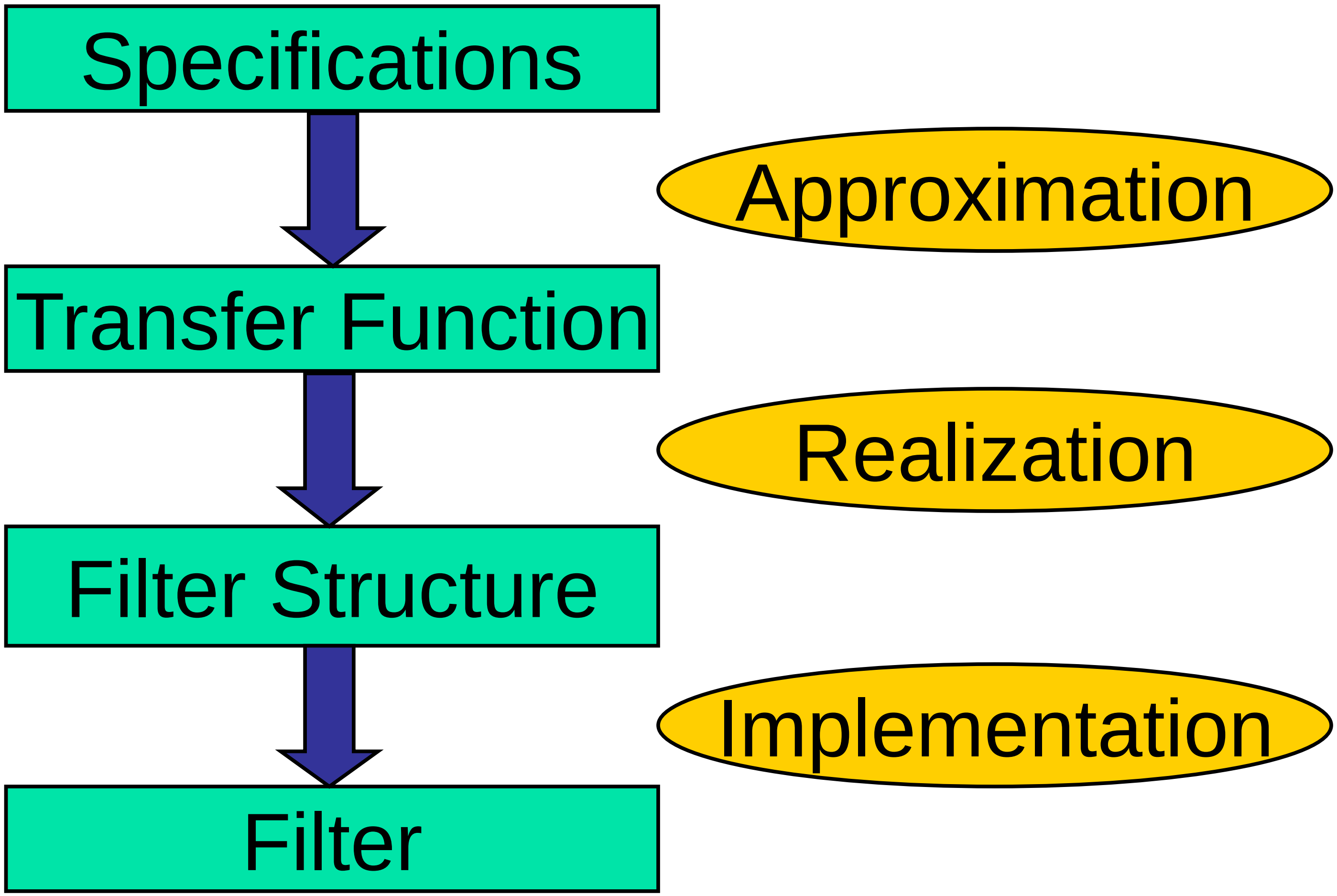
Digital Filter Design Procedure



Digital Filter Design

- FIR
  - possibly linear-phase response
  - always stable
- IIR
  - sharp cutoff with lower order

Different Design Methods

Design of FIR Filters

having Linear-Phase

- N-tap (N-1[th] order) FIR Filter

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-jn\omega}$$

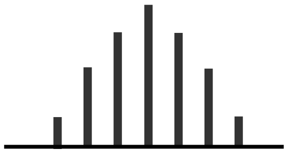
has linear-phase if it can be written as $H_1(\omega)e^{-j(\alpha\omega+\beta)}$, where H_1 is a real even function in ω

