

8.3 Contra-directional mode coupling

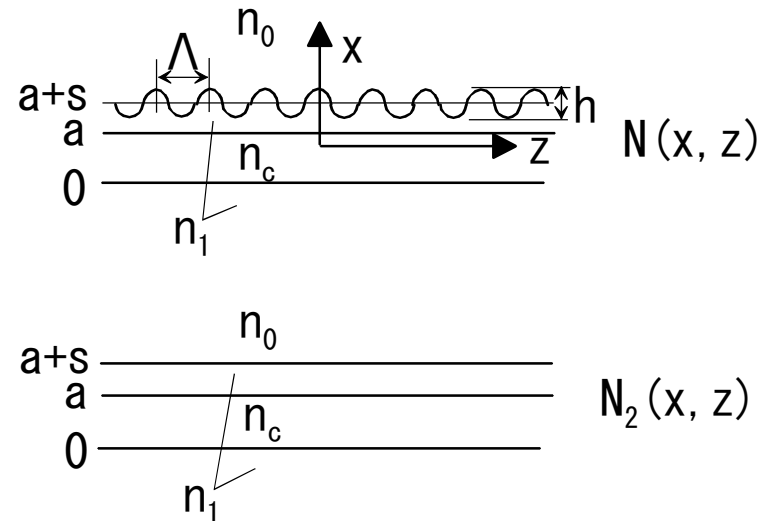
EM waves in a corrugated (grating) waveguide



coupling between forward and backward traveling waves

→ same mode,
but opposite propagation direction

$$E_1 = E_2 = E \quad \beta_1 = -\beta_2 = n_e k_0$$



difference in refractive index

→ periodic perturbation → can be expressed in Fourier series

$$N^2(x, z) - N_2^2(x, z) = \begin{cases} 0 & : x < a + s - \frac{h}{2} \\ \sum_{m=-\infty}^{\infty} A_m(x) e^{-j \frac{2\pi}{\Lambda} m z} & : a + s - \frac{h}{2} \leq x \leq a + s + \frac{h}{2} \\ 0 & : a + s + \frac{h}{2} < x \end{cases}$$

$$A_m(x) = \frac{1}{\Lambda} \int_0^{\Lambda} (N^2(x, z) - N_2^2(x, z)) e^{j \frac{2\pi}{\Lambda} m z} dz$$

Coupling in periodic structure : coupling coefficient

$$\begin{aligned}
 \kappa_{12} &= \frac{\omega \varepsilon_0 \iint (N^2 - N_2^2) \mathbf{E}_1^* \cdot \mathbf{E}_2 dS}{\iint \mathbf{u}_z \cdot (\mathbf{E}_1^* \times \mathbf{H}_1 + \mathbf{E}_1 \times \mathbf{H}_1^*) dS} = \frac{\omega \varepsilon_0 \int (\sum_m A_m(x) E_1^* \cdot E_2) dx e^{-j \frac{2\pi}{\Lambda} mz}}{\int_{-\infty}^{\infty} \mathbf{u}_z \cdot (\mathbf{E}_1^* \times \mathbf{H}_1 + \mathbf{E}_1 \times \mathbf{H}_1^*) dx} \\
 &= \sum_{m=-\infty}^{\infty} \frac{\omega \varepsilon_0 \int A_m(x) E_1^* \cdot E_2 dx}{\int_{-\infty}^{\infty} \mathbf{u}_z \cdot (\mathbf{E}_1^* \times \mathbf{H}_1 + \mathbf{E}_1 \times \mathbf{H}_1^*) dx} e^{-j \frac{2\pi}{\Lambda} mz} = \sum_{m=-\infty}^{\infty} \kappa_{Gm} e^{-j \frac{2\pi}{\Lambda} mz}
 \end{aligned}$$

