数値解析への入り口
Entrance to numerical analysis

For vibration simulation of complicated structures, numerical methods are used.

0.Preparation : vibration in a string

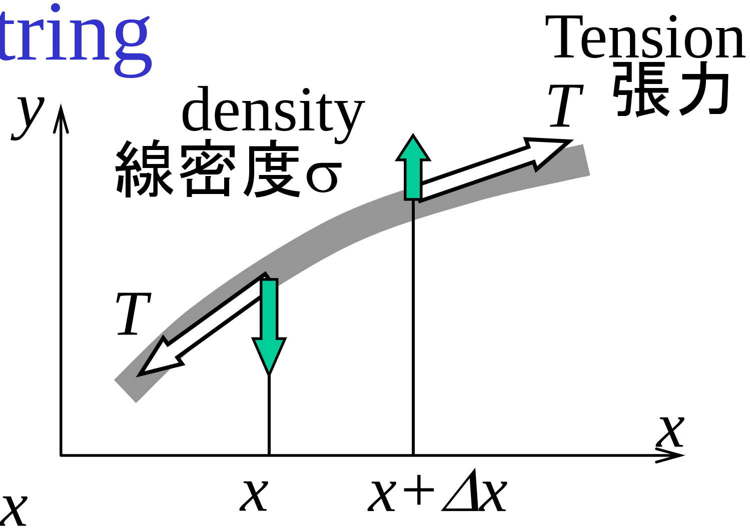
1.Energy methods

 1) Rayleigh method

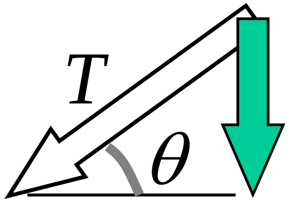
 2) Ritz method

2.Discrete methods

弦の振動1 Vibration of a string



Vertical components of tension at $x=x$



$$T\sin\theta \approx T\tan\theta = T\frac{dy}{dx}$$

Vertical components of tension at $x=x+\Delta x$

$$\left.\frac{dy}{dx}\right|_{x=x+\Delta x} = \frac{dy}{dx} + \frac{d^2y}{dx^2}\Delta x \quad \text{より} \quad T\frac{dy}{dx} + T\frac{d^2y}{dx^2}\Delta x$$

From the equilibrium between the vertical component of the tension and the inertial force acting on the small part:

$$T\frac{d^2y}{dx^2} = \sigma\frac{d^2y}{dt^2}$$

Wave equation for vibration of a string

弦の振動2 Vibration of a string

弦の波動方程式
Wave equation

$$\frac{d^2y}{dx^2} = \frac{1}{c^2}\frac{d^2y}{dt^2}$$

弦振動の伝搬速度
Propagation speed

$$c = \sqrt{\frac{T}{\sigma}}$$

T, 張力 [N]
σ, 線密度 [kg/m]

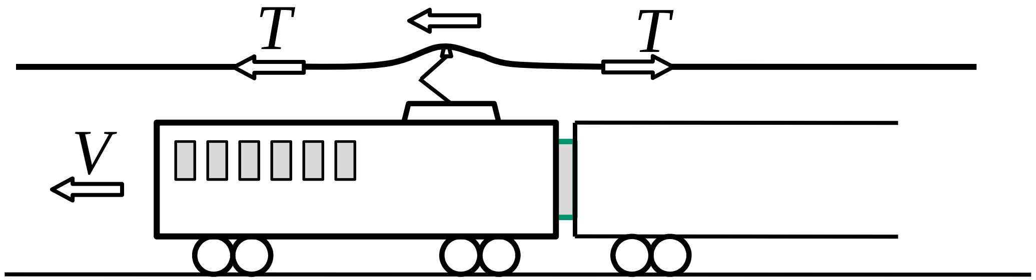
弦の最大許容応力を$P_{\max}$として、最大伝搬速度は $c_{\max} = \sqrt{\frac{P_{\max}}{\rho}}$

Assuming the maximum stress without failure $P_{\max}$, the maximum propagation speed of $c_{\max}$ can be defined.

Maximum speed of electric train

Max. safety stress in the feeding line, $P_{\max}$

Speed of the train should be smaller than the wave propagation to keep stable contact between the feeding line and the pantograph.



$$V < c_{\max} = \sqrt{\frac{P_{\max}}{\rho}}$$

Strength of materials

Duralumin A2017 ＝500 MPa、 S-duralumin A2024＝323 MPa

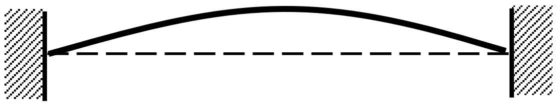
SS-duralumin A7075＝505 MPa、 Steel S45C＝727 MPa

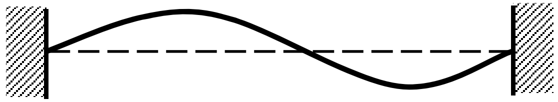
Stainless steel SUS304＝205 MPa、 Ti＝800 MPa