When $H_1(\omega)$ is positive arg$H(e^{j\omega}) = -\alpha\omega - \beta$

When $H_1(\omega)$ is negative arg$H(e^{j\omega}) = -\alpha\omega - \beta - \pi$

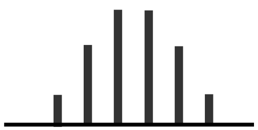
Four Types of Linear-Phase FIR Filters (1)

- $N = 2M + 1, h(n) = h(N-1-n)$, for all n



$$H(e^{j\omega}) = \sum_{n=-M}^{M} h(M+n)e^{-j(n+M)\omega} = e^{-jM\omega}\{h(M) + 2\sum_{n=1}^{M} h(M+n)\cos n\omega\}$$

$$\begin{cases} \alpha = M, \beta = 0 \\ H_1(\omega) = h(M) + 2\sum_{n=1}^{M} h(M+n)\cos n\omega \end{cases}$$

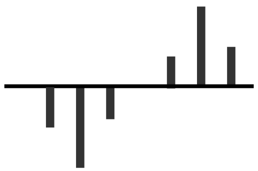


- $N = 2M, h(n) = h(N-1-n)$

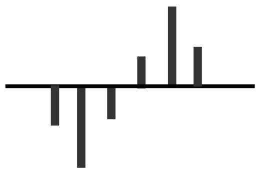
$$\begin{cases} \alpha = M - \frac{1}{2}, \beta = 0 \\ H_1(\omega) = 2\sum_{n=1}^{M} h(M+n-1)\cos\left(n - \frac{1}{2}\right)\omega \end{cases}$$

Four Types of Linear-Phase FIR Filters (2)

- $N = 2M + 1, h(n) = -h(N-1-n)$



$$\begin{cases} \alpha = M, \beta = \frac{\pi}{2} \\ H_1(\omega) = 2\sum_{n=1}^{M} h(M+n)\sin n\omega \end{cases}$$



- $N = 2M, h(n) = -h(N-1-n)$

$$\begin{cases} \alpha = M - \frac{1}{2}, \beta = \frac{\pi}{2} \\ H_1(\omega) = 2\sum_{n=1}^{M} h(M+n-1)\sin\left(n-\frac{1}{2}\right)\omega \end{cases}$$

Fourier Series Design

$H_d(\omega)$: desired response

periodic function with period 2π

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expressed using DTFT

$$H_d(\omega) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-jn\omega}\ , h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{jn\omega}\, d\omega$$

$h_d(n)$: real $\Leftrightarrow$ $H_d(-\omega) = H_d^*(\omega)$

$H_d(\omega)$: real $\Leftrightarrow$ $h_d(n) = h_d(-n)$ (e.g. ideal lowpass)

$H_d(\omega)$: pure imaginary $\Leftrightarrow$ $h_d(n) = -h_d(-n)$

(e.g. differentiator $H_d(\omega) = j\omega$)

Mean Squared Error

$H_1(e^{j\omega}) = \sum_{n=n_1}^{n_2} h_1(n) e^{-jn\omega}$ is designed so that mean squared error

$\varepsilon^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_d(\omega) - H_1(e^{j\omega})|^2 \, d\omega$ is minimized

