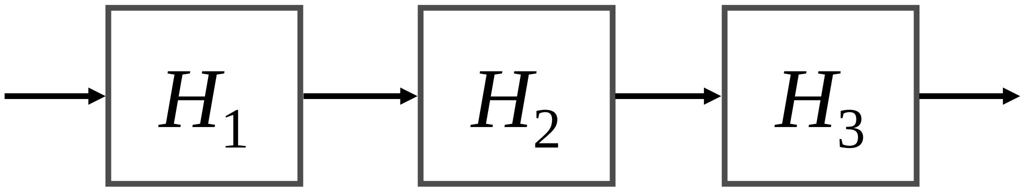
Multirate Systems



f_1 f_2 f_3 : sampling rate

applications

- communication systems
- speech and audio processing systems
- antenna systems
- radar systems
- etc.

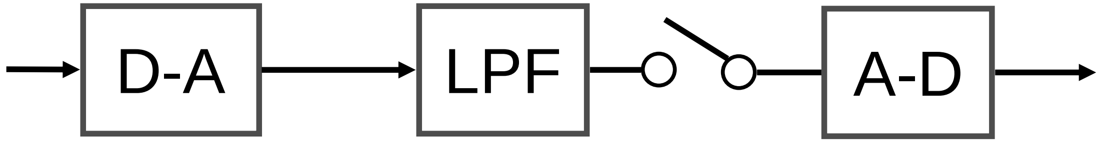
e.q. digital audio systems

CD	44.1kHz
studio	48kHz
broadcast	32kHz
DVD audio	192kHz

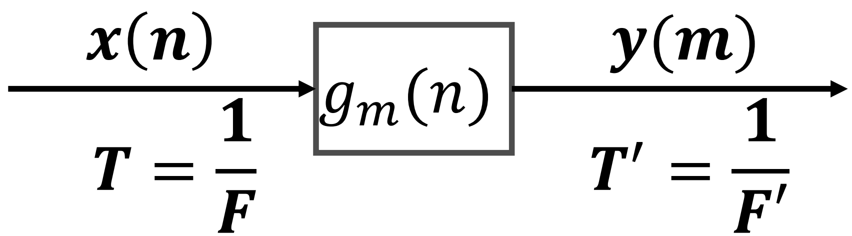
conversion

Sampling Rate Conversion

analog method



direct digital method



linear time-varying system

assume

$$\frac{T'}{T} = \frac{F}{F'} = \frac{M}{L}$$

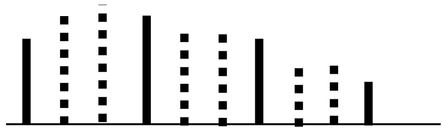
$L \, , M$ **:** mutually prime integers

Sampling Rate Reduction

Decimation by an Integer factor M

$$\frac{T'}{T} = M \qquad\qquad F' = \frac{F}{M}$$

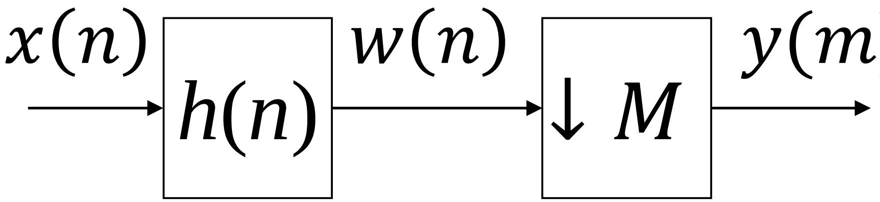
$x(n)$: fullband (0~π)

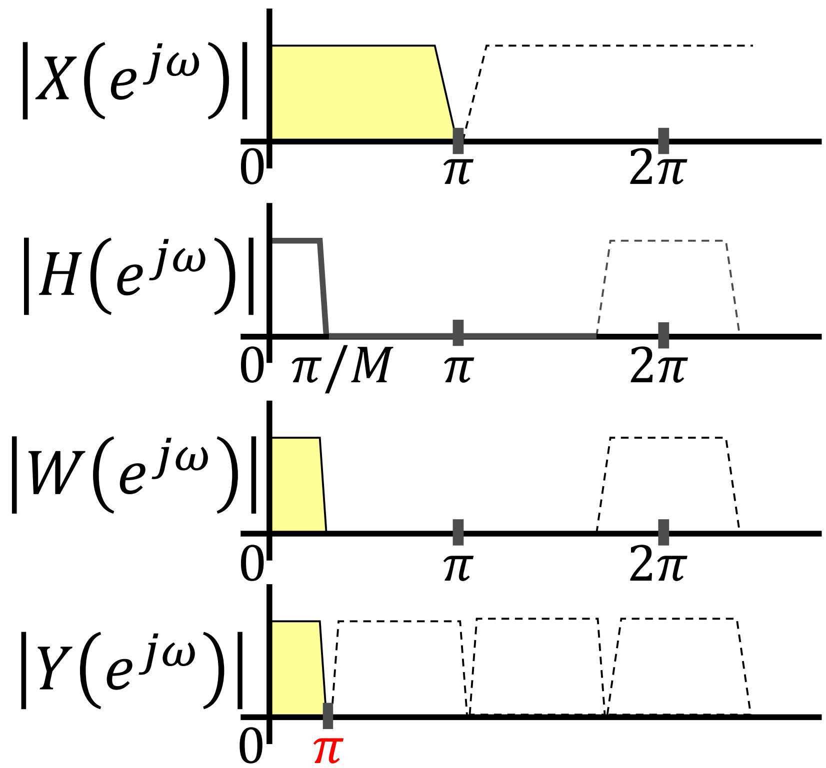


to lower the sampling rate and to avoid aliasing
LPF is necessary

$$h(n) \leftrightarrow H\left(e^{j\omega}\right) = \begin{cases} 1 & |\omega| \leq \frac{2\pi F' T}{2} = \frac{\pi}{M} \\ 0 & \text{otherwise} \end{cases}$$

