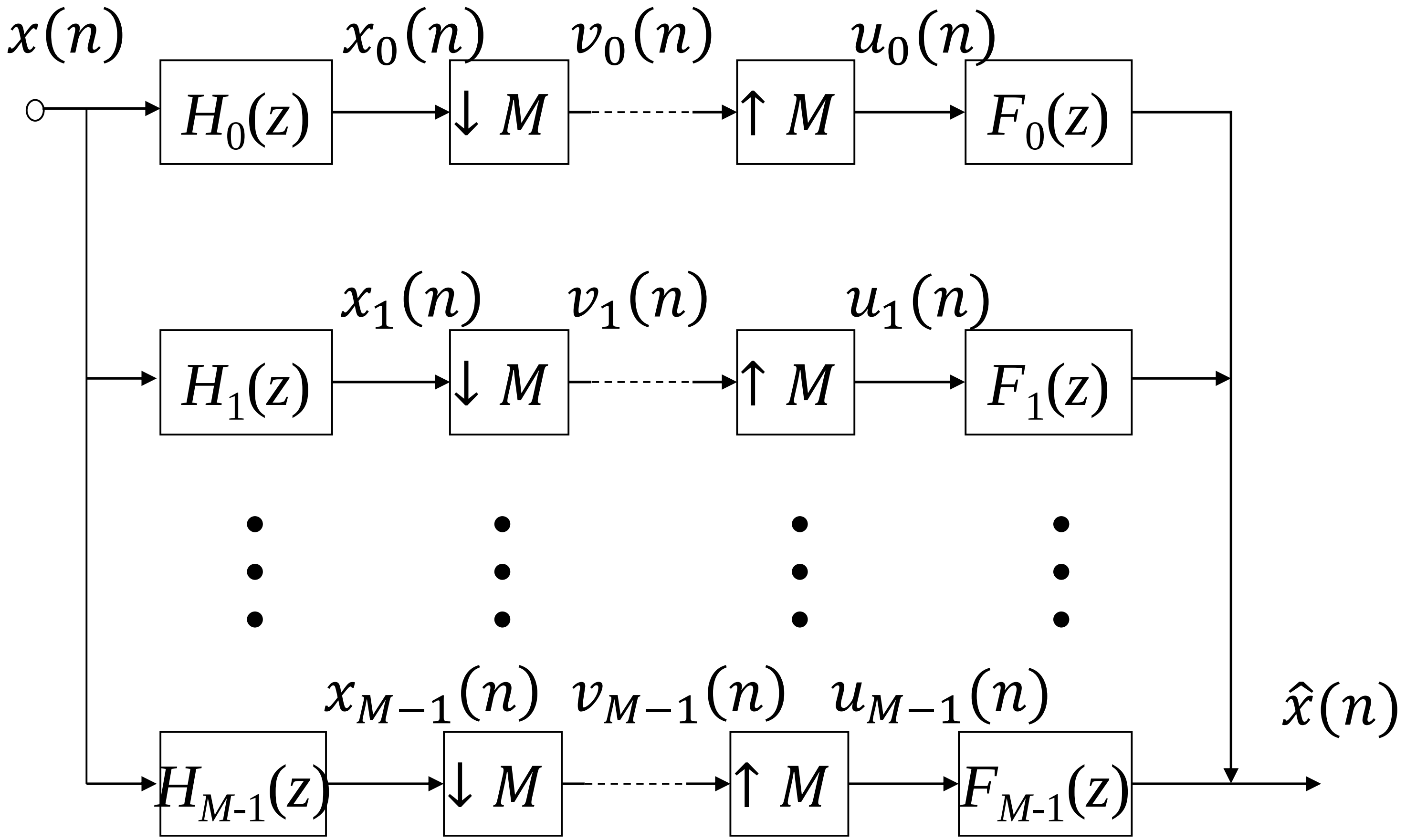
M-channel Filter Banks



z-Domain Analysis

$$X_k(z) = H_k(z)X(z)$$

$$V_k(z) = \frac{1}{M}\sum_{\ell=0}^{M-1} H_k\left(z^{1/M}W^{\ell}\right) X\left(z^{1/M}W^{\ell}\right)$$

$$U_k(z) = V_k(z^M) = \frac{1}{M}\sum_{\ell=0}^{M-1} H_k\left(zW^{\ell}\right) X\left(zW^{\ell}\right)$$

$$\hat{X}(z) = \sum_{k=0}^{M-1} F_k(z)U_k(z)$$

$$= \frac{1}{M}\sum_{\ell=0}^{M-1} X\left(zW^{\ell}\right) \sum_{k=0}^{M-1} H_k\left(zW^{\ell}\right) F_k(z)$$