Parseval's equation

$$\varepsilon^2 = \sum_{n=-\infty}^{\infty} |h_d(n) - h_1(n)|^2$$

The solution is $h_1(n) = h_d(n) \ \ (n_1 \le n \le n_2)$

If $H_d\ (\omega)$ is a real even function and $n_1 = -M, n_2 = M$

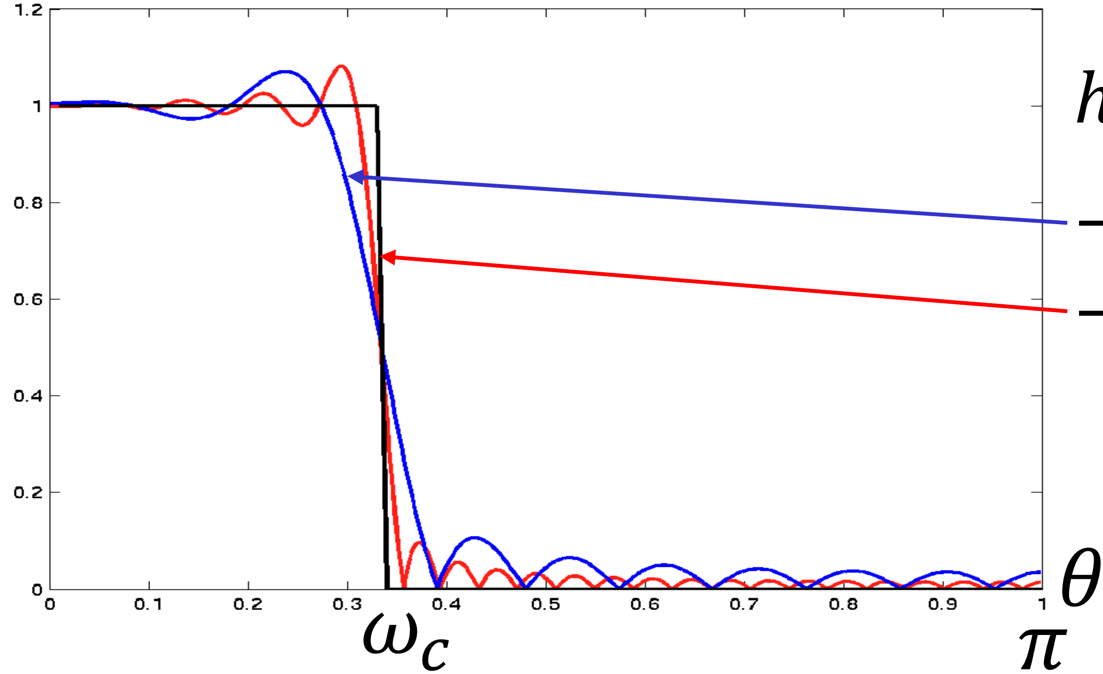
$H_1(e^{j\omega})$ is also real and even

A causal filter is obtained as

$$h(n) = h_1(n-M) \overset{DTFT}{\longleftrightarrow} H(e^{j\omega}) = e^{-jM\omega} H_1(e^{j\omega})$$

Gibbs Phenomenon

$$|H_1(e^{j\omega})|$$
$$H_d(e^{j\omega})$$



$$h(n) = \frac{\sin n\omega_c}{n\pi}$$

$$-10 \le n \le 10$$

$$-25 \le n \le 25$$

- Truncation of Fourier Coefficients

- oscillation in frequency response
- sharp over- and under-shoot around discontinuous point

Gibbs Phenomenon

J.W.Gibbs, “Fourier Series”, Nature, 1899

Window Design

Fourier coefficients are smoothly truncated using a window function $w(n)$

$$h_1(n) = h_d(n)w(n) \overset{DTFT}{\longleftrightarrow} H_1(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\phi)W\left(e^{j(\omega-\phi)}\right)d\phi$$

Rectangular window $w(n) = \begin{cases} 1 & |n| \le M \\ 0 & otherwise \end{cases}$

Hann window $w(n) = \begin{cases} 0.5 + 0.5\cos\left(\frac{\pi}{M}n\right) & |n| \le M \\ 0 & otherwise \end{cases}$

Hamming window $w(n) = \begin{cases} 0.54 + 0.46\cos\left(\frac{\pi}{M}n\right) & |n| \le M \\ 0 & otherwise \end{cases}$