Observation in Time and Frequency Domain





$$w(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$y(m) = w(Mm)$$

$$= \sum_{-\infty}^{\infty} h(k)x(Mm-k)$$

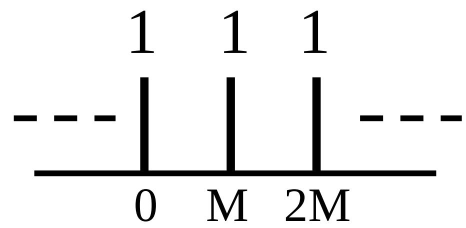
not time-invariant

consider an integer shift (not multiple of M)

z-domain relationship

new signal $w'(n) = \begin{cases} w(n) & n = rM \\ 0 & \text{otherwise} \end{cases}$

product of $w(n)$ and



Fourier series representation

$$w'(n) = w(n)\frac{1}{M}\sum_{\ell=0}^{M-1} e^{j2\pi\ell n/M}$$

z-domain relationship (cont'd)

$$y(m) = w'(Mm) = w(Mm)$$

$$Y(z) = \sum_{m=-\infty}^{\infty} y(m)\, z^{-m} = \sum_{m=-\infty}^{\infty} w'(Mm)\, z^{-m}$$

$w'(m)$ is zero except at integer multiples of M

$$= \sum_{m=-\infty}^{\infty} w'(m)\, z^{-m/M} = \sum_{m=-\infty}^{\infty} w(m) \frac{1}{M} \sum_{\ell=0}^{M-1} e^{j2\pi\ell m/M}\, z^{-m/M}$$

$$= \frac{1}{M} \sum_{\ell=0}^{M-1} \sum_{m=-\infty}^{\infty} w(m)\, e^{j2\pi\ell m/M}\, z^{-m/M} = \frac{1}{M} \sum_{\ell=0}^{M-1} W\left(e^{-j2\pi\ell/M} z^{1/M}\right)$$

z-domain relationship (cont'd)

Since $W(z) = H(z)X(z)$

$$Y(z) = \frac{1}{M}\sum_{\ell=0}^{M-1} H\left(e^{-j2\pi\ell/m} z^{1/M}\right) X\left(e^{-j2\pi\ell/m} z^{1/M}\right)$$

on the unit circle $z = e^{j\omega'}$ $(\omega' = 2\pi f T')$

$$Y\left(e^{j\omega'}\right) = \frac{1}{M}\sum_{\ell=0}^{M-1} H\left(e^{j(\omega'-2\pi\ell)/M}\right) X\left(e^{j(\omega'-2\pi\ell)/M}\right)$$

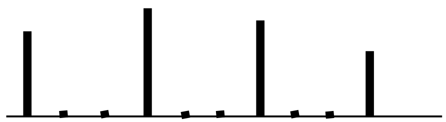
If $H\left(e^{j\omega}\right)$ can filter out the components above $\frac{\pi}{M}$,

$$Y\left(e^{j\omega'}\right) \cong \frac{1}{M} X\left(e^{j\omega'/M}\right) \quad \text{for } |\omega'| < \pi$$

Sampling Rate Increase

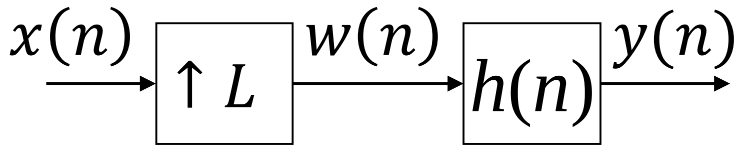
Interpolation by an Integer Factor L

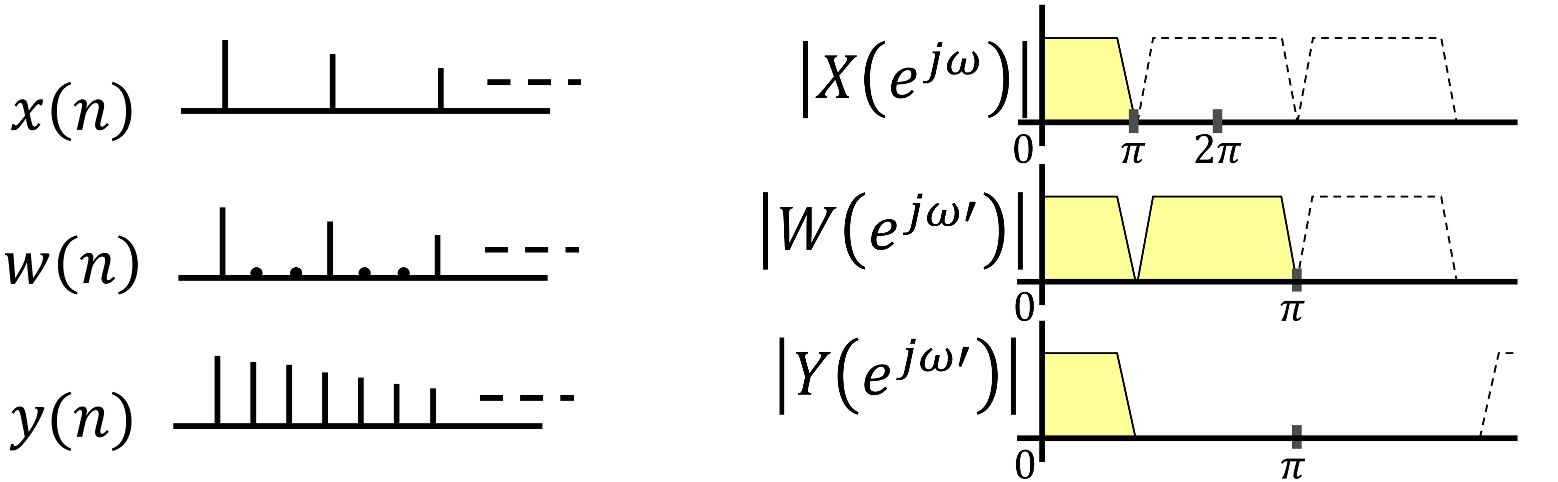
$$\frac{T'}{T} = \frac{1}{L} \qquad F' = LF$$