Example : TE mode in a slab waveguide

$$(\mathbf{E}_1^* \times \mathbf{H}_1 + \mathbf{E}_1 \times \mathbf{H}_1^*) \cdot \mathbf{u}_z = \frac{2\beta}{\omega \mu_0} |E_y|^2$$

$$(\mathbf{E}_2^* \times \mathbf{H}_2 + \mathbf{E}_2 \times \mathbf{H}_2^*) \cdot \mathbf{u}_z = -\frac{2\beta}{\omega \mu_0} |E_y|^2$$

$$\mathbf{E}_1^* \cdot \mathbf{E}_2 = \mathbf{E}_2^* \cdot \mathbf{E}_1 = |E_y|^2$$

$$\begin{aligned}
 \kappa_{12} &= \frac{\omega \varepsilon_0 \iint (N^2 - N_2^2) |E_y|^2 dS}{\frac{2\beta}{\omega \mu_0} \iint |E_y|^2 dS} = \frac{k_0^2 \int_{a+s-\frac{h}{2}}^{a+s+\frac{h}{2}} (N^2 - N_2^2) |E_y|^2 dx}{2\beta \int_{-\infty}^{\infty} |E_y|^2 dx} \\
 &= \frac{k_0^2}{2\beta \int_{-\infty}^{\infty} |E_y|^2 dx} \int_{a+s-\frac{h}{2}}^{a+s+\frac{h}{2}} \sum_m A_m(x) e^{-j \frac{2\pi}{\Lambda} mz} |E_y|^2 dx \\
 &= \sum_{m=-\infty}^{\infty} \frac{k_0^2}{2\beta \int_{-\infty}^{\infty} |E_y|^2 dx} \int_{a+s-\frac{h}{2}}^{a+s+\frac{h}{2}} A_m(x) |E_y|^2 dx \times e^{-j \frac{2\pi}{\Lambda} mz} = \sum_{m=-\infty}^{\infty} \kappa_{Gm} e^{-j \frac{2\pi}{\Lambda} mz}
 \end{aligned}$$

Coupling in periodic structure

1-st order coupling (largest coupling coefficient)

$$\kappa_{12} = \kappa_G e^{-j\frac{2\pi}{\Lambda}z}$$

power conservation $\frac{d}{dz}|A|^2 + \frac{d}{d(-z)}|B|^2 = 0$

$$-j(\kappa_{12} + \kappa_{21}^*)A^*Be^{-j2\delta z} + j(\kappa_{12}^* + \kappa_{21})AB^*e^{j2\delta z} = 0$$

$$\therefore \kappa_{21} = -\kappa_{12}^* = -\kappa_G e^{j\frac{2\pi}{\Lambda}z}$$

$$\frac{dA}{dz} = (-j\kappa_G e^{-j\frac{2\pi}{\Lambda}z} B e^{-j(\beta_2 - \beta_1)z}) = -j\kappa_G B(z) e^{j(\beta_1 - \beta_2 - \frac{2\pi}{\Lambda})z}$$

$$\frac{dB}{dz} = j\kappa_G A(z) e^{-j(\beta_1 - \beta_2 - \frac{2\pi}{\Lambda})z}$$

$$\phi = (\beta_1 - \beta_2 - \frac{2\pi}{\Lambda})/2 = \beta - \frac{\pi}{\Lambda}$$

Coupling in periodic structure

A grating is formed in $0 \leq z \leq L$

Only a forward propagation mode is incident at $z=0$. $A(0) = A_0$ $B(L) = 0$

$$\frac{d^2 A}{dz^2} - j2\phi \frac{dA}{dz} - \kappa_G^2 A = 0 \quad A(z) = e^{\gamma z} \longrightarrow \gamma^2 - j2\phi\gamma - \kappa_G^2 = 0$$

$$\therefore \gamma = j\phi \pm \sqrt{\kappa_G^2 - \phi^2}$$

$$(1) |\phi| > \kappa_G \quad \left(\beta > \frac{\pi}{\Lambda} + \kappa_G, \beta < \frac{\pi}{\Lambda} - \kappa_G \right)$$

$$\gamma = j(\phi \pm \rho) \quad \text{with} \quad \rho = \sqrt{\phi^2 - \kappa_G^2}$$

$$A(z) = a_1 e^{j(\phi+\rho)z} + a_2 e^{j(\phi-\rho)z}$$

$$B(z) = -\frac{1}{\kappa_G} (a_1(\phi + \rho)e^{j(-\phi+\rho)z} + a_2(\phi - \rho)e^{-j(\phi+\rho)z})$$