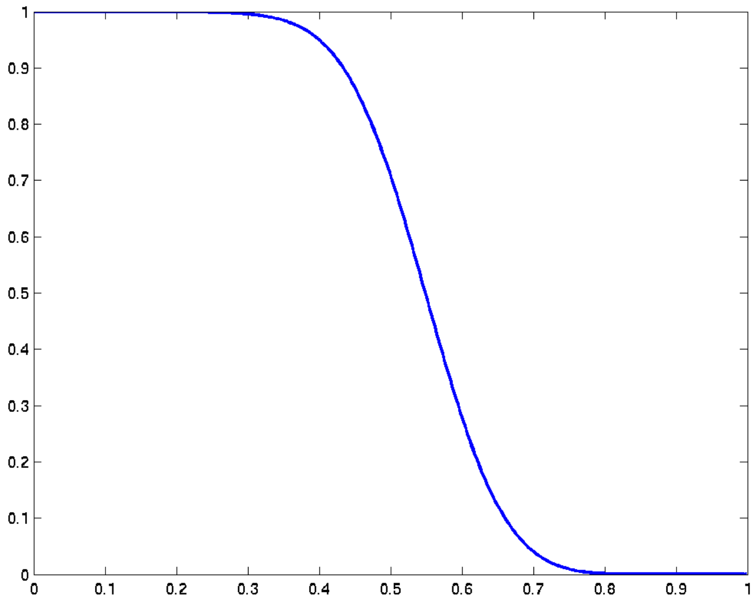
Kaiser window $w(n) = \begin{cases} \dfrac{I_0\left(\beta\sqrt{1-\left(\frac{n}{M}\right)^2}\right)}{I_0(\beta)} & |n| \le M \\ 0 & otherwise \end{cases}$

I_0 is the zero-order modified Bessel function of the first kind

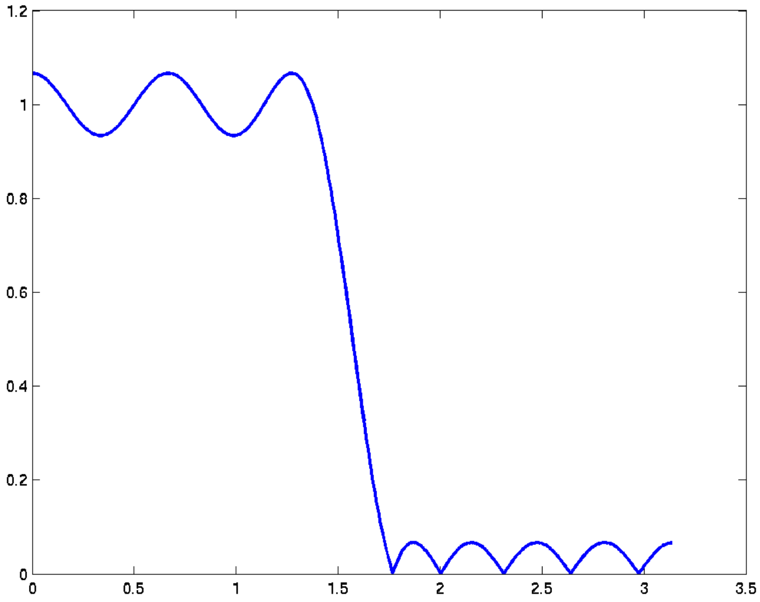
Square errors increases but
Ripple decreases, not optimum in any sense

Optimum Design

- maximally flat



- mini max



Maximally Flat Filters

Maximum possible vanishing derivatives of $H(e^{j\omega})$ at $\omega = 0$ and π
Analytical solution available;

$$H_{N,K,d}(z) = \left(\frac{1+z^{-1}}{2}\right)^{K} \sum_{j=0}^{N-K} \sum_{i=0}^{j} (-1)^{j-i} \binom{\frac{N}{2}-d}{i} \binom{\frac{N}{2}+d}{j-i} \left(\frac{1-z^{-1}}{2}\right)^{j} \left(\frac{1+z^{-1}}{2}\right)^{N-K-j}$$

N: filter order
K: the number of zeros at $z = -1$
d: group delay is $\frac{N}{2} + d$ at $\omega = 0$

JAVA applet available at
http://www.nh.cradle.titech.ac.jp/old/maxflat/

Minimax Design

Chebyshev error

$\max_{\omega} \left| H_d(\omega) - H_1\left(e^{j\omega}\right) \right|$ is minimized

for odd *N* and even symmetry, for example

$$H_1\left(e^{j\omega}\right) = h_1(0) + 2\sum_{n=1}^{M} h_1(n) \cos n\omega = p(\cos\omega)$$

$p(x)$: M-th order polynomial in x

$\cos n\omega = T_n(\cos\omega)$ Chebyshev polynomial

Chebyshev's Theorem

For a continuous function $d(x)$ in $-1 \le x \le 1$

1. There exist a unique function $p(x)$ with the order at most N such that
   $\max_x |d(x) - p(x)|$ is minimized.

2. A function is that $p(x)$ if and only if

   $d(x_i) - p(x_i) = (-1)^i \varepsilon, i = 0, 1, \cdots, N+1$ (equiripple)

   holds at $N + 2$ points, x_i where
   $-1 \le x_0 < x_1 < \cdots < x_{N+1} \le 1$

Minimax Solution

Not analytically solved.