interpolate L-1 new zero-valued samples between each pair

Observation in Time and Frequency Domain





$$w(n) = \begin{cases} x\left(\frac{m}{L}\right) & m = rL \\ 0 & \text{otherwise} \end{cases}$$

$$W(z) = \sum_{m=-\infty}^{\infty} w(m) z^{-m} = \sum_{m=-\infty}^{\infty} x(m) z^{-mL} = X(z^L)$$

$$W\left(e^{j\omega'}\right) = X\left(e^{j\omega L}\right) \quad (\omega' = 2\pi f T' \quad \omega = 2\pi f T)$$

eliminate the unwanted spectra

$$H\left(e^{j\omega}\right) \cong \begin{cases} G & |\omega'| \leq \dfrac{2\pi F T'}{2} = \dfrac{\pi}{L} \\ 0 & \text{otherwise} \end{cases}$$

then

$$Y\left(e^{j\omega'}\right) = H\left(e^{j\omega'}\right) X\left(e^{j\omega' L}\right) = \begin{cases} G X\left(e^{j\omega' L}\right) & |\omega'| \leq \dfrac{\pi}{L} \\ 0 & \text{otherwise} \end{cases}$$

Why gain *G* ?

$$\left[\begin{array}{l}\text{in the case of decimation,}\\ \text{If } \left|H\left(e^{j\omega}\right)\right| = 1 \text{ for } |\omega| \le \frac{\pi}{M},\\ y(0) = w(0) = x(0)\end{array}\right]$$

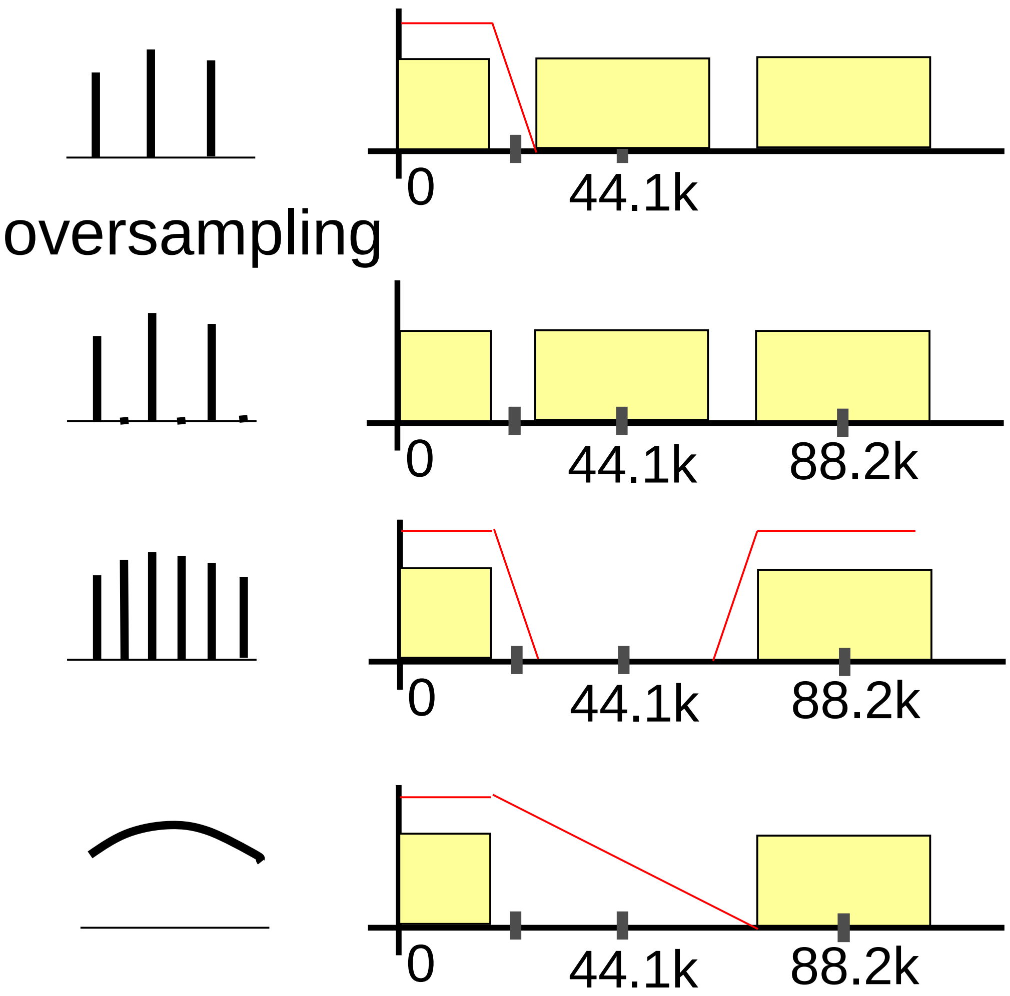
for interpolator ,

$$y(0) = \int_{-\pi}^{\pi} Y\left(e^{j\omega'}\right) d\omega' = \int_{-\pi}^{\pi} H\left(e^{j\omega'}\right) X\left(e^{j\omega' L}\right) d\omega'$$

$$= G \int_{-\pi/L}^{\pi/L} X\left(e^{j\omega' L}\right) d\omega' = G \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) \frac{d\omega}{L} = \frac{G}{L} x(0)$$

so $G=L$ is required to match the amplitude of the envelopes of $y(n)$ and $x(n)$.