Reconstructed Signal

$$\hat{X}(z) = \sum_{\ell=0}^{M-1} A_\ell(z) X\left(zW^\ell\right)$$

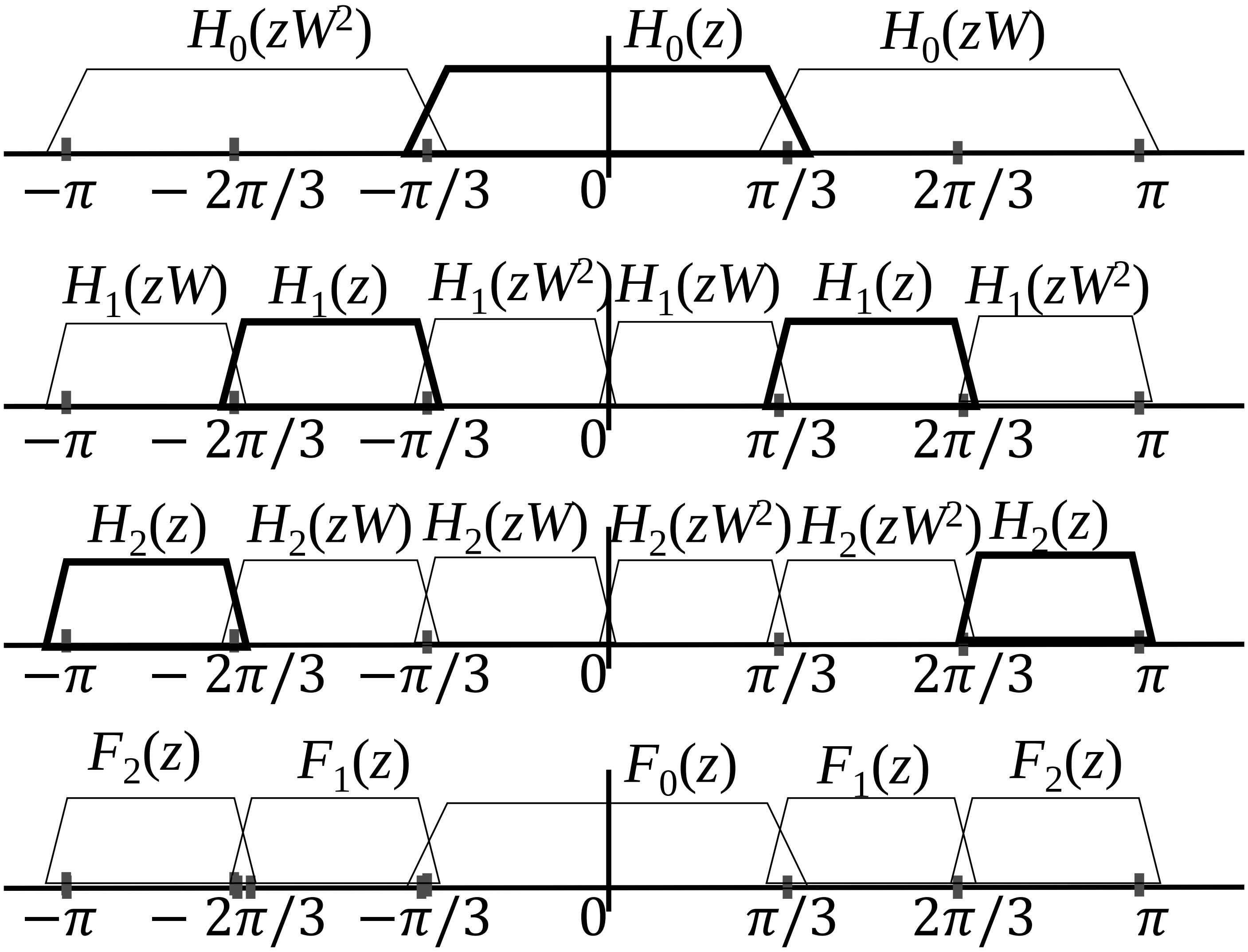
where $$A_\ell(z) = \frac{1}{M} \sum_{k=0}^{M-1} H_k\left(zW^\ell\right) F_k(z)$$

for $z = e^{j\omega}$

$$X\left(e^{j\omega} W^\ell\right) = X\left(e^{j(\omega - 2\pi\ell/M)}\right)$$

for $\ell \neq 0$, this represents a shifted version of $X\left(e^{j\omega}\right)$

Analysis Filters, Shifted Versions and Synthesis Filters



Alias Component Matrix

$$M\begin{bmatrix} A_0(z) \\ A_1(z) \\ \vdots \\ A_{M-1}(z) \end{bmatrix} =$$

$$\begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW) & H_1(zW) & \cdots & H_{M-1}(zW) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(zW^{M-1}) & H_1(zW^{M-1}) & \cdots & H_{M-1}(zW^{M-1}) \end{bmatrix}\begin{bmatrix} F_0(z) \\ F_1(z) \\ \vdots \\ F_{M-1}(z) \end{bmatrix}$$

$$\boldsymbol{A}(z) = \boldsymbol{H}(z)\boldsymbol{F}(z) = \begin{bmatrix} A_0(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$ to cancel aliasing

M-channel PR Filter Banks

in principle by

$$\boldsymbol{F}(z) = \boldsymbol{H}^{-1}(z) \begin{bmatrix} z^{-n_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

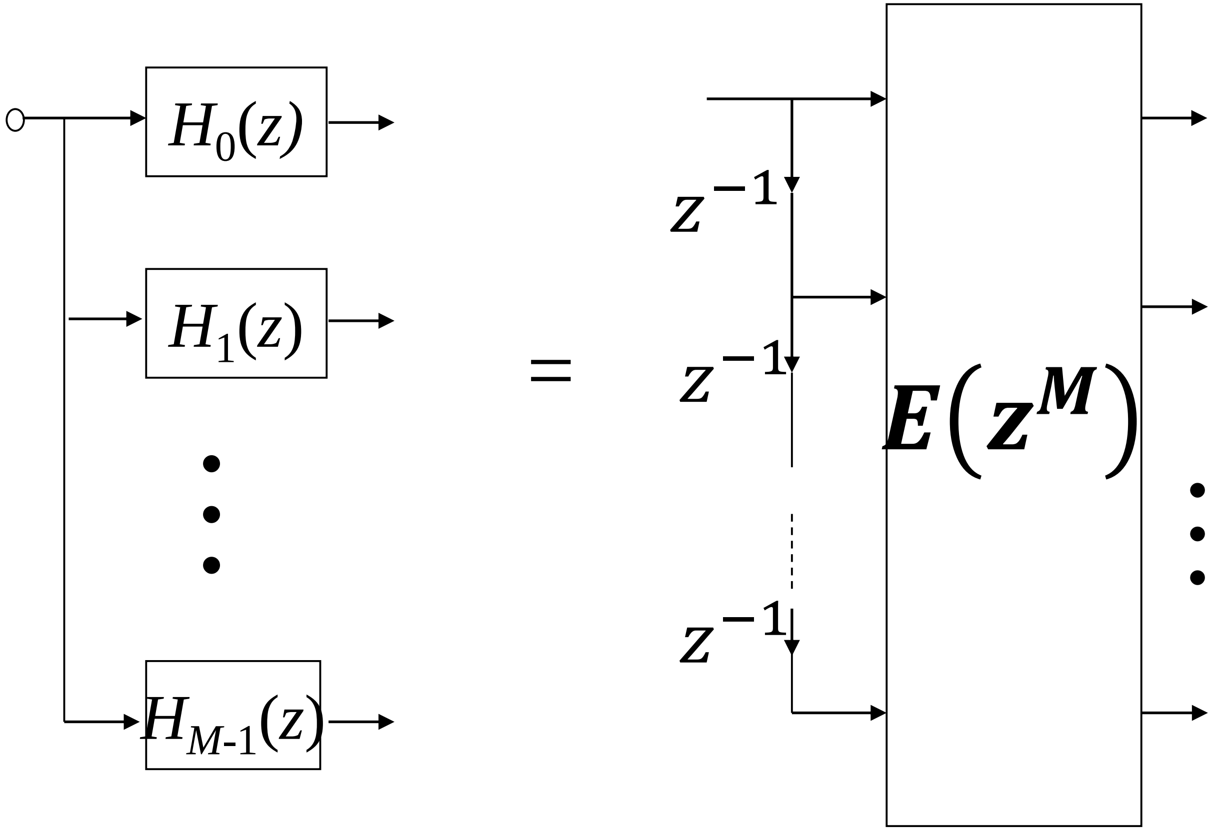
- Matrix inversion is difficult
- $F_k(z)$ may be IIR even if $H_k(z)$ is FIR
- $F_k(z)$ may be unstable