Max speed of the train: SUS304→580 km/h; A7075→1500 km/h

Energy method 1

両端固定の弦





厳密解 $\omega_1^{\ 2} = \dfrac{2.46740}{l^2}\left(\dfrac{T}{\sigma}\right)$ $\qquad \omega_2^{\ 2} = \dfrac{22.207}{l^2}\left(\dfrac{T}{\sigma}\right)$

Analytic solution

張力T, 線密度 σ, 長さ$2l$

Rayleigh法

$$T_{\max} = U_{\max} \quad \begin{cases} T: \text{運動エネルギー Kinetic energy} \\ U: \text{ポテンシャルエネルギー Potential energy} \end{cases}$$

変位分布を仮定する。例えば、$y = a(l^2 - x^2)$

Assume a displacement function:

$$\left.\begin{aligned} T_{\max} &= \int_{-l}^{l} \frac{1}{2}\rho\left(\frac{dy}{dt}\right)^2 dx = \int_{-l}^{l} \frac{1}{2}\rho\omega^2 y^2 dx \\ U_{\max} &= \int_{-l}^{l} \frac{1}{2} T\left(\frac{dy}{dx}\right)^2 dx \end{aligned}\right\} \Longrightarrow \omega_1^{\ 2} = \frac{2.5}{l^2}\left(\frac{T}{\sigma}\right)$$

誤差 +1.3%

Energy method 2:

Ritz法 $L = T_{\max} - U_{\max}$ を最小にする Minimize L

Assume the displacement as below:

変位分布を $y = a_1\varphi_1(x) + a_2\varphi_2(x) + \cdots + a_i\varphi_i(x) + \cdots$

として、 $\dfrac{\partial L}{\partial a_i} = 0$ より a_i を定める。

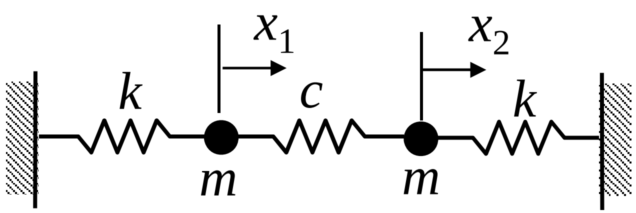
ここでは、例えば、 $y = a_1(l^2 - x^2) + a_2 x^2 (l^2 - x^2)$

境界条件を満たす関数形の線形結合

$$\begin{cases} \omega_1^{\ 2} = \dfrac{2.46744}{l^2}\left(\dfrac{T}{\sigma}\right) & \text{誤差 } +0.0016\% \\[2ex] \omega_2^{\ 2} = \dfrac{25.6}{l^2}\left(\dfrac{T}{\sigma}\right) & \text{誤差 } +15\% \end{cases}$$

Discrete method

Example: deg. of freedom = 2



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{x}$$

変位ベクトル

Displacement vector

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} = \mathbf{M}$$

質量マトリックス

Mass matrix

$$\begin{pmatrix} k+c & -c \\ -c & k+c \end{pmatrix} = \mathbf{k}$$

弾性マトリックス

Elastic matrix

左側の質点に関する運動方程式

$$m\frac{d^2 x_1}{dt^2} = -kx_1 + c(x_2 - x_1)$$

右側の質点に関する運動方程式

$$m\frac{d^2 x_2}{dt^2} = -kx_2 - c(x_2 - x_1)$$

調和振動を考える＋行列表示

Harmonic case, matrix expression

$$-\omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} k+c & -c \\ -c & k+c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$(\omega^2 \mathbf{M} - \mathbf{k})\mathbf{x} = 0$$

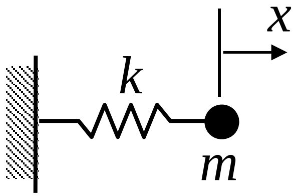
共振周波数の式

$$\left| \omega^2 \mathbf{M} - \mathbf{k} \right| = 0$$

Frequency equation

Discrete method

単振り子との比較



$$m\frac{d^2x}{dt^2} = -kx$$

$$(\omega^2 m - k)x = 0$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$(\omega^2\mathbf{M} - \mathbf{k})\mathbf{x} = 0$$ 共振周波数の式

$$\left|\omega^2\mathbf{M} - \mathbf{k}\right| = 0$$

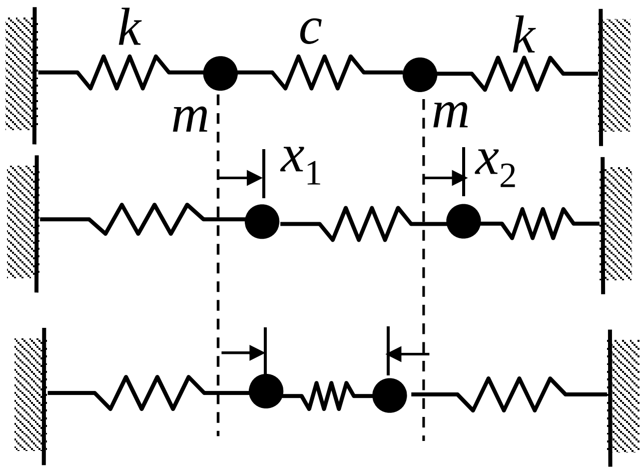
$$\begin{vmatrix} \omega^2 m - (k+c) & -c \\ -c & \omega^2 m - (k+c) \end{vmatrix} = 0$$