Oversampling DF for CD Players



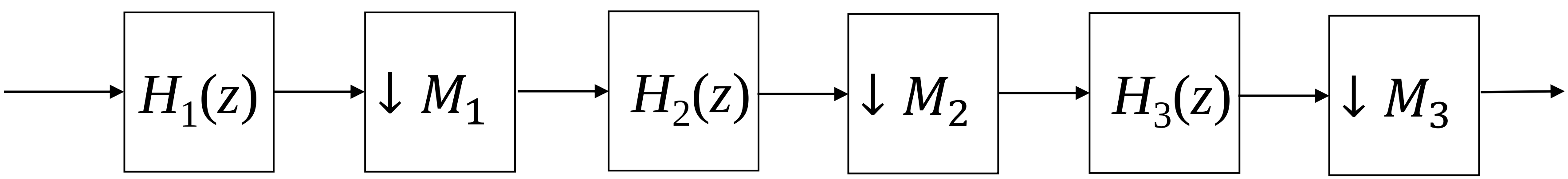
for direct D-A conversion
sharp CT LPF is needed

sharp digital filter
is easier to implement

this gentle CT LPF is
easy to implement

Multistage Implementations

for a large integer factor of
decimation/interpolation



more efficient in terms of
the number of computations per unit time

Polyphase Representation

(Bellanger 1976)

$$H(z)=\sum_{-\infty}^{\infty}h(n)z^{-n}=\sum_{-\infty}^{\infty}h(2n)z^{-2n}+\sum_{-\infty}^{\infty}h(2n+1)z^{-(2n+1)}$$

definition

$$\begin{cases} E_0(z)=\sum_{-\infty}^{\infty}h(2n)z^{-n} \\ E_1(z)=\sum_{-\infty}^{\infty}h(2n+1)z^{-n} \end{cases}$$

$$H(z)=E_0(z^2)+z^{-1}E_1(z^2)$$

examples

FIR

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4}$$
$$= 1 + 3z^{-2} + 5z^{-4} \quad + z^{-1}(2 + 4z^{-2})$$

$$E_0(z) = 1 + 3z^{-1} + 5z^{-2} \qquad E_1(z) = 2 + 4z^{-1}$$

IIR

$$H(z) = \frac{1}{1 - az^{-1}} = \frac{1}{1 - a^2z^{-2}} + \frac{az^{-1}}{1 - a^2z^{-2}}$$