Polyphase Representation

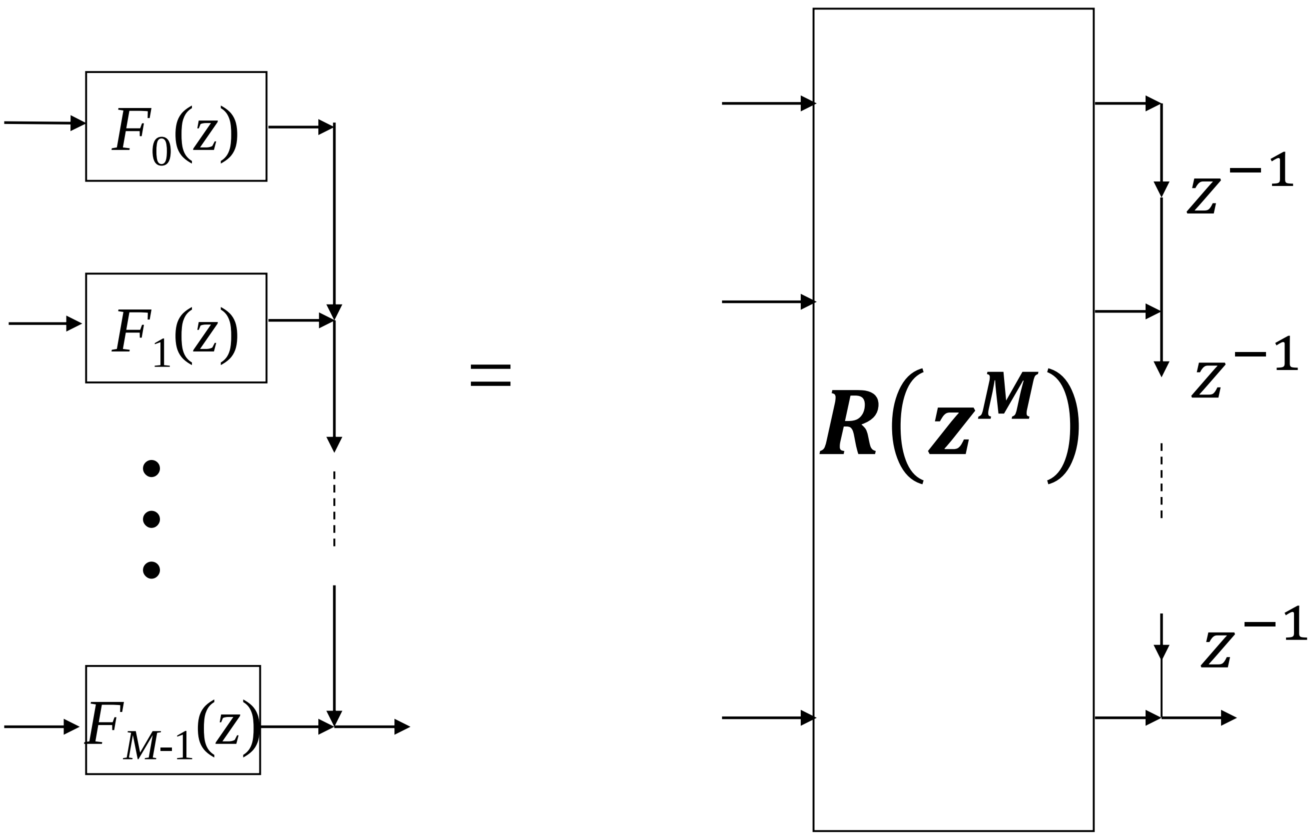
$$H_k(z) = \sum_{\ell=0}^{M-1} z^{-\ell} E_{k\ell}(z^M)$$

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = \begin{bmatrix} E_{00}(z^M) & E_{01}(z^M) & \cdots & E_{0,M-1}(z^M) \\ E_{10}(z^M) & E_{11}(z^M) & \cdots & E_{1,M-1}(z^M) \\ \vdots & \vdots & \ddots & \vdots \\ H_{M-1,0}(z^M) & E_{M-1,1}(z^M) & \cdots & E_{M-1,M-1}(z^M) \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix}$$

