# Lecture 4

# 4 Geometric Constructuion

## 4.1 Naive Thought in Euclidean Age

- What is a geometric construction ?

- Tools : **Compass** and **Ruler** without marking.

- **Example :** How to draw a bisector for a given angle !

- **Question :** Is it possible to construct a trisector for a give angle ?

## 4.2 Formulate Problem

- Start with two points on the plane, say $(0,0)$ and $(1,0)$ in euclidean coordinates. What we can do is

    1. To draw a line through two exsiting points on the plane.

    2. To record the distance between two exsiting points by compass.

    3. To draw a circle on the plane where the center is an existing point and a radius is a recorded distance.

    4. To record the intersecton of two curves (line and/or circle) on the plane.

- **Exercise :**

    1. Draw an equilatelal tiangle !

    2. Draw a square !

    3. Draw a regular pentagon (Homework 3)!

- **Exercise :**

    1. If $a, b \in \mathbb{R}$ are recorded, then $a \pm b$ can be recorded.

    2. If $a, b \in \mathbb{R}$ are recorded, then $ab$ can be recorded.

    3. If $a, b \in \mathbb{R}$ are recorded, then $a/b$ can be recorded.

    4. If $a \in \mathbb{R}$ is recorded, then $\sqrt{a}$ can be recorded.

- Again start with two initial points $p_0 = (0, 0)$ and $p_1 = (1, 0)$ on the plane. Suppose we recorded $m$ points $P_m$ on the plane by the construciton above, say,

$$P_m = \{p_0, p_1, \cdots, p_m \in \mathbb{R}^2\}.$$

  Using the coordinates of $p_i = (x_i, y_i)$, arrange the numbers such as

$$x_0, \ y_0, \ x_1, \ y_1, \ x_2, \ x_3, \cdots .$$

  We then let $Q_n$ be the set consisting of the first $n$ distinct numbers in the above ordered sequence. Thus, we obtained an increasing siquence

$$\{0, 1\} = Q_2 \subset Q_3 \subset \cdots \subset Q_n \subset \cdots$$

  of the sets of recorded real numbers.

- Letting $K_n$ be the field generated by $Q_n$ over $\mathbb{Q}$, and we obtain a sequence of simple field extensions :

$$\mathbb{Q} = K_2 \subset K_3 \subset \cdots \subset K_n \subset \cdots$$

- **Theorem 4.1.1 :** $[K_{n+1} : K_n] = 1$ or 2.

  *Proof.* The equations of a line and a circle on the plane are

$$ax + by = c,$$
$$(x - d)^2 + (y - e)^2 = r^2$$

  in general. If these are drawn in terms of points in $P_m$, then $a, b, c, d, e, r^2$ can be choosen from $K_n$ for some $n$. Each coordinate of an upcoming point is a root $u$ of a quadratic equation over $K_n$. If $u \in K_n$, then $[K_{n+1} : K_n] = 1$. If $u \notin K_n$, then $[K_{n+1} : K_n] = 2$. □

- **Corollary 4.1.2 :** $[K_n : \mathbb{Q}] = 2^m$ for some $m \leq n$.

## 4.3  Three Problems in Greek Age

- **Squaring the Circle :** Construct a aquare with the same area as a given circle.

  *Proof.* Impossible because $\pi$ is transcendental ! □

- **Dubling the Cube :** Construct a side of a cube that has twice the volume of a cube with a given side.

*Proof.* Impossible because $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$. □

- **Trisecting the Angle :** construct an angle that is one-third of a given arbitrary angle.

  *Proof.* Will see impossible for $2\pi/3$ ! □

## 4.4   Homework

1. Is it always possible to construct $\alpha \in \mathbb{R}$ if $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^n$ where $n \geq 2$ ?

2. Show that the equilateral $n$-gon can be constructed if and only if $\varphi(n) = 2^k$ for some $k \geq 1$, and if and only if $n = 2^m p_1 p_2 \cdots p_k$ so that each $p_i$ is a distinct prime of the form $2^{2^s} + 1$ (called Fermat prime).

3. Construct the regular pentagon.

4. Contsruct the regular 17-gon.

5. Is it possible to construct a square from a triangle with the same area ?

6. Find an angle for which the construction of trisector is possible with respect to the initial set $P_1 = \{(0,0), (1,0)\}$.