$$E_0(z) = \frac{1}{1 - a^2z^{-1}} \qquad E_1(z) = \frac{a}{1 - a^2z^{-1}}$$

Generalization

$$H(z) = \sum_{k=0}^{M-1} z^{-k} E_k(z^M)$$

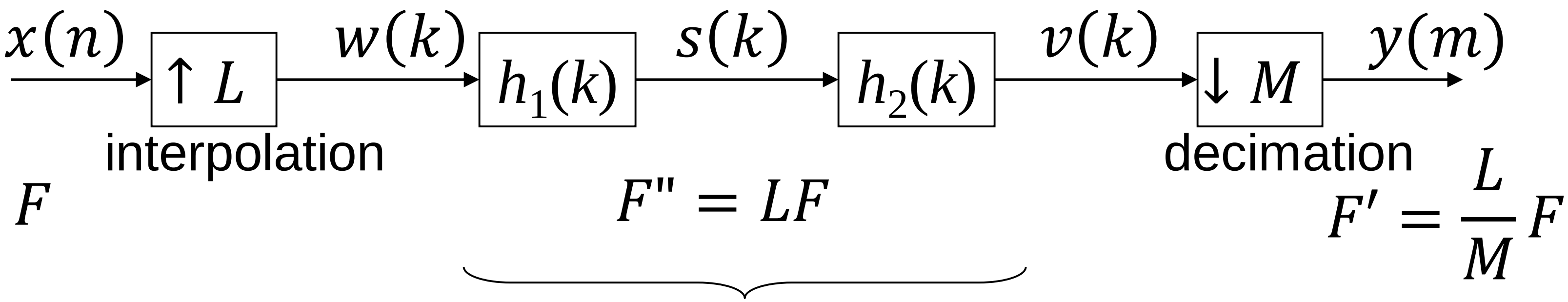
$E_k(z)$: polyphase component

coefficient $e_k(n) = h(nM + k)$

M-fold decimated version of $h(n+k)$

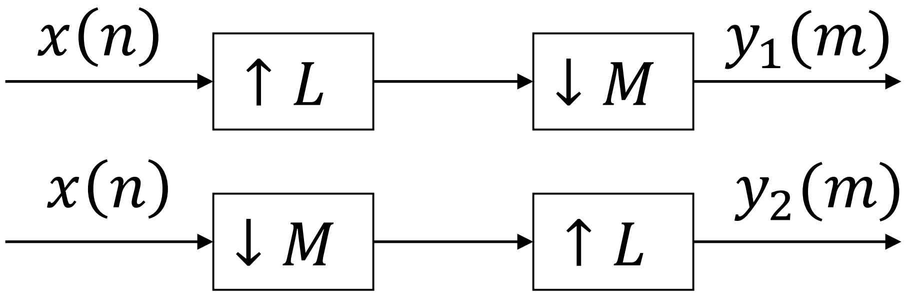
Fractional Sampling Rate Conversion

$$\frac{T'}{T} = \frac{M}{L} \quad \text{or} \quad F' = \frac{L}{M}F$$



$$H\left(e^{j\omega''}\right) \cong \begin{cases} L & |\omega''| = \dfrac{2\pi F T''}{2} \leq \min\left(\dfrac{\pi}{L}, \dfrac{\pi}{M}\right) \\ 0 & \text{otherwise} \end{cases}$$

Upsamper and Downsampler



$$Y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{\frac{L}{M}} e^{-\frac{j2\pi kL}{M}}\right)$$

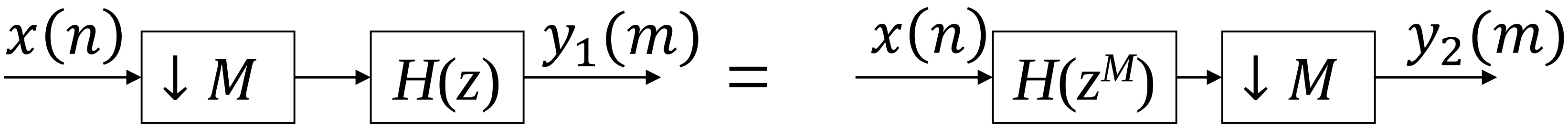
$$Y_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{\frac{L}{M}} e^{-\frac{j2\pi k}{M}}\right)$$

$$\begin{cases} e^{-\frac{j2\pi kL}{M}} & k = 0, 1, \cdots, M-1 \\ e^{-\frac{j2\pi k}{M}} & k = 0, 1, \cdots, M-1 \end{cases}$$

identical set if and only if *M* and *L* are relatively prime integers

The Noble Identities

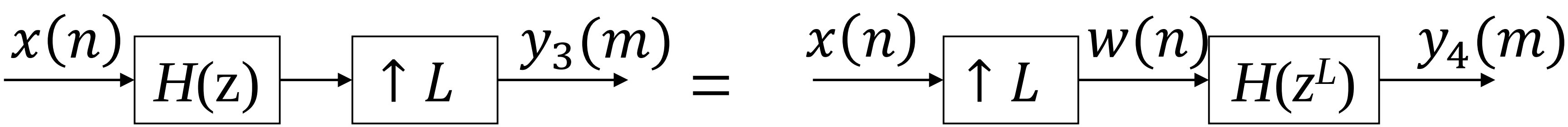
$H(z)$: rational transfer function



$$Y_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{-j2\pi k/M} z^{1/M}\right) H\left(\left(e^{-j2\pi k/M} z^{1/M}\right)^M\right)$$

$$= \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{-j2\pi k/M} z^{1/M}\right) H(z)$$