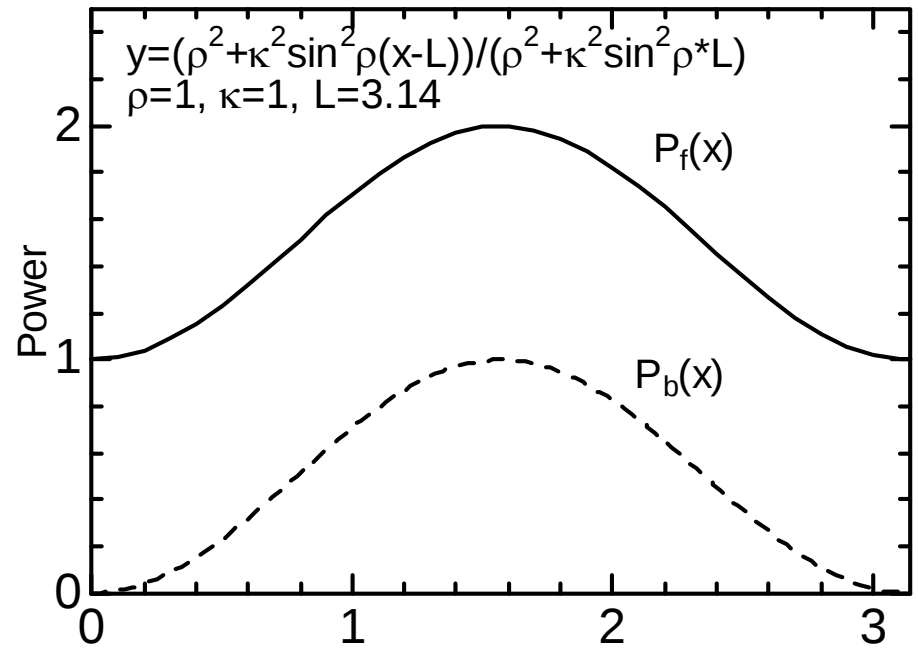
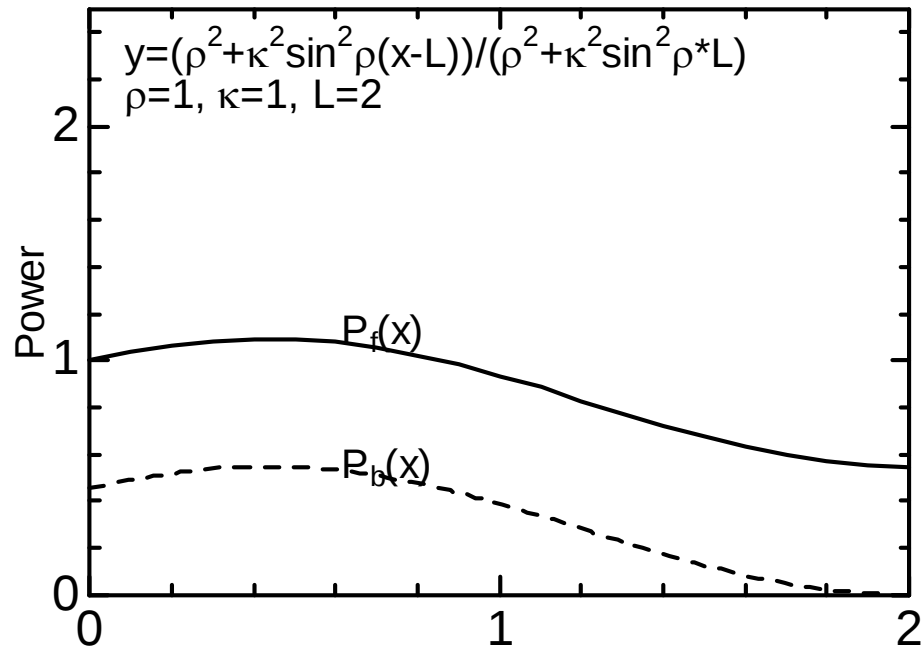
$$A(z) = A_0 \frac{\rho \cos \rho(z-L) - j\phi \sin \rho(z-L)}{\rho \cos \rho L + j\phi \sin \rho L} e^{j\phi z} \rightarrow P_f(z) = \frac{|A(z)|^2}{|A(0)|^2} = \frac{\rho^2 + \kappa_G^2 \sin^2 \rho(z-L)}{\rho^2 + \kappa_G^2 \sin^2 \rho L}$$

