$(\beta_c^2 = \gamma^2 + \omega^2 \epsilon \mu)$$

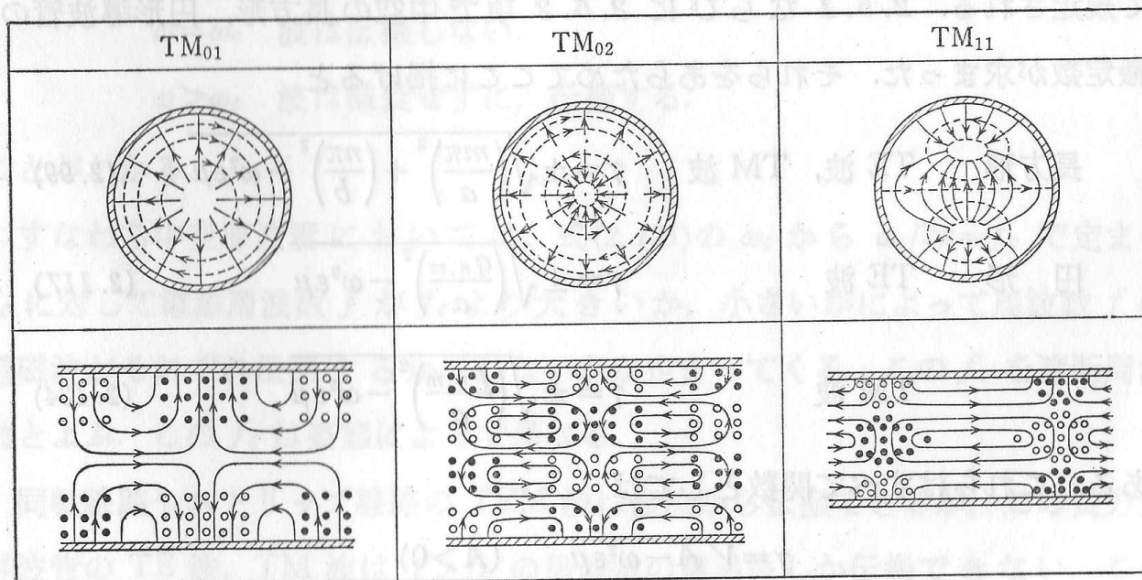
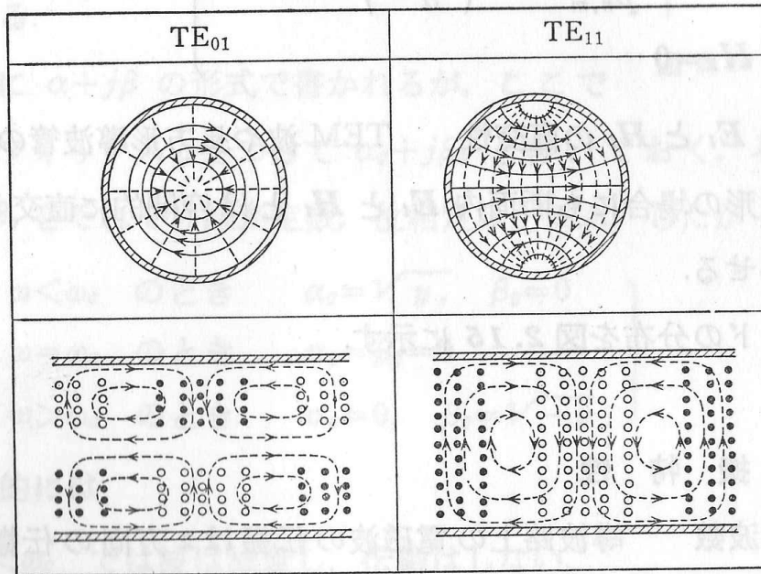
$$J_n(\beta_c a) = 0 \rightarrow p_{nm} = \beta_c a \quad \therefore \beta_c = \frac{p_{nm}}{a}, \quad \gamma = \pm \sqrt{\left(\frac{p_{nm}}{a}\right)^2 - \omega^2 \epsilon \mu}$$

Bessel function $J_n(x)$



Reference: Formula in Mathematics

Field distribution



Reference: Y. Naito
Micro and Millimeter Wave Engineering

5.3 Cut-off

cut-off frequency:

- The frequency at which the propagation constant of the mode concerned γ becomes 0.
- The mode is evanescent below that frequency.

Rectangular ($a=2b$): listed in the order of lower cut-off frequency

TE_{10}

TE_{20}, TE_{01} (same cut-off frequency)

TE_{11}, TM_{11}

$TE_{21}, TM_{21}, \dots$

$$\gamma = \pm \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \epsilon \mu}$$

Circular : listed in the order of lower cut-off frequency

TE_{11}

TM_{01}

TE_{21}

$TE_{01}, TM_{11}, \dots$

$$\text{TE - mode: } \gamma = \pm \sqrt{\left(\frac{q_{nm}}{a}\right)^2 - \omega^2 \epsilon \mu}$$

$$\text{TM - mode: } \gamma = \pm \sqrt{\left(\frac{p_{nm}}{a}\right)^2 - \omega^2 \epsilon \mu}$$