

9. Scattering Matrix

The matrix representations of circuit are addressed focused on the scattering matrix. The topics include:

9.1 Definition of scattering matrix

9.2 Scattering matrix and transmission matrix

9.3 Relation between scattering matrix and impedance matrix

(9.4 Method for determination of scattering matrix)

Definition

Along a transmission line

$$V(y) = V_i e^{j\beta y} + V_r e^{-j\beta y} = \sqrt{R_c} \left(\frac{V_i}{\sqrt{R_c}} e^{j\beta y} + \frac{V_r}{\sqrt{R_c}} e^{-j\beta y} \right)$$
$$I(y) = \frac{1}{R_c} (V_i e^{j\beta y} - V_r e^{-j\beta y}) = \frac{1}{\sqrt{R_c}} \left(\frac{V_i}{\sqrt{R_c}} e^{j\beta y} - \frac{V_r}{\sqrt{R_c}} e^{-j\beta y} \right) \quad (9.1)$$

Let's define the following quantity.

$$a(y) = \frac{V_i}{\sqrt{R_c}} e^{j\beta y} \rightarrow |a(y)|^2 = \frac{|V_i|^2}{R_c} : \text{incident power}$$
$$b(y) = \frac{V_r}{\sqrt{R_c}} e^{-j\beta y} \rightarrow |b(y)|^2 = \frac{|V_r|^2}{R_c} : \text{reflected power}$$

$$\begin{aligned} V(y) &= \sqrt{R_c} (a(y) + b(y)) \\ I(y) &= \frac{1}{\sqrt{R_c}} (a(y) - b(y)) \end{aligned} \quad (9.2) \quad \Rightarrow \quad \begin{aligned} a(y) &= \frac{V(y) + R_c I(y)}{2\sqrt{R_c}} \\ b(y) &= \frac{V(y) - R_c I(y)}{2\sqrt{R_c}} \end{aligned} \quad (9.3)$$

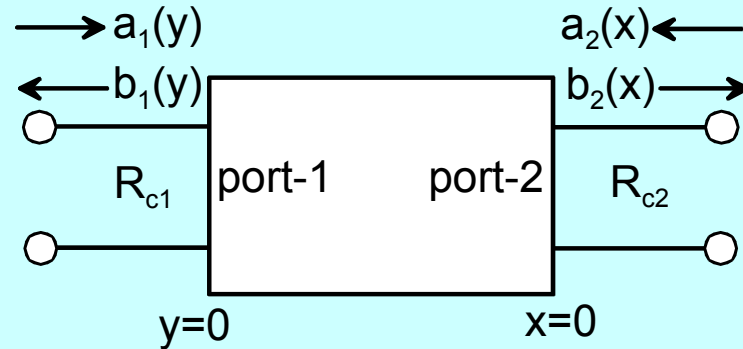
2-port circuit

$$a_1(y) = \frac{V_1(y) + R_{c1}I_1(y)}{2\sqrt{R_{c1}}}$$

$$b_1(y) = \frac{V_1(y) - R_{c1}I_1(y)}{2\sqrt{R_{c1}}}$$

$$a_2(x) = \frac{V_2(x) + R_{c2}I_2(x)}{2\sqrt{R_{c2}}}$$

$$b_2(x) = \frac{V_2(x) - R_{c2}I_2(x)}{2\sqrt{R_{c2}}}$$



Definition of scattering matrix: $a_1 = a_1(0)$, etc

$$b_1 = S_{11}a_1 + S_{12}a_2$$

$$b_2 = S_{21}a_1 + S_{22}a_2 \quad (9.4)$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\mathbf{b} = \mathbf{S}\mathbf{a} \quad (9.5)$$

Loss-less properties

Input power is given by

$$|a_1|^2 + |a_2|^2 + \dots + |a_n|^2 = (a_1^*, a_1^*, \dots, a_n^*) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \tilde{\mathbf{a}} \mathbf{a}$$

$\tilde{\mathbf{a}}$: transpose conjugate of $\mathbf{a}$

Loss-less : input power = output power

$$\tilde{\mathbf{a}} \mathbf{a} = \tilde{\mathbf{b}} \mathbf{b}$$

$$\mathbf{b} = \mathbf{S} \mathbf{a}$$

$$\tilde{\mathbf{b}} = \tilde{\mathbf{a}} \tilde{\mathbf{S}}$$

$$\tilde{\mathbf{a}} \mathbf{a} - \tilde{\mathbf{b}} \mathbf{b} = \tilde{\mathbf{a}} \mathbf{a} - \tilde{\mathbf{a}} \tilde{\mathbf{S}} \mathbf{S} \mathbf{a} = \tilde{\mathbf{a}} (\mathbf{I} - \tilde{\mathbf{S}} \mathbf{S}) \mathbf{a} = 0$$

$$\therefore \tilde{\mathbf{S}} \mathbf{S} = \mathbf{S} \tilde{\mathbf{S}} = \mathbf{I} \quad (9.6)$$