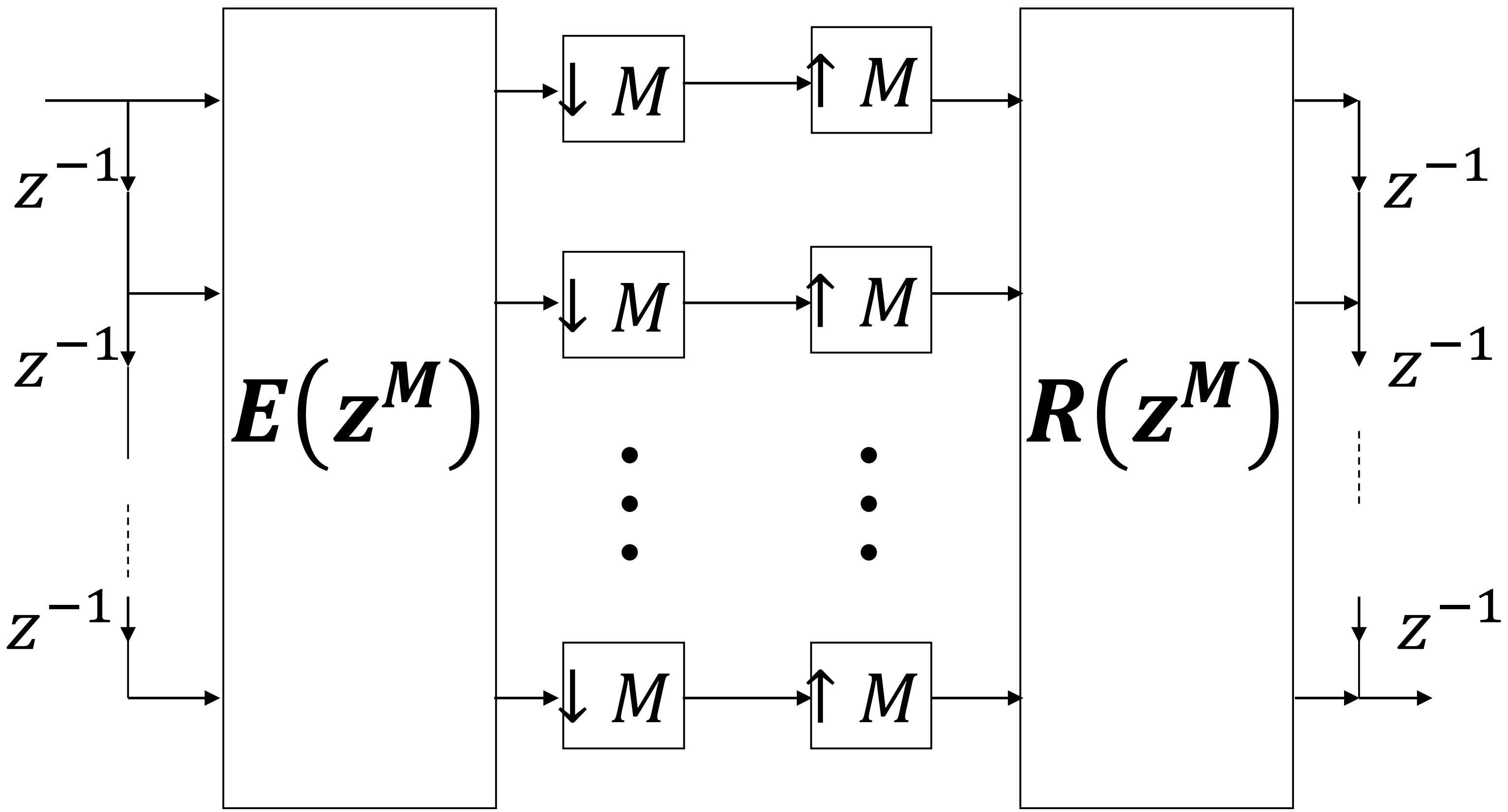
Analysis Bank



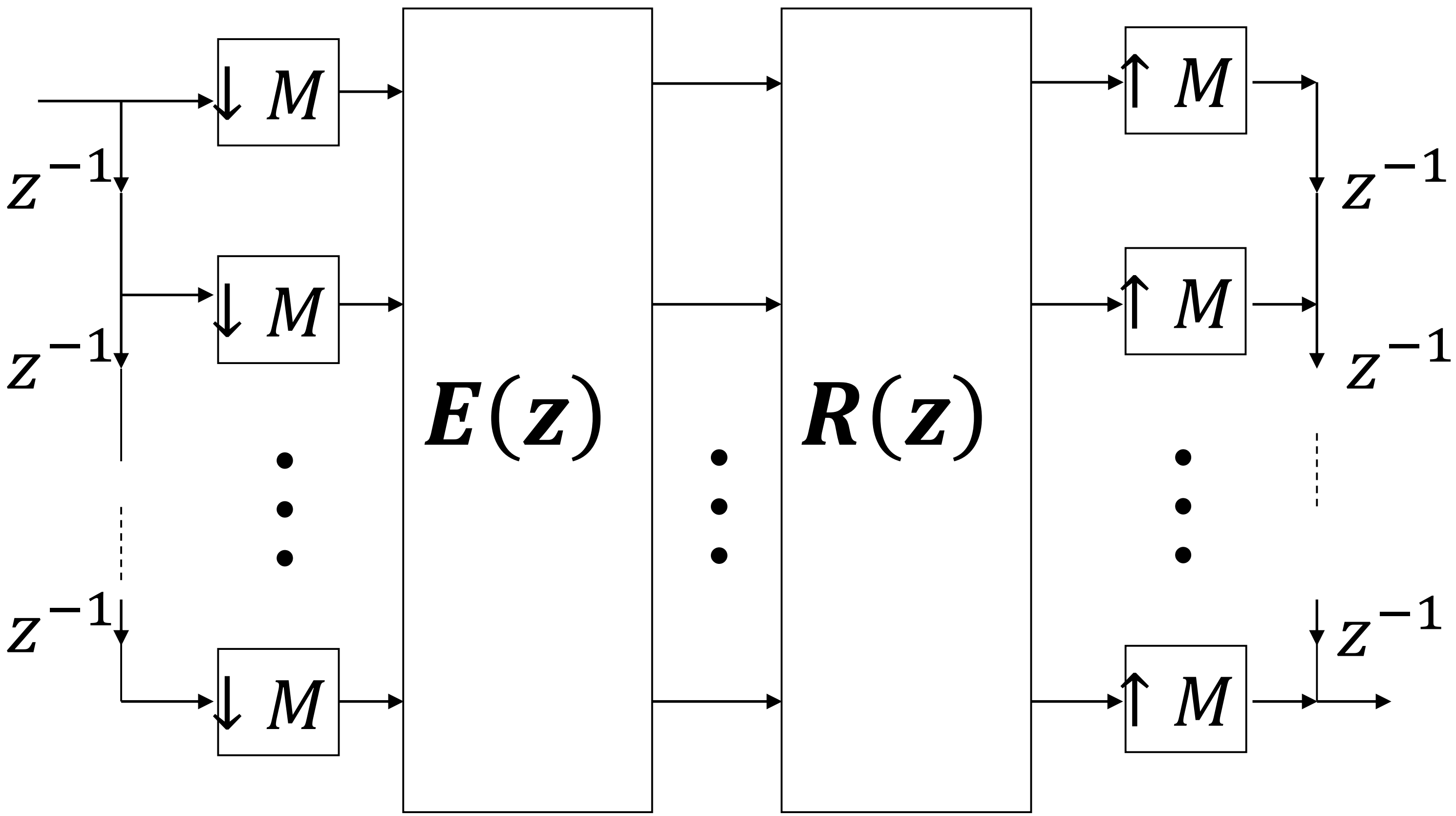
Synthesis Bank



Polyphase Representation of *M*-ch FB



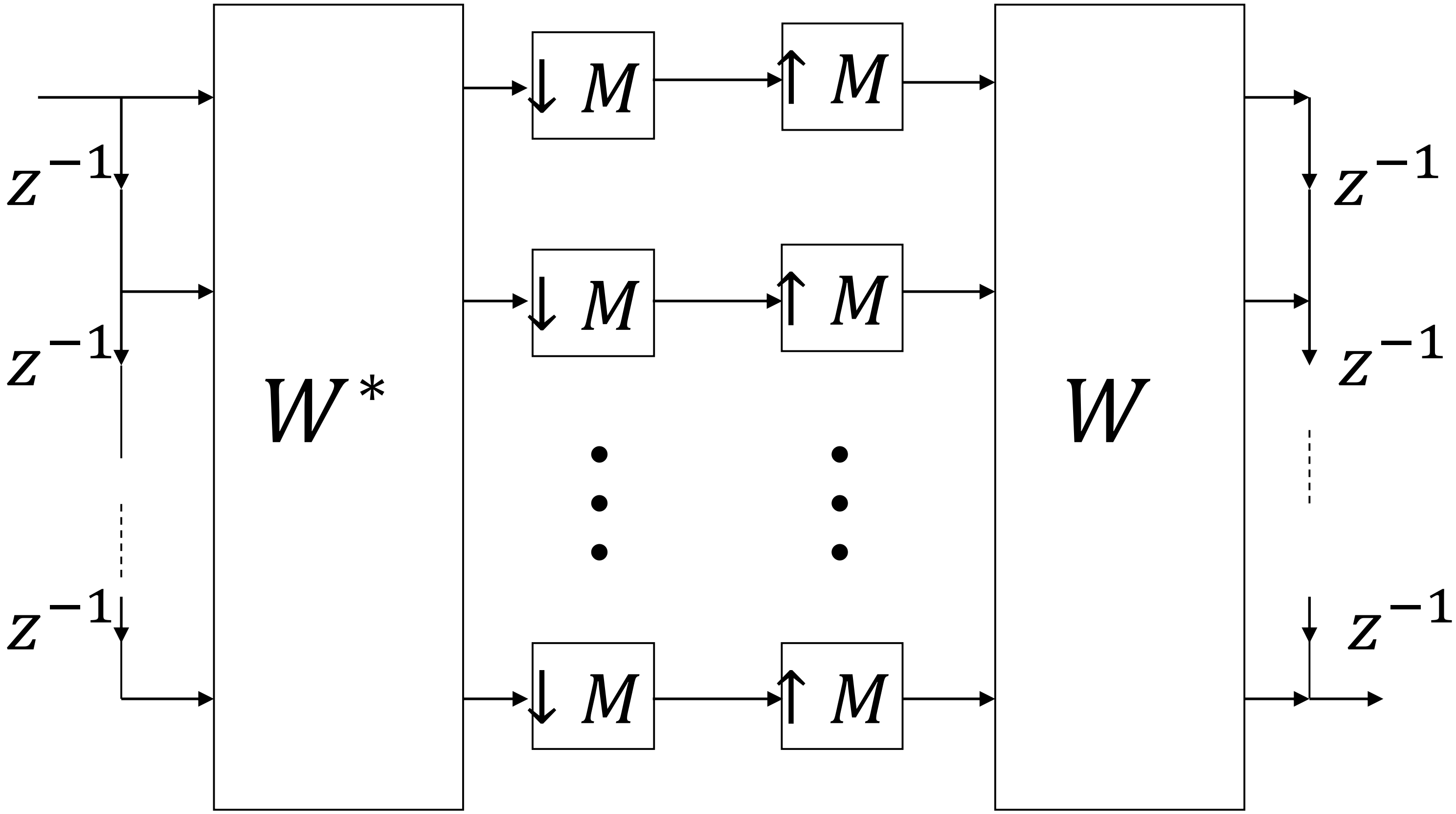
