Complex Networks
random graphs

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network models

- “If I know a network as some particular property, such as a particular degree distribution, what effect will that have on the wider behavior of the system?”
- building mathematical models of networks
  - mimic the patterns of connections in real networks
  - understand the implications of the patterns

random graph

- a model network in which some specific set of parameters take fixed values, but the network is random in other respects
- simplest example: G(n, m)
  - take n vertices and place m edges at random
  - simple graph (no multiedges or self-edges)

⇨ a probability distribution P(G) over possible networks

$$P(G) = \begin{cases} 1/\Omega & \text{if G is a simple graph with n vertices and m edges} \\ 0 & \text{otherwise} \end{cases}$$

Ω:the total number of simple graphs with n vertices and m edges

random graph model = an ensemble of networks

- properties of random graphs = the average properties of the ensemble
- diameter of G(n,m) : $\langle l \rangle = \sum_G P(G) l(G) = \frac{1}{\Omega} \sum_G l(G)$
- this is a useful definition for several reasons:
  - many average properties can be calculated exactly
  - we are often interested in typical properties of the networks
  - distributions of values for many network measures is sharply peaked
- average degree : $\langle k \rangle = 2m/n$

another random graph model

- G(n,p)
  - n : the number of vertices
  - p : the probability of edges between vertices
- G(n,p) is the ensemble of networks with n vertices in which each simple G appears with probability $P(G) = p^m (1-p)^{\binom{n}{2}-m}$

  m: the number of edges
- often called as “Erdos-Renyi model”, “Poisson random graph”, “Bernoulli random graph”, or “the random graph”

mean number of edges and mean degree of G(n,p)

- total probability of drawing a graph with m edges from the ensemble is

$$P(m) = \binom{\binom{n}{2}}{m} p^m (1-p)^{\binom{n}{2}-m}$$

the expected value of binomially distributed variable (see the next page)

- the mean value of m is $\langle m \rangle = \sum_{m=0}^{\binom{n}{2}} mP(m) = \binom{n}{2} p$
- the mean degree is $\langle k \rangle = \sum_{m=0}^{\binom{n}{2}} \frac{2m}{n} P(m) = \frac{2}{n}\binom{n}{2} p = (n-1)p$

often denoted as c

n(n-1)/2

$$c = (n-1)p$$

the expected value of binomially distributed variable

$$\langle m\rangle = \sum_{m=0}^{\binom{n}{2}} mP(m) \underset{0\cdot P(0)=0}{=} \sum_{m=1}^{\binom{n}{2}} mP(m) = \sum_{m=1}^{\binom{n}{2}} m \binom{\binom{n}{2}}{m} p^{m}(1-p)^{\binom{n}{2}-m}$$

$$\binom{N}{k} = \frac{N!}{k!(N-k)!} = \frac{N\cdot(N-1)!}{k\cdot(k-1)!((N-1)-(k-1))!}$$

$$= \frac{N}{k}\cdot\frac{(N-1)!}{(k-1)!((N-1)-(k-1))!} = \frac{N}{k}\cdot\binom{N-1}{k-1}$$

$$N = \binom{n}{2},\, k = m$$

$$\langle m\rangle = \sum_{m=1}^{\binom{n}{2}} m \binom{\binom{n}{2}}{m} p^{m}(1-p)^{\binom{n}{2}-m} = \binom{n}{2} p \sum_{m=1}^{\binom{n}{2}} \binom{\binom{n}{2}-1}{m-1} p^{m-1}(1-p)^{\binom{n}{2}-m}$$

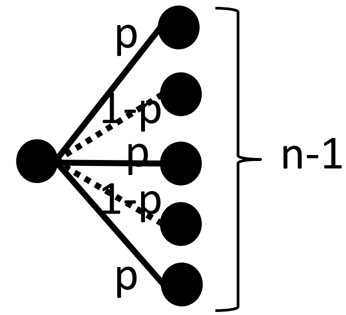
$$j = m-1$$

$$\langle m\rangle = \binom{n}{2} p \sum_{j=0}^{\binom{n}{2}-1} \binom{\binom{n}{2}-1}{j} p^{j}(1-p)^{\left(\binom{n}{2}-1\right)-j} \underset{\text{binomial theorem}}{=} \binom{n}{2} p \underbrace{\left(p+(1-p)\right)}_{1}^{\binom{n}{2}-1} = \binom{n}{2} p$$

degree distribution of G(n,p) (1)