Reciprocity:

$$\left. \frac{b_2}{a_1} \right|_{a_2=0} = \left. \frac{b_1}{a_2} \right|_{a_1=0}$$

$$\therefore S_{21} = S_{12}$$



$$S_{ij} = S_{ji}$$

$$(9.7)$$

S-matrix and T-matrix

T-matrix

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} b_2 \\ a_2 \end{pmatrix} \quad (9.8)$$

suitable for cascaded connection

$$\begin{aligned} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} &= \mathbf{T}_1 \begin{pmatrix} b_2' \\ a_2' \end{pmatrix} \\ \begin{pmatrix} b_2' \\ a_2' \end{pmatrix} &= \mathbf{T}_2 \begin{pmatrix} b_2 \\ a_2 \end{pmatrix} \quad \therefore \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \mathbf{T}_1 \mathbf{T}_2 \begin{pmatrix} b_2 \\ a_2 \end{pmatrix} \end{aligned} \quad (9.9)$$

From the definition of S-matrix

$$a_1 = \frac{1}{S_{21}} b_2 - \frac{S_{22}}{S_{21}} a_2$$

$$b_1 = \frac{S_{11}}{S_{21}} b_2 + \left(S_{12} - \frac{S_{11} S_{22}}{S_{21}} \right) a_2$$

$$\begin{aligned} T_{11} &= \frac{1}{S_{21}}, & T_{12} &= -\frac{S_{22}}{S_{21}} \\ T_{21} &= \frac{S_{11}}{S_{21}}, & T_{22} &= S_{12} - \frac{S_{11} S_{22}}{S_{21}} \end{aligned} \quad (9.10)$$

S-matrix and Z-matrix

$$a_j = \frac{V_j + R_{cj} I_j}{2\sqrt{R_{cj}}} \quad \Rightarrow \quad \boxed{V_j} = \sqrt{R_{cj}} (a_j + b_j)$$

$$b_j = \frac{V_j - R_{cj} I_j}{2\sqrt{R_{cj}}} \quad \Rightarrow \quad \boxed{I_j} = \frac{1}{\sqrt{R_{cj}}} (a_j - b_j) \quad (9.11)$$

Then,

$$\boxed{\begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix}} = \begin{pmatrix} \sqrt{R_{c1}} & & & 0 \\ & \sqrt{R_{c2}} & & \\ & & \ddots & \\ & & & \sqrt{R_{cn}} \\ 0 & & & & \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} \sqrt{R_{c1}} & & & 0 \\ & \sqrt{R_{c2}} & & \\ & & \ddots & \\ & & & \sqrt{R_{cn}} \\ 0 & & & & \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\boxed{[V]} = [\sqrt{R}]([a] + [b]) \quad (9.12)$$

$$[\sqrt{R}] = \begin{pmatrix} \sqrt{R_{c1}} & & & 0 \\ & \sqrt{R_{c2}} & & \\ & & \ddots & \\ & & & \sqrt{R_{cn}} \\ 0 & & & & \end{pmatrix}$$

S-matrix and Z-matrix

Similarly,

$$[I] = [\sqrt{R}]^{-1}([a] - [b]) \quad (9.13)$$

$$[\sqrt{R}]^{-1} = \begin{pmatrix} \sqrt{R_{c1}} & & & 0 \\ & \sqrt{R_{c2}} & & \\ & & \ddots & \\ 0 & & & \sqrt{R_{cn}} \end{pmatrix}^{-1} = \begin{pmatrix} \sqrt{R_{c1}}^{-1} & & & 0 \\ & \sqrt{R_{c2}}^{-1} & & \\ & & \ddots & \\ 0 & & & \sqrt{R_{cn}}^{-1} \end{pmatrix}$$

Since $[V] = [Z][I]$

$$[V] = [V_i] + [V_r]$$

$$[I] = [R]^{-1}([V_i] - [V_r]) \quad (9.14)$$

$$[V_i] + [V_r] = [Z][R]^{-1}([V_i] - [V_r])$$

$$(1 + [Z][R]^{-1})[V_r] = ([Z][R]^{-1} - 1)[V_i] \quad (9.15)$$

S-matrix and Z-matrix

$$[V_i] = [\sqrt{R}][a]$$

$$[V_r] = [\sqrt{R}][b]$$

$$[b] = [S][a]$$

Thus,

$$(1 + [Z][R]^{-1})[\sqrt{R}][S] = ([Z][R]^{-1} - 1)[\sqrt{R}]$$

$$\therefore [S] = [\sqrt{R}]^{-1} (1 + [Z][R]^{-1})^{-1} ([Z][R]^{-1} - 1)[\sqrt{R}] \quad (9.16)$$