Noble Identity



$$R(z)E(z) = z^{-m}I$$

necessary and sufficient condition for PR

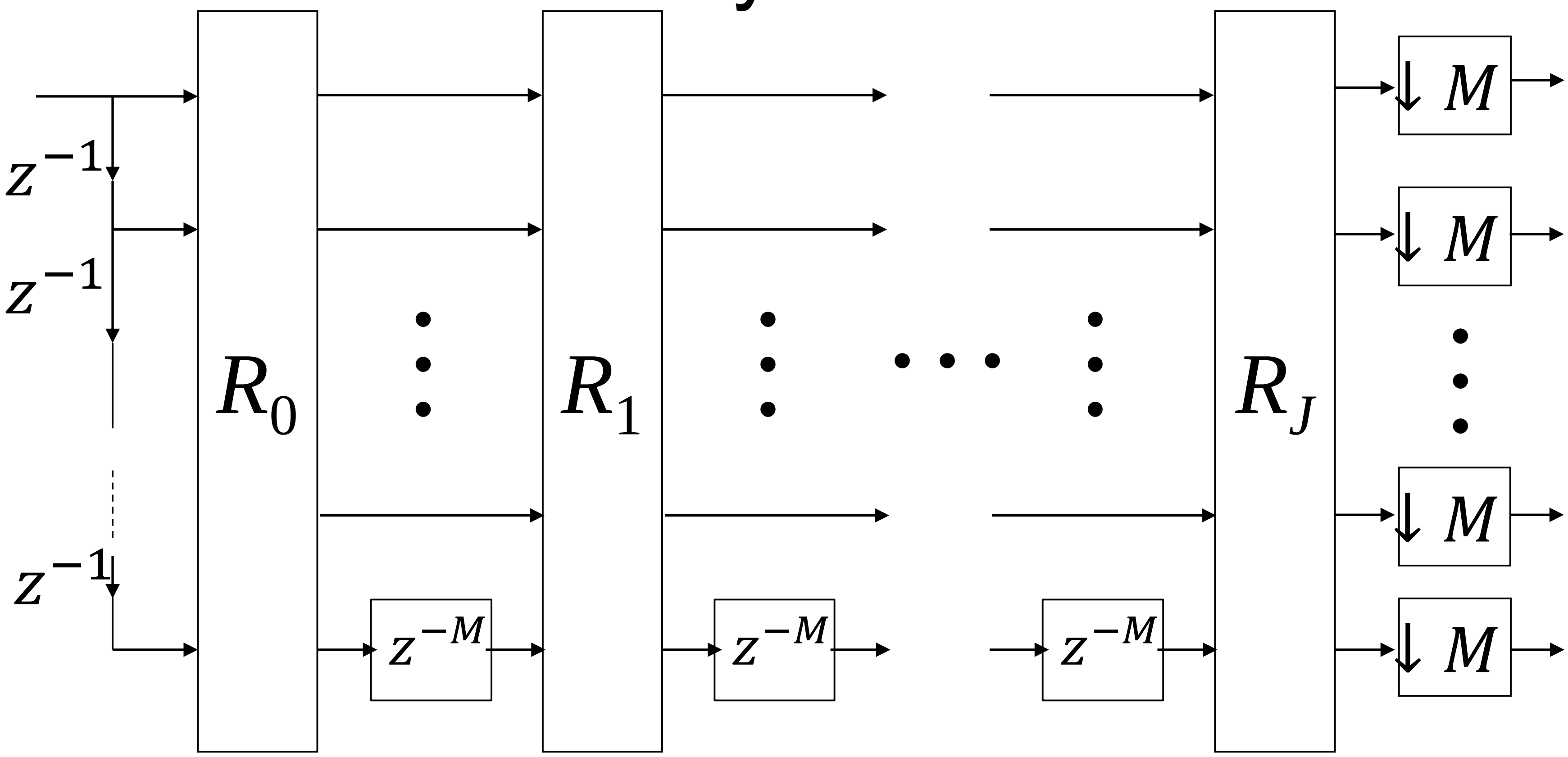
DFT Filter Bank



$$H_0(z) = 1 + z^{-1} + \cdots + z^{-(M-1)}$$