- a vertex is connected with probability p to each of the n-1 other vertices

$$p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$



- we are interested in large networks
  - a mean degree is approximately constant
  - $p = c/(n-1)$ becomes vanishingly small as $n \to \infty$

$$\ln\left[(1-p)^{n-1-k}\right] = (n-1-k)\ln\left(1-\frac{c}{n-1}\right) \cong -(n-1-k)\frac{c}{n-1} \cong -c$$

$$(1-p)^{n-1-k} = e^{-c}$$

for large n

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \qquad for all |x| < 1$$

Taylor expansion

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$$

degree distribution of G(n,p) (2)

- for large n

(n-1)(n-2)...(n-k)/k!

$$\binom{n-1}{k} = \frac{(n-1)!}{(n-1-k)!k!} \cong \frac{(n-1)^k}{k!}$$

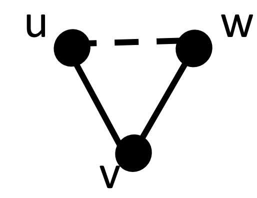
- p_k becomes as follows in the limit of large n

$$p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k} = \frac{(n-1)^k}{k!} p^k e^{-c} = \frac{(n-1)^k}{k!} \left(\frac{c}{n-1}\right)^k e^{-c} = e^{-c} \frac{c^k}{k!}$$

Poisson distribution

clustering coefficient of G(n,p)

- clustering coefficient : the probability that two network neighbors of a vertex are also neighbors of each other



- in a random graph, the probability is $p = c/(n-1)$

$$C = \frac{c}{n-1}$$

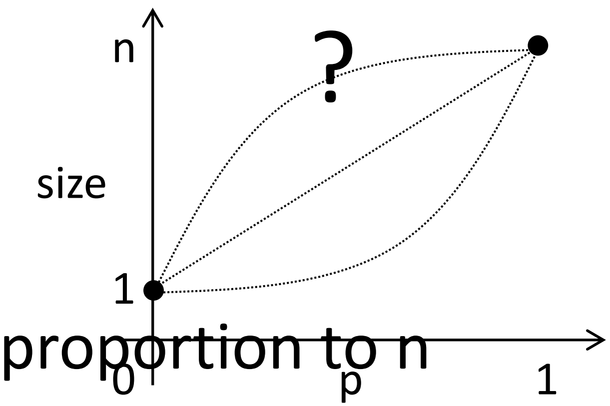
- tends to zero in the limit $n \to \infty$
- differs sharply from most of the real-world networks (quite high clustering coefficient)

Giant component (1)

- the size of the largest component in a network
  - p=0 → size=1
  - p=1 → size=n

independent of the size of the network

it will grow with the network



- giant component
  - a component whose size grows in proportion to n
- u : average fraction of vertices that do not belong to the giant component
  - u = 1 if there is no giant component
  - u is the probability that a randomly chosen vertex does not belong to the giant component

Giant component (2)

- if vertex i does not belong to the giant component, for every other vertex j

  probability of not being connected to g.c. via j: 1-p+pu
  - i is not connected to j by an edge

    probability: 1-p
  - I is connected to j, but j is not a member of the giant component

    probability: pu
- total probability of not being connected to g.c. via any of n-1 vertices:

$$u = (1 - p + pu)^{n-1} = \left[1 - \frac{c}{n-1}(1-u)\right]^{u-1}$$

Giant component (3)

- taking logs of both sides

$$\ln u = (u-1)\ln\left[1-\frac{c}{n-1}(1-u)\right] \cong -(n-1)\frac{c}{n-1}(1-u) = -c(1-u)$$