Especially, for $R_j = R_0$ ($j = 1, 2, 3 \dots n$)

$$[\sqrt{R}]^{-1} = \frac{1}{\sqrt{R_0}}[1], \quad [R]^{-1} = \frac{1}{R_0}[1], \quad [\sqrt{R}] = \sqrt{R_0}[1]$$

$$[S] = \left(1 + \frac{1}{R_0}[Z]\right)^{-1} \left(\frac{1}{R_0}[Z] - 1\right) = ([Z] + R_0)^{-1}([Z] - R_0) \quad (9.17)$$

How to find S-matrix elements ?

Simple method:

$$\begin{aligned} b_1 &= S_{11}a_1 \rightarrow S_{11} = \left. \frac{b_1}{a_1} \right|_{a_2=0} \\ b_2 &= S_{21}a_1 \rightarrow S_{21} = \left. \frac{b_2}{a_1} \right|_{a_2=0} \end{aligned} \quad (9.18) , \text{ but } a_2=0 \text{ is hard to realize.}$$

Deshamp's method

Terminate port-2 with an impedance Z_{L2} .

→ Reflection by the impedance Z_{L2} is given by $S_2 = \frac{a_2}{b_2} = \frac{Z_{L2} - R_{c2}}{Z_{L2} + R_{c2}}$

$$\begin{aligned} \frac{b_2}{a_1} &= \frac{b_2}{a_2} \frac{a_2}{a_1} = \frac{1}{S_2} \frac{a_2}{a_1} = S_{21} + S_{22} \frac{a_2}{a_1} \\ \rightarrow \frac{a_2}{a_1} \left(\frac{1}{S_2} - S_{22} \right) &= S_{21}, \quad \frac{a_2}{a_1} = \frac{S_{21}S_2}{1 - S_{22}S_2} \end{aligned}$$

$$\text{Reflection at port-1: } S_1 = \frac{b_1}{a_1} = S_{11} + S_{12} \frac{a_2}{a_1} = \frac{S_{11} + (S_{12}S_{21} - S_{11}S_{22})S_2}{1 - S_{22}S_2} \quad (9.20)$$

How to find S-matrix elements ?

S_1 and S_2 : linear transformation

When S_2 moves along a circle $S_2 = \exp(-2j\beta x)$
 S_1 moves also along another circle.

$$\begin{aligned}(1 - S_{22}S_2)S_1 &= S_{11} + (S_{12}S_{21} - S_{11}S_{22})S_2 \\ ((S_{12}S_{21} - S_{11}S_{22}) + S_{22}S_1)S_2 &= S_1 - S_{11} \\ S_2 &= \frac{S_1 - S_{11}}{S_{22}S_1 + (S_{12}S_{21} - S_{11}S_{22})}\end{aligned}\quad (9.21)$$

$|S_2| = 1$ (S_2 is located on a unit circle) gives the following relation.

$$\begin{aligned}(1 - S_{22}\bar{S}_{22})S_1\bar{S}_1 - ((S_{12}S_{21} - S_{11}S_{22})\bar{S}_{22} + S_{11})\bar{S}_1 - ((\overline{S_{12}S_{21} - S_{11}S_{22}})S_{22} + \bar{S}_{11})S_1 \\ + (S_{11}\bar{S}_{11} - (S_{12}S_{21} - S_{11}S_{22})(\overline{S_{12}S_{21} - S_{11}S_{22}})) = 0\end{aligned}\quad (9.22)$$

$$\text{center: } \frac{D\bar{S}_{22} + S_{11}}{1 - S_{22}\bar{S}_{22}} = \frac{S_{12}S_{21}\bar{S}_{22}}{1 - S_{22}\bar{S}_{22}} + S_{11} \quad (9.23a)$$

$$\text{radius: } \frac{\sqrt{|D\bar{S}_{22} + S_{11}|^2 - (1 - S_{22}\bar{S}_{22})(S_{11}\bar{S}_{11} - D\bar{D})}}{1 - S_{22}\bar{S}_{22}} = \frac{|S_{12}S_{21}|}{1 - S_{22}\bar{S}_{22}} \quad (9.23b)$$

where $D = (S_{12}S_{21} - S_{11}S_{22})$

How to find S-matrix elements ?

By changing the position of movable short, x , we can measure the corresponding reflection S_1 .

(S_1 moves also along another circle.)

e.g.

$$\text{for } S_2 = \pm 1 \quad S_1 = \frac{S_{11} \pm (S_{12}S_{21} - S_{11}S_{22})}{1 \mp S_{22}}$$

By the correspondence between S_1 and S_2 , we can determine S_{ij} .