$$(\omega^2 m - k)\left\{\omega^2 m - (k+2c)\right\} = 0$$

$$\omega = \sqrt{\frac{k}{m}},\quad \sqrt{\frac{k+2c}{m}}$$

共振周波数（固有値）

Eigen values= resonance frequencies



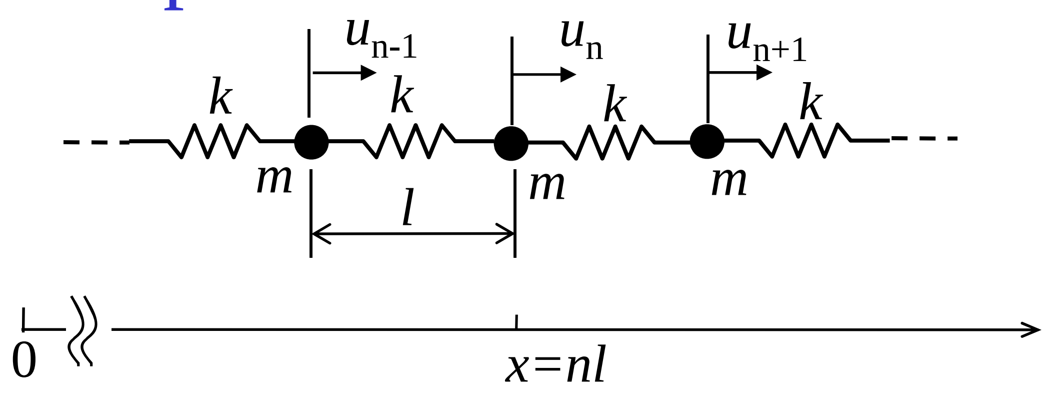
<u>固有ベクトル＝振動モード</u>

Eigen vector = vibration mode

$x_1 = x_2$ Lower mode

$x_1 = -x_2$ Higher mode

Sound speed in the discrete method (1/2)



$$u_n = A\sin(\omega t - n\varphi)$$

Wave equation for the n-th mass
$$m\frac{d^2u_n}{dt^2} = -k(u_n - u_{n-1}) + k(u_{n+1} - u_n)$$

$$m\frac{d^2u_n}{dt^2} = -k\{\sin(\omega t - n\varphi) - \sin(\omega t - n\varphi + \varphi)\} + k\{\sin(\omega t - n\varphi - \varphi) - \sin(\omega t - n\varphi)\}$$

$$= -2kA\left\{\cos(\omega t - n\varphi + \frac{\varphi}{2}) - \cos(\omega t - n\varphi - \frac{\varphi}{2})\right\}\sin\frac{-\varphi}{2}$$

$$= -2kA\left\{-2\sin(\omega t - n\varphi)\sin\frac{-\varphi}{2}\right\}\sin\frac{-\varphi}{2}$$

$$= -4kA\sin(\omega t - n\varphi)\sin^2\frac{\varphi}{2}$$

$$m\frac{d^2u_n}{dt^2} = -\omega^2 A\sin(\omega t - n\varphi)$$

Sound speed in the discrete method (2/2)

$$\begin{cases} \omega_0 = \sqrt{k/m} \\ \omega << \omega_0 \end{cases} \Rightarrow \frac{\omega}{\omega_0} = 2\sin\frac{\varphi}{2} \approx \varphi$$

$$\frac{\varphi}{2\pi} = \frac{l}{\lambda}$$

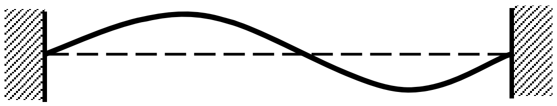
$$c = f\lambda = \frac{\omega}{2\pi}\lambda = \frac{\omega_0 \varphi}{2\pi}\lambda = \omega_0 l = l\sqrt{\frac{k}{m}} = \sqrt{\frac{kl}{m/l}}$$

$$c = \sqrt{\frac{kl/S}{m/lS}}$$

Elastic constant

Density

Question:



$$\omega_2^{\ 2} = \frac{22.207}{l^2}\left(\frac{T}{\sigma}\right)$$ Analytic solution

張力T，線密度 σ，長さ$2l$

Assuming the displacement function as below, find the second resonance frequency using the Rayleigh method.

Displacement function: $y = ax(l^2 - x^2)$

$$T_{\max} = U_{\max}$$

$$T_{\max} = \int_{-l}^{l} \frac{1}{2}\rho\left(\frac{dy}{dt}\right)^2 dx = \int_{-l}^{l} \frac{1}{2}\rho\omega^2 y^2 dx$$

$$U_{\max} = \int_{-l}^{l} \frac{1}{2}T\left(\frac{dy}{dx}\right)^2 dx$$

$\Rightarrow$ $\omega_2^{\ 2}$?