$$H_k(z) = H_0(zW^k)$$

High-Order FIR PR System

Analysis Bank



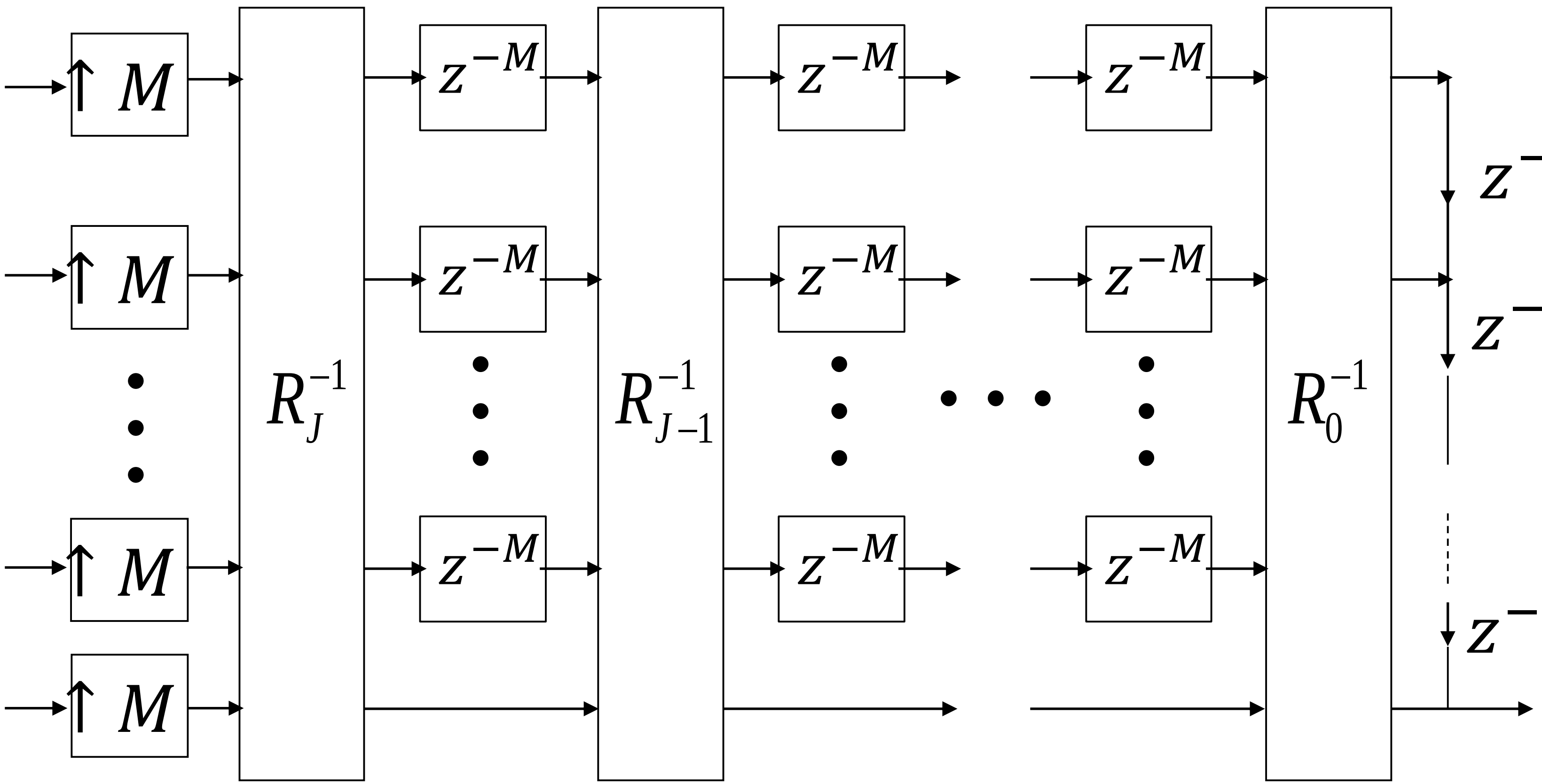
$$E(z) = R_J \Lambda(z) R_{J-1} \cdots \Lambda(z) R_0$$