$$= Y_1(z)$$



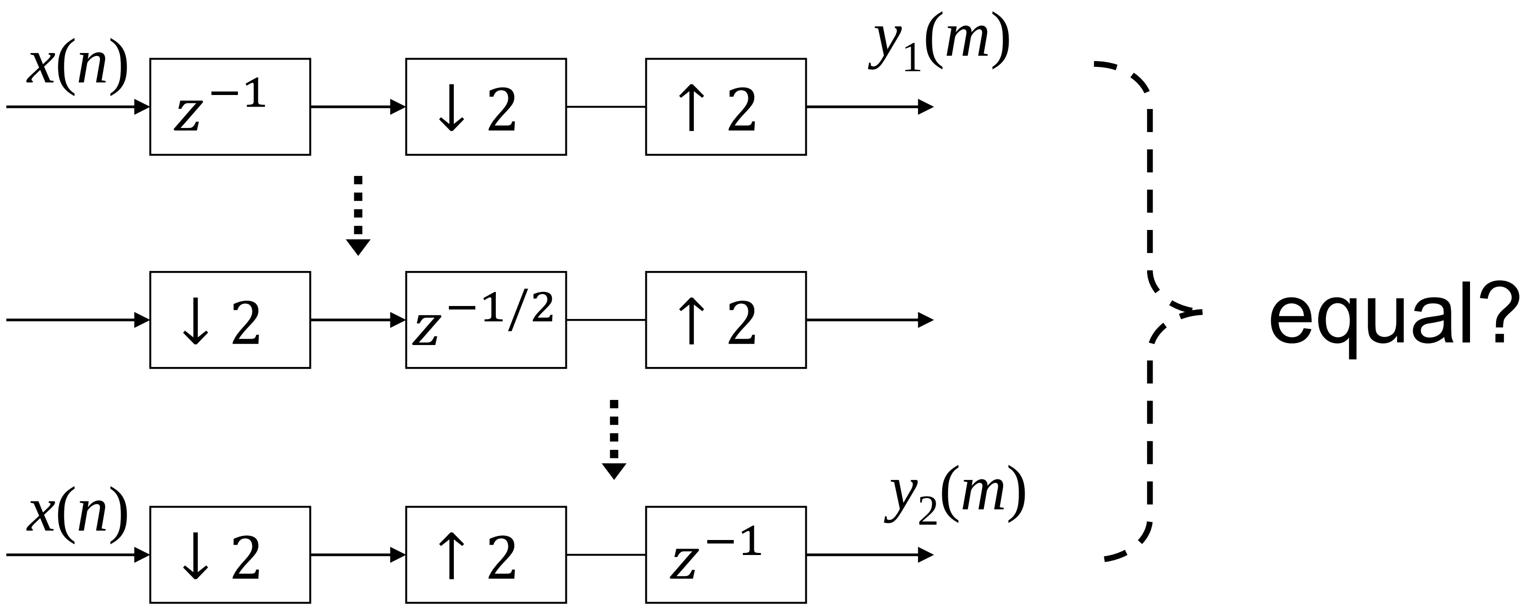
$$Y_4(z) = H(z^L) W(z)$$

$$= H(z^L) X(z^L)$$

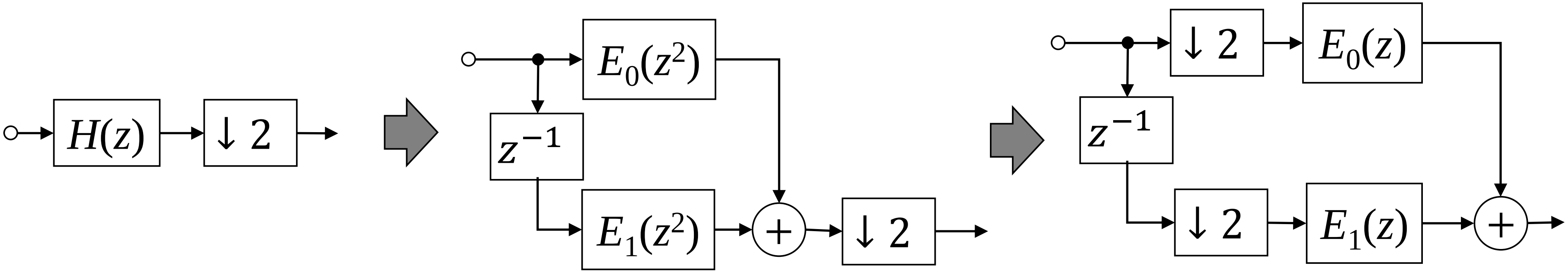
$$= Y_3(z)$$

The Noble Identity Does Not Hold

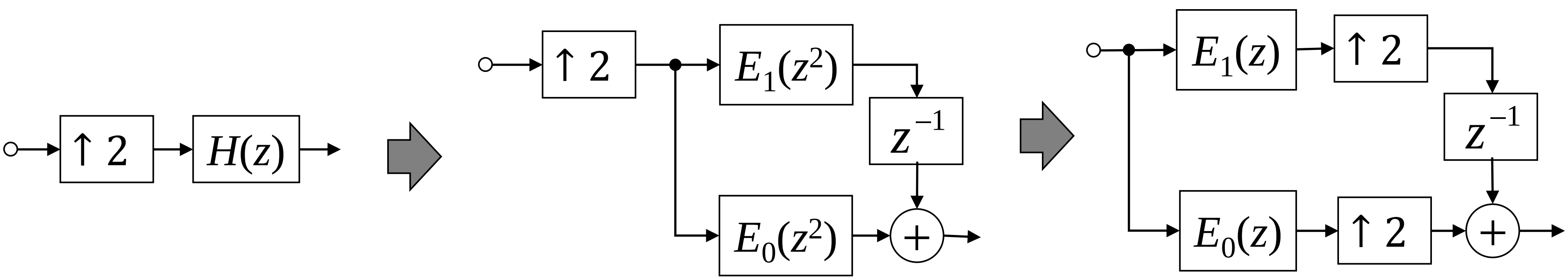
when $H(z)$ is irrational



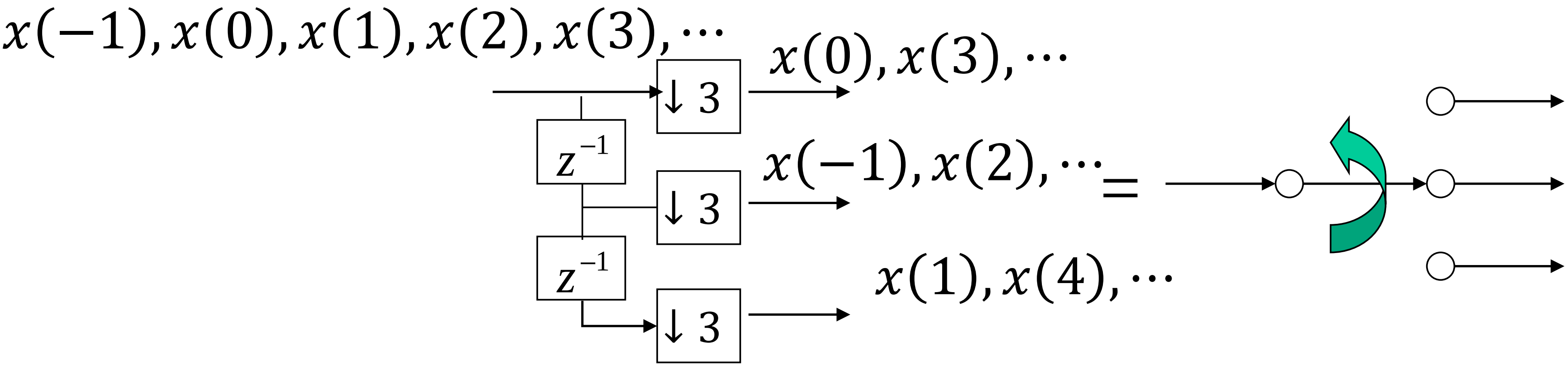
Decimation Filters

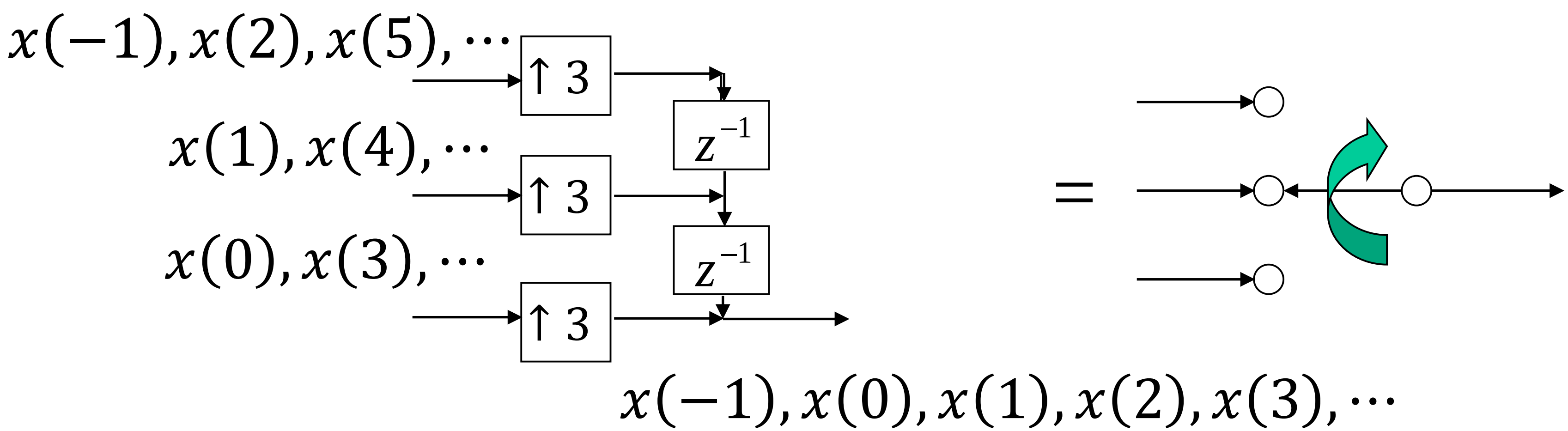


Interpolation Filters

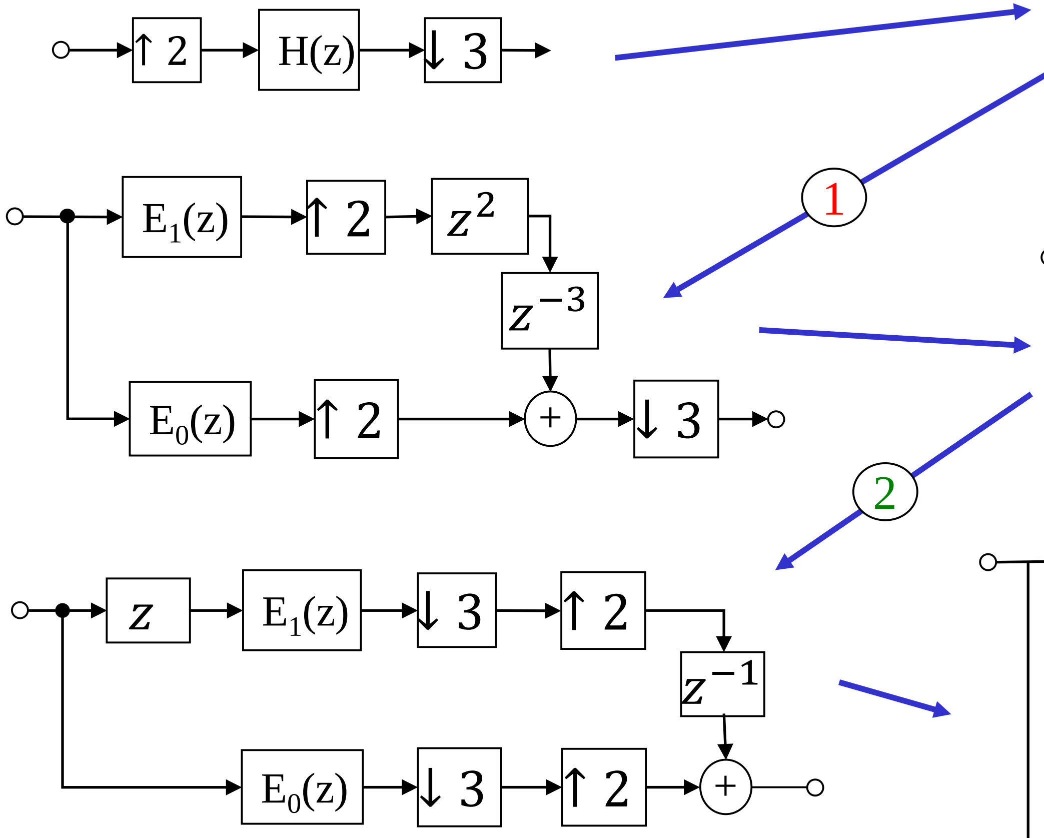


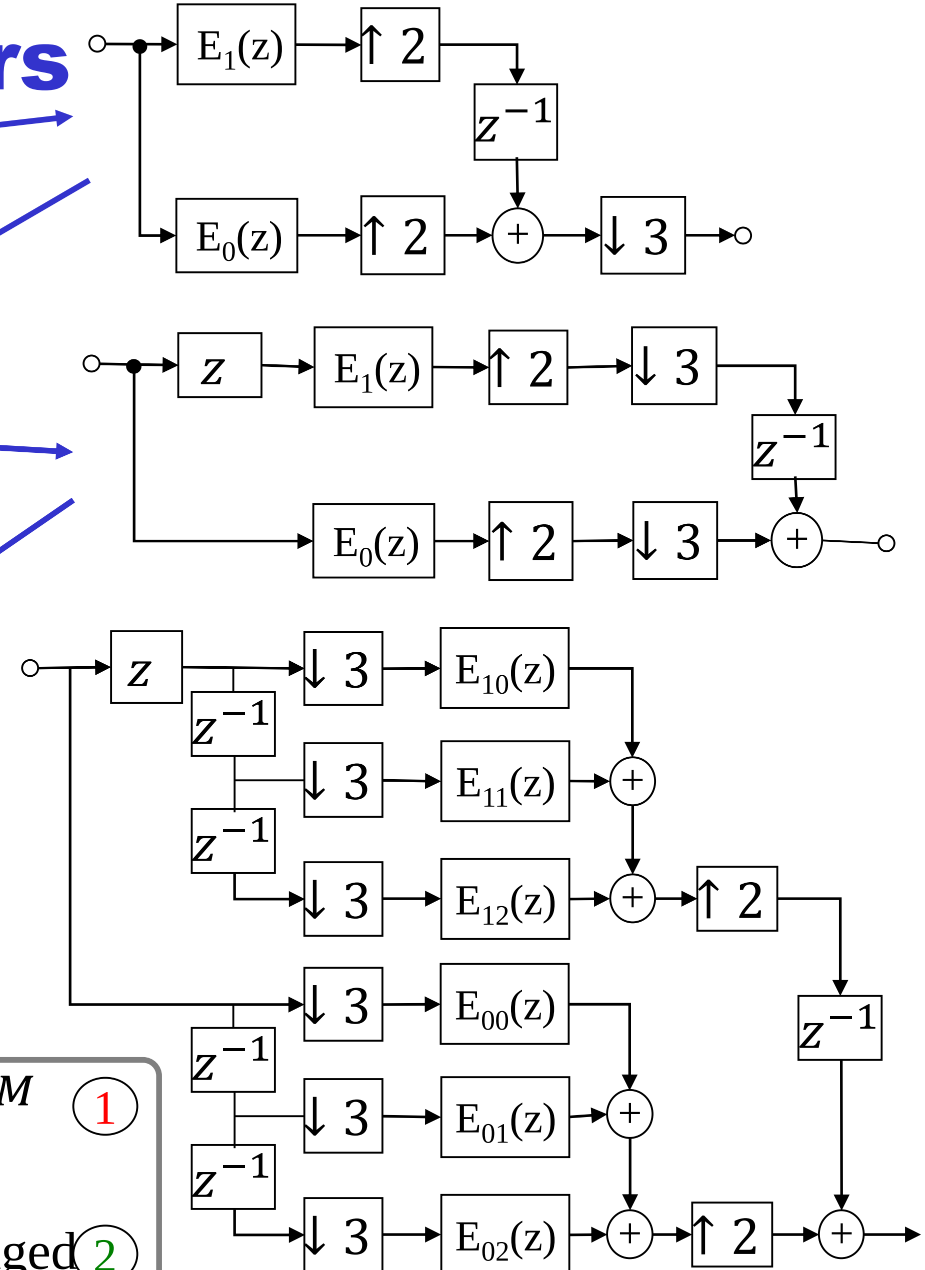
Commutator Model





Fractional Decimators





L and *M* are mutually prime

$$-n_0L + n_1M = 1 \qquad z^{-1} = z^{n_0L}z^{-n_1M}$$ (1)