- taking exponentials of both sides

$$u = e^{-c(1-u)}$$

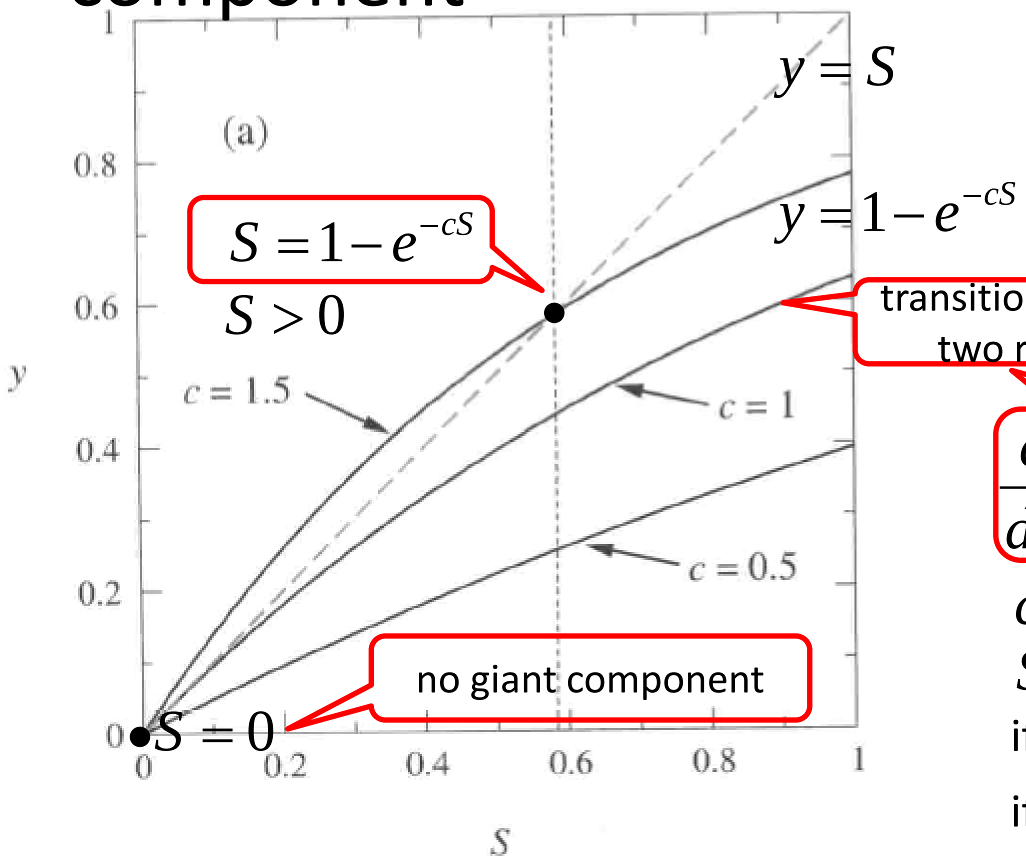
- u is the fraction of vertices not in the giant component
- the fraction of vertices that are in the giant component is $S = 1-u$

$$S = 1 - e^{-cS}$$

it doesn't have a simple solution for S

Giant component (4)