$$\Lambda(z) = \begin{bmatrix} I_{M-1} & 0 \\ 0 & z^{-1} \end{bmatrix}$$

R_m: MxM constant non-singular matrix

High-Order FIR PR System

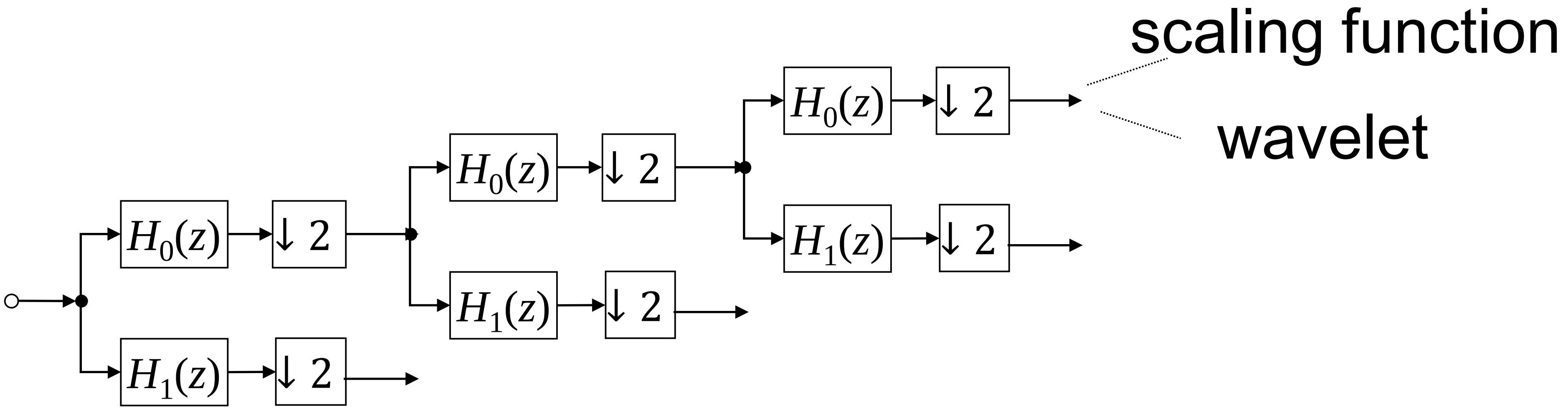
Synthesis Bank



Applications of Filter Banks

- Subband coding
- Audio compression
  - MP3, AAC
- Image compression
  - JPEG2000

Wavelet



Filter Bank Tree

Wavelets are localized waves.

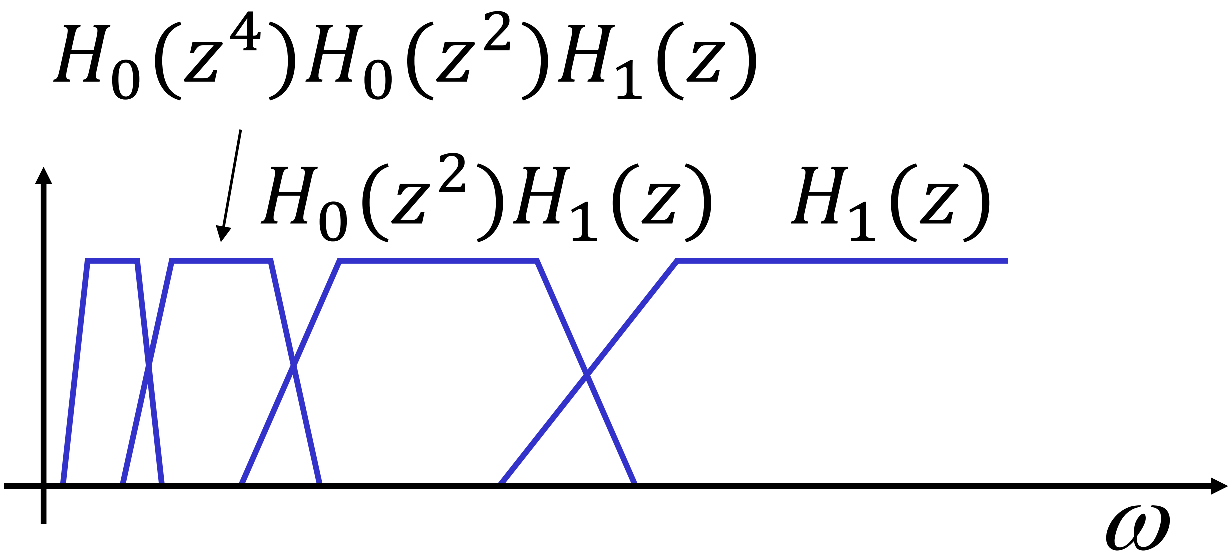
The lowest frequency branch

$$H_0^i(z) = \prod_{k=0}^{i-1} H_0\left(z^{2^k}\right)$$

its impulse response $h_0^i(n)$

become infinitely long for $i \to \infty$

Wavelet



constant relative band width

Multiresolution

Limit Function

$f^i(x)$: piecewise constant function
with value $2^{i/2}h_0^i(n)$ in the interval $\left[\frac{n}{2^i}, \frac{n+1}{2^i}\right]$

support of $f^i(x)$ is $[0, L-1]$, L is the length of $h_0(n)$

$i \rightarrow \infty$, $f^i(x) \rightarrow$ continuous function $\phi(x)$
or a function with many discontinuities