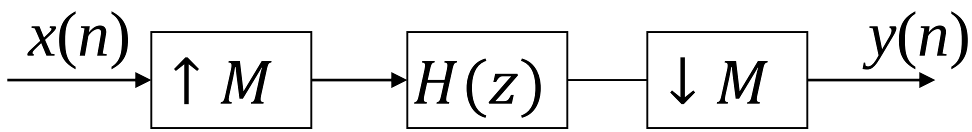
(Euclid Theorem)

decimation and interpolation can be interchanged (2)

For length N FIR $H(z)$, N multiplications at $\frac{1}{M}$ rate

Exercise 4

1. Show that the system shown in the figure is linear and shift-invariant. How is the overall transfer function related to $H(z)$?



2. Consider the two sets of M numbers given by W^k and W^{kL}, where $W=e^{-j2\pi/M}$ and k is an integer from 1 to M. Show that these sets are identical if and only if L and M are relatively prime.
3. Given a software (or equipment) for 1024-point *DFT*, we would like to compute the *DFT* of a 512-point sequence $f(n)$, for n from 0 to 511. Compare the following three methods to fill the vacancy of the input sequence.
   a. use $f(n)$ for n from 0 to 511 and fill 0 for n 512 to 1023,
   b. place $f(n)$ in the middle and fill 0 before and after the sequence, or
   c. repeat $f(n)$ twice.
4. Read the following paper and study how an IIR filter is decomposed into polyphase components.

   M. Bellanger, G. Bonnerot, M. Coudreuse, Digital filtering by polyphase network: Application to sample-rate alteration and filter banks, IEEE Trans. on Acoustics, Speech and Signal Processing, vol.24, 2, pp. 109- 114, Apr. 1976