$$B(z) = A_0 \frac{j\kappa_G \sin \rho(z-L)}{\rho \cos \rho L + j\phi \sin \rho L} e^{-j\phi z} \rightarrow P_b(z) = \frac{|B(z)|^2}{|A(0)|^2} = \frac{\kappa_G^2 \sin^2 \rho(z-L)}{\rho^2 + \kappa_G^2 \sin^2 \rho L}$$

$$\text{reflection at } z=0: \quad R = \frac{\kappa_G^2 \sin^2 \rho L}{\rho^2 + \kappa_G^2 \sin^2 \rho L} \quad (\rho = \sqrt{(\beta - \frac{\pi}{\Lambda})^2 - \kappa_G^2})$$

Coupling in periodic structure

$$(1) \quad |\phi| > \kappa_G \quad \left(\beta > \frac{\pi}{\Lambda} + \kappa_G, \beta < \frac{\pi}{\Lambda} - \kappa_G \right)$$



Coupling in periodic structure

$$(2) \quad \begin{aligned} |\phi| &= \kappa_G \\ \gamma &= j\phi \end{aligned} \quad \left(\beta = \frac{\pi}{\Lambda} \pm \kappa_G \right)$$

$$A(z) = a_1 e^{j\phi z} + a_2 z e^{j\phi z}$$

$$B(z) = \frac{e^{-j2\phi z}}{-j\kappa_G} (j\phi a_1 e^{j\phi z} + (1 + j\phi z) a_2 e^{j\phi z})$$

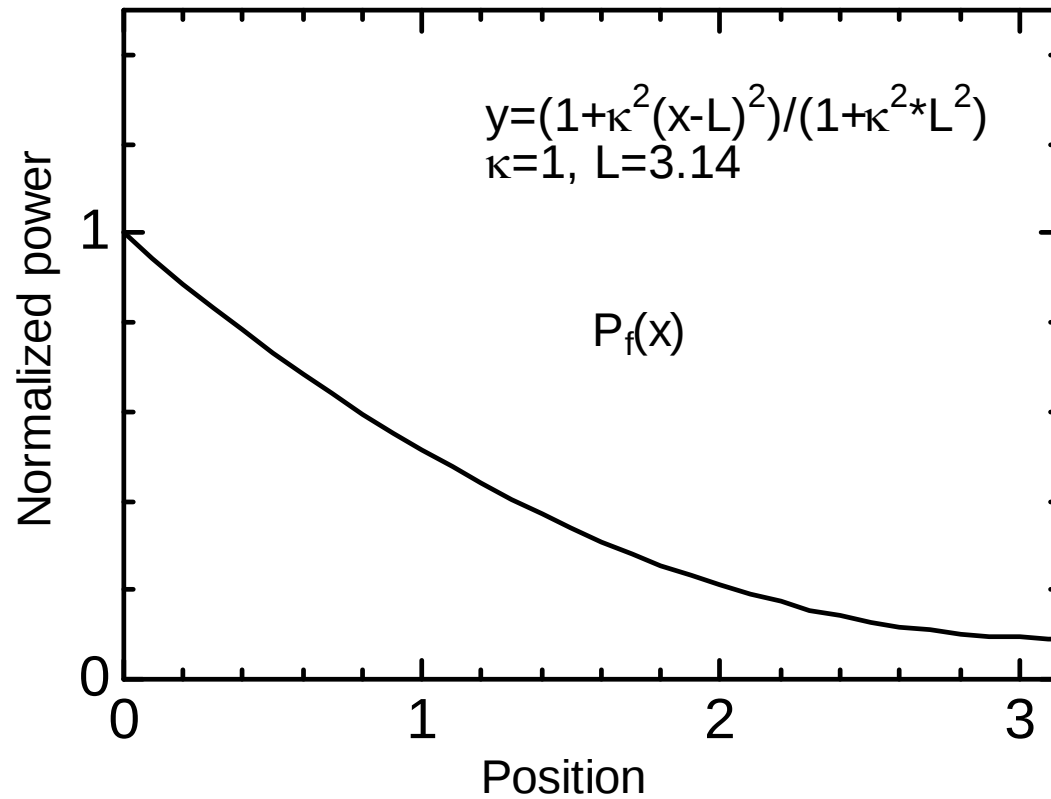
$$A(z) = A_0 \frac{1 - j\phi(z - L)}{1 + j\phi L} e^{j\phi z} \rightarrow P_f(z) = \frac{1 + \phi^2 (z - L)^2}{1 + \phi^2 L^2} = \frac{1 + \kappa_G^2 (z - L)^2}{1 + \kappa_G^2 L^2}$$