- graphical solution for the size of the giant component



$$\frac{d}{dS}(1 - e^{-cS}) = 1$$

$$ce^{-cS} = 1$$

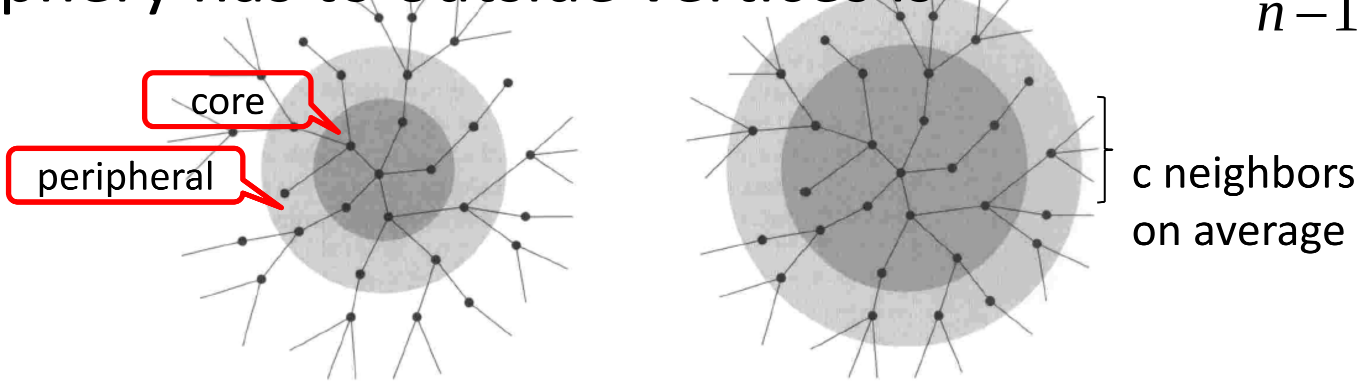
$$S = 0 \rightarrow c = 1$$

if c<=1, no giant comonent

if c>1, two solutions for S (S=0 & S>o)

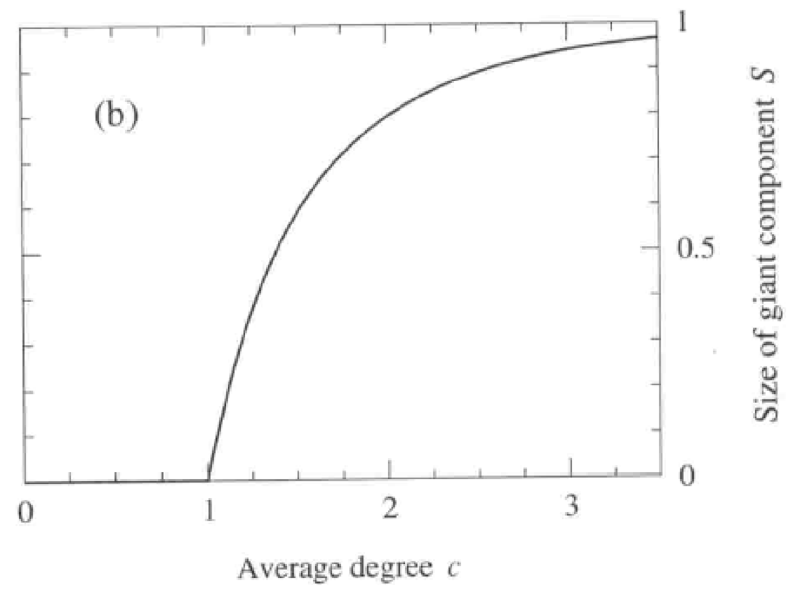
the value of c and the growth of a set (1)

- core : all neighbor are inside the set
- peripheral : at least one neighbor is outside
- enlarging the set by adding immediate neighbor
  - s vertices in the set, n-s vertices outside the set
  - the average number of connections a vertex in the periphery has to outside vertices is $p(n-s) = c\dfrac{n-s}{n-1} \cong c$



the value of c and the growth of a set (2)

- each peripheral has c neighbors outside
  - (# of new peripheral) = c X (# of old peripheral)
- if c > 1, the average size of the periphery will grow exponentially → giant component
- the size of the giant component is the larger solution of $S = 1 - e^{-cS}$



small components (1)

- when c > 1, there exist a giant component
- what is the structure of the remainder of the network?
  - it is made up of many small components whose average size is constant and doesn't increase with the size of the network
- there is only one giant component
  - suppose there are two giant components which have size S_1n and S_2n
    - the number of distinct pairs of vertices between the two is $S_1S_2n^2$
    - the probability that there is no edge between the two component is

$$q = (1-p)^{S_1S_2n^2} = \left(1 - \frac{c}{n-1}\right)^{S_1S_2n^2}$$

small components (2)

- taking logs of both sides and going to the limit $n \to \infty$

$$\ln q = S_1 S_2 \lim_{n\to\infty}\left[n^2 \ln\left(1-\frac{c}{n-1}\right)\right] = S_1 S_2\left[-c(n+1)+\frac{1}{2}c^2\right]$$

$$= cS_1S_2\left[-n+\left(\frac{1}{2}c-1\right)\right]$$

Taylor expansion

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n \qquad for all |x|<1$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$$