Wavelet

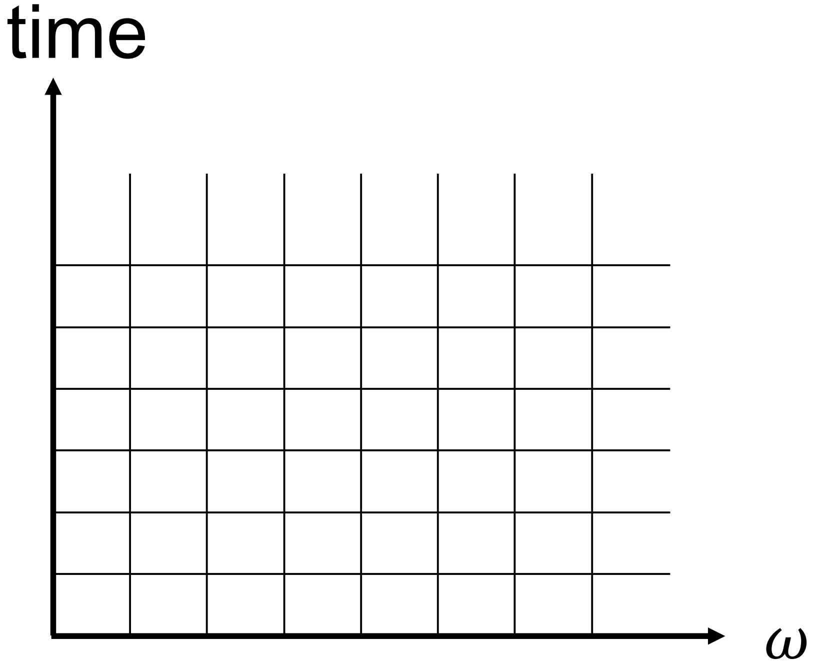
necessary condition for convergence:

$H_0(z)$ has a sufficient number of zeros at $z = -1$

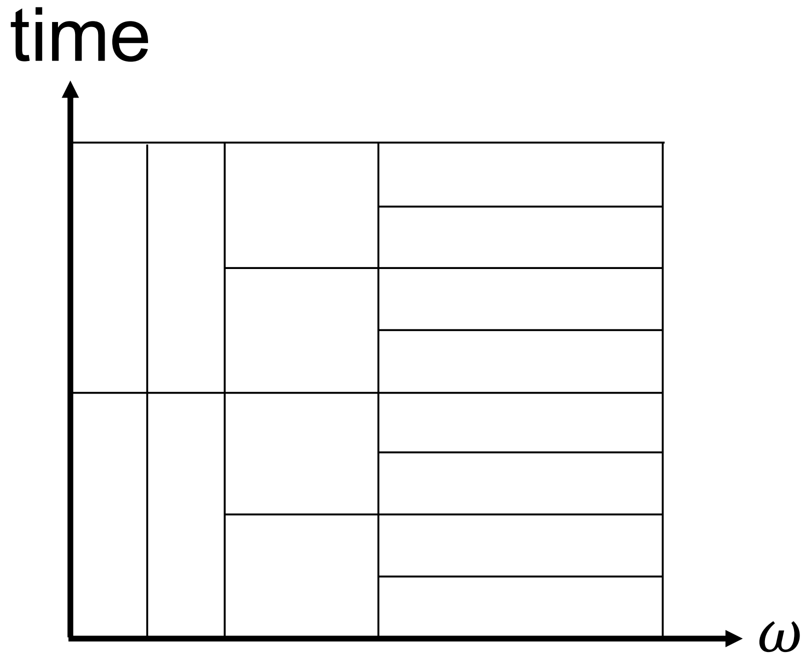
$$\phi(x) = \sum_{n=-\infty}^{\infty} h_0(n)\phi(2x-n)$$ scaling function

$$w(x) = \sum_{n=-\infty}^{\infty} h_1(n)\phi(2x-n)$$ wavelet

Time-Frequency Grid



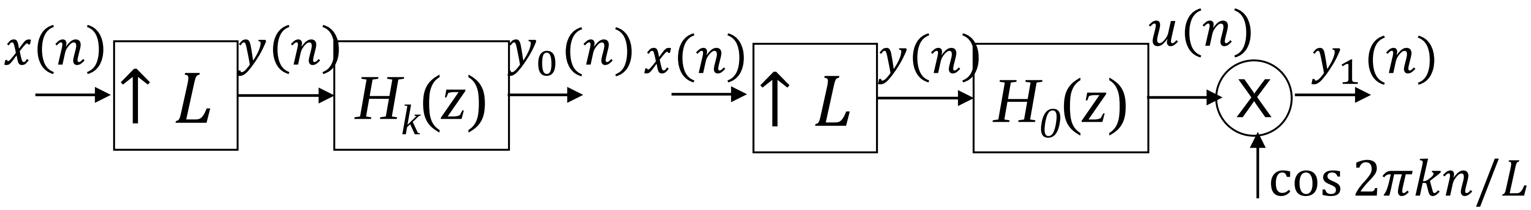
Short Time Fourier Transform
Windowed Fourier Transform



Wavelet Transform

Exercise 7

1. For an alias-free M-channel filter bank let the distortion function be $T(z)$. Show that a new filter bank whose analysis filter $H_k(z)$ and synthesis filter $F_k(z)$ are interchanged is also alias-free. What is the distortion function of that new filter bank?
2. When synthesis filter $F_k(z)$ is replaced by $F_k(zW^{\ell})$ (ℓ is an integer independent of k) for a perfect reconstruction *M*-channel filter bank, derive its output. Can you recover its input from that?
3. Prove that the following two systems are equivalent ($y_0(n) = y_1(n)$) when $h_k(n) = h_0(n)\cos 2\pi kn/L$ holds for arbitrary integer k.



Are the two systems still equivalent if the *L*-fold upsamplers are omitted?

4. Read the following paper;

P. P. Vaidyanathan, “Quadrature Mirror Filter Banks, M-Band Extensions and Perfect-Reconstruction Techniques”, IEEE ASSP Magazine, Vol.4, 3, pp.4- 20, July1987