$$B(z) = A_0 \frac{j\kappa_G (z - L)}{1 + j\phi L} e^{-j\phi z} \rightarrow P_b(z) = \frac{\kappa_G^2 (z - L)^2}{1 + \kappa_G^2 L^2}$$

$$R = \frac{\kappa_G^2 L^2}{1 + \kappa_G^2 L^2}$$

Coupling in periodic structure

$$(2) \quad |\phi| = \kappa_G \quad \left(\beta = \frac{\pi}{\Lambda} \pm \kappa_G \right)$$



Coupling in periodic structure (Bragg reflection)

$$(3) \quad |\phi| < \kappa_G \quad \left(\frac{\pi}{\Lambda} - \kappa_G < \beta < \frac{\pi}{\Lambda} + \kappa_G \right)$$

$$\gamma = j\phi \pm \alpha$$

$$\alpha = \sqrt{\kappa_G^2 - \phi^2}$$

$$A(z) = a_1 e^{(j\phi + \alpha)z} + a_2 e^{(j\phi - \alpha)z} = e^{j\phi z} (a_1 e^{\alpha z} + a_2 e^{-\alpha z})$$

$$B(z) = \frac{e^{-j2\phi z}}{-j\kappa_G} \frac{dA(z)}{dz} = \frac{e^{-j\phi z}}{-j\kappa_G} (a_1 (j\phi + \alpha) e^{\alpha z} + a_2 (j\phi - \alpha) e^{-\alpha z})$$

$$A(z) = A_0 \frac{\alpha \cosh \alpha(z-L) - j\phi \sinh \alpha(z-L)}{\alpha \cosh \alpha L + j\phi \sinh \alpha L} e^{j\phi z} \rightarrow P_f(z) = \frac{\alpha^2 + \kappa_G^2 \sinh^2 \alpha(z-L)}{\alpha^2 + \kappa_G^2 \sinh^2 \alpha L}$$

$$B(z) = A_0 \frac{j\kappa_G \sinh \alpha(z-L)}{\alpha \cosh \alpha L + j\phi \sinh \alpha L} e^{-j\phi z} \rightarrow P_b(z) = \frac{\kappa_G^2 \sinh^2 \alpha(z-L)}{\alpha^2 + \kappa_G^2 \sinh^2 \alpha L}$$

$$R = \frac{\kappa_G^2 \sinh^2 \alpha L}{\alpha^2 + \kappa_G^2 \sinh^2 \alpha L}, \quad \alpha = \sqrt{\kappa_G^2 - (\beta - \frac{\pi}{\Lambda})^2}$$

$$\text{Bragg reflection:} \quad \beta = \frac{\pi}{\Lambda} \quad (\lambda_B = 2\Lambda) \quad R = \tanh^2 \kappa_G L$$

Coupling in periodic structure

$$(3) \quad |\phi| < \kappa_G \quad \left(\frac{\pi}{\Lambda} - \kappa_G < \beta < \frac{\pi}{\Lambda} + \kappa_G \right)$$