- taking the exponential again

$$q = q_0 e^{-cS_1S_2n} \qquad q_0 = e^{c(c/2-1)S_1S_2}$$

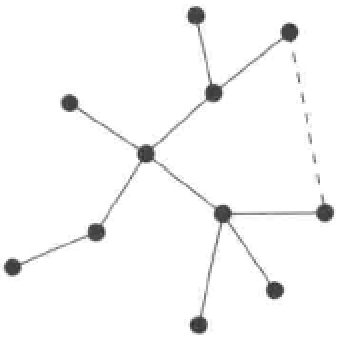
constant

- the probability that the two giant components are really components dwindles exponentially with increasing n

sizes of small components (1)

- π_s : probability that a randomly chosen vertex belongs to a small component of size s

  $$\sum_{s=0}^{\infty} \pi_s = 1 - S$$

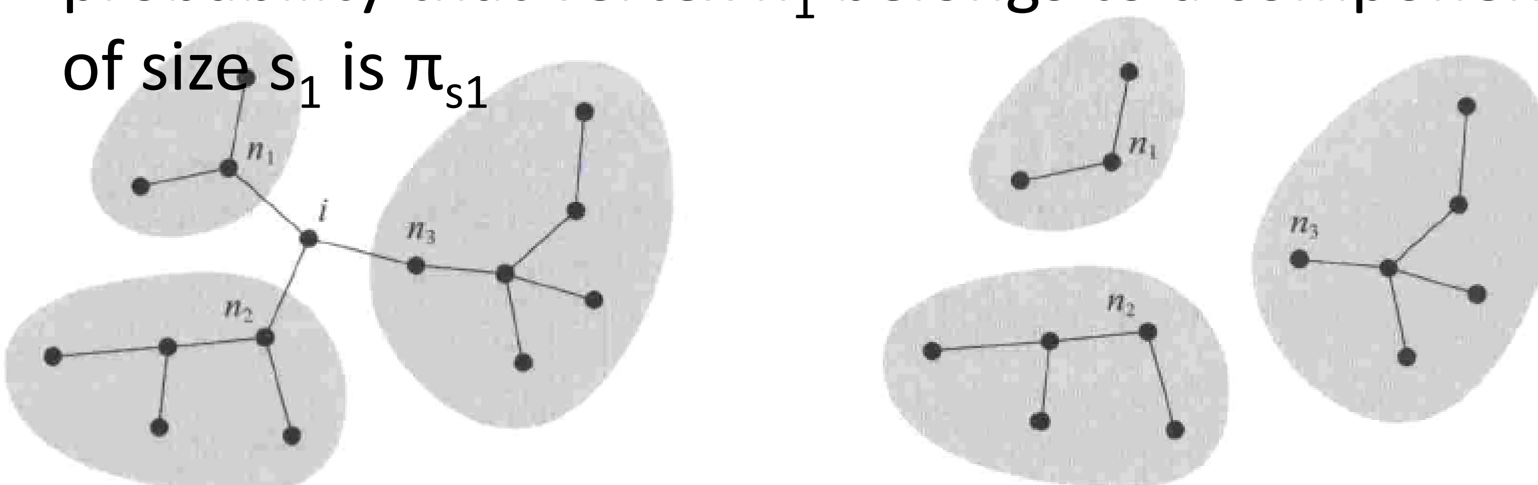


- small components are trees
  - a tree of s vertices contains s-1 edges
  - the total number of places where we could add an extra edge to the tree: $\binom{s}{2} - (s-1) = \frac{1}{2}(s-1)(s-2)$
  - the total number of extra edges :

    $$\frac{1}{2}(s-1)(s-2)p = \frac{1}{2}(s-1)(s-2)\frac{c}{n-1}$$
  - s increases more slowly than $\sqrt{n}$ ⇨ extra edges $\rightarrow$ 0

sizes of small components (2)

- because the component is a tree,
  - (size of the component) = $\sum$(size of n_i) + 1
- if vertex i is removed, the subcomponents become components in their own right
  - probability that vertex n_1 belongs to a component of size s_1 is π_{s1}



size of small components (3)