(it is easy if x_i's are known)

- Numerical solution

  Remez exchange algorithm
- free software
- MATLAB

Design of IIR Filters

$$H(z) = \frac{\sum_{i=0}^{N} a_i z^{-i}}{1 + \sum_{i=1}^{N} b_i z^{-i}}$$

Use known theories for continuous filters

Mapping $s \to z$

Certain index must be preserved

Impulse Invariance

Impulse response of a digital filter
= samples of IR of a model continuous-time filter

$h(n) = T\hat{h}(nT)$
$\hat{H}(s)$: *N*-th order transfer function

- Numerator order < denominator order
- No multiple roots in denominator

$$\hat{H}(s) = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \cdots + \frac{c_N}{s - p_N} = \sum_{i=1}^{N} \frac{c_i}{s - p_i}$$

Impulse Response

$\hat{h}(t) = c_1 e^{p_1 t} + c_2 e^{p_2 t} + \cdots + c_N e^{p_N t}, \qquad t > 0$

is sampled at $t = nT$

$h(n) = Tc_1 e^{p_1 nT} + Tc_2 e^{p_2 nT} + \cdots + Tc_N e^{p_N nT}$

Its *z*-transform is

$$H(z) = \sum_{i=1}^{N} \frac{Tc_i}{1 - e^{p_i T} z^{-1}}$$

Frequency response is expressed as

$H\left(e^{j\omega T}\right) = \sum_{k=-\infty}^{\infty} \hat{H}\left(j\omega + jk\frac{2\pi}{T}\right)$ (because of sampling)

which includes aliases

The method is valid only when $|\hat{H}\left(j\frac{\pi}{T}\right)|$ is small enough

s-z Transform

$H\left(e^{j\omega}\right) = \widehat{H}(j\Omega(\omega))$

No alias effect occurs if

$-\pi \leq \omega \leq \pi$ corresponds to entire $-\infty \leq \Omega \leq \infty$

Mapping $z = \varphi(s)$ or

$\widehat{H}(s) = H(\varphi(s))$

Conditions for the mapping

1. $j\Omega$ axis corresponds to the unit circle: $\varphi(s) = e^{j\omega}$
2. Inverse transform $s = \varphi^{-1}(z)$ exists: $H(z) = \widehat{H}(\varphi^{-1}(z))$
3. Stability preserved: $Re(s) < 0 \Leftrightarrow |\varphi(s)| < 1$
4. DC response preserved: $\varphi(0) = 1$

Mapping which satisfies all the above

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Bilinear Transform

Bilinear Transform

$$z = \varphi(s) = \frac{1+s}{1-s}$$

$$s = \frac{z-1}{z+1}$$

substitution of $z = e^{j\omega}$ leads to

$$j\Omega(\omega) = \frac{e^{j\omega}-1}{e^{j\omega}+1} = j\tan\frac{\omega}{2} \Rightarrow \Omega(\omega) = \tan\frac{\omega}{2}$$

$$H(z) = \widehat{H}\left(\frac{z-1}{z+1}\right)$$

http://momiji.i.ishikawa-nct.ac.jp/dfdesign_eng/iir/i_lpf.shtml

Other Design Method

- Eigen filter
- Time-domain design
- Linear Programming
- Semi definite Programming
- etc.

Exercise 2

1. If $h(n) = h^*(-n)$, what is the specific feature of its frequency response $H(e^{j\omega})$?
2. What is the advantage of linear phase property in digital filters?
3. Obtain the filter coefficients using the program at

   http://www.nh.cradle.titech.ac.jp/old/maxflat/PlotMFnew.htm

   with $N = 10, K = 5, d = 0$, and discuss about the results.
4. In some textbooks the bilinear transformation is defined by

   $$s = \frac{2}{T}\frac{z-1}{z+1},$$

   which is different from the one in this lecture. Explain the physical meaning of the difference.
5. Read the following paper;

   O. Hermann and W.Schuessler, Design of nonrecursive digital filters with minimum phase, Electronics Letters, vol. 6, Issue 11, pp.329-330, May 1970