- suppose that vertex i has degree k
- probability P(s|k) that vertex i belongs to small component of size s is

$$P(s|k) = \sum_{s_1=1}^{\infty} \cdots \sum_{s_k=1}^{\infty} \left[\prod_{j=1}^{k} \pi_{s_j} \right] \delta(s-1, \sum_j s_j)$$

- to get π_s, just average P(s|k) over the distribution p_k of the degree

$$\pi_s = \sum_{k=1}^{\infty} p_k P(s|k) = \sum_{k=0}^{\infty} p_k \sum_{s_1=1}^{\infty} \cdots \sum_{s_k=1}^{\infty} \left[\prod_{j=1}^{k} \pi_{s_j} \right] \delta(s-1, \sum_j s_j)$$

$$= e^{-c} \sum_{k=0}^{\infty} \frac{c^k}{k!} \sum_{s_1=1}^{\infty} \cdots \sum_{s_k=1}^{\infty} \left[\prod_{j=1}^{k} \pi_{s_j} \right] \delta(s-1, \sum_j s_j)$$

$$\because p_k = e^{-c} \frac{c^k}{k!}$$

in the limit of large k

size of small components (4)

- generating function encapsulates all of the information about the degree distribution in a single function

$$h(z) = \pi_1 z + \pi_2 z^2 + \pi_3 z^3 + \ldots = \sum_{s=1}^{\infty} \pi_s z^s$$

- we can recover the probabilities by differentiating $\pi_s = \frac{1}{s!} \left. \frac{d^s h}{dz^s} \right|_{z=0}$
- substituting π_s into the equation of h(z)

$$h(z) = \sum_{s=1}^{\infty} z^s e^{-c} \sum_{k=0}^{\infty} \frac{c^k}{k!} \sum_{s_1=1}^{\infty} \ldots \sum_{s_k=1}^{\infty} \left[\prod_{j=1}^{k} \pi_{s_j} \right] \delta(s-1, \sum_j s_j)$$

size of small components (5)

$$h(z) = \sum_{s=1}^{\infty} z^s e^{-c} \sum_{k=0}^{\infty} \frac{c^k}{k!} \sum_{s_1=1}^{\infty} \cdots \sum_{s_k=1}^{\infty} \left[\prod_{j=1}^{k} \pi_{s_j} \right] \delta(s-1, \textstyle\sum_j s_j)$$

δ=1 only when
s-1=$\sum_j s_j$

$$= e^{-c} \sum_{k=0}^{\infty} \frac{c^k}{k!} \sum_{s_1=1}^{\infty} \cdots \sum_{s_k=1}^{\infty} \left[\prod_{j=1}^{k} \pi_{s_j} \right] z^{1+\sum_j s_j}$$

$$= z e^{-c} \sum_{k=0}^{\infty} \frac{c^k}{k!} \sum_{s_1=1}^{\infty} \cdots \sum_{s_k=1}^{\infty} \left[\prod_{j=1}^{k} \pi_{s_j} z^{s_j} \right] \qquad \because \exp x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$= z e^{-c} \sum_{k=0}^{\infty} \frac{c^k}{k!} \left[\sum_{s=1}^{\infty} \pi_s z^s \right]^k = z e^{-c} \sum_{k=0}^{\infty} \frac{c^k}{k!} [h(z)]^k = z \exp[c(h(z)-1)]$$

- it doesn't have a known closed-form solution for h(z), but we can calculate many useful things from it without solving for h(z) explicitly

size of small components (6)

- mean size of the component to which a randomly chosen vertex belongs

$$\langle s \rangle = \frac{\sum_s s\pi_s}{\sum_s \pi_s} = \frac{h'(1)}{1-S}$$

h'(z) : the first derivative of h(z)

- from the equation of h(z)

$$h'(z) = \exp[c(h(z)-1)] + czh'(z)\exp[c(h(z)-1)] = \frac{h(z)}{z} + ch(z)h'(z)$$

$$h'(z) = \frac{h(z)}{z[1-ch(z)]}$$

$$h(1) = \sum_s \pi_s = 1-S$$

$$h'(1) = \frac{h(1)}{1-ch(1)} = \frac{1-S}{1-c+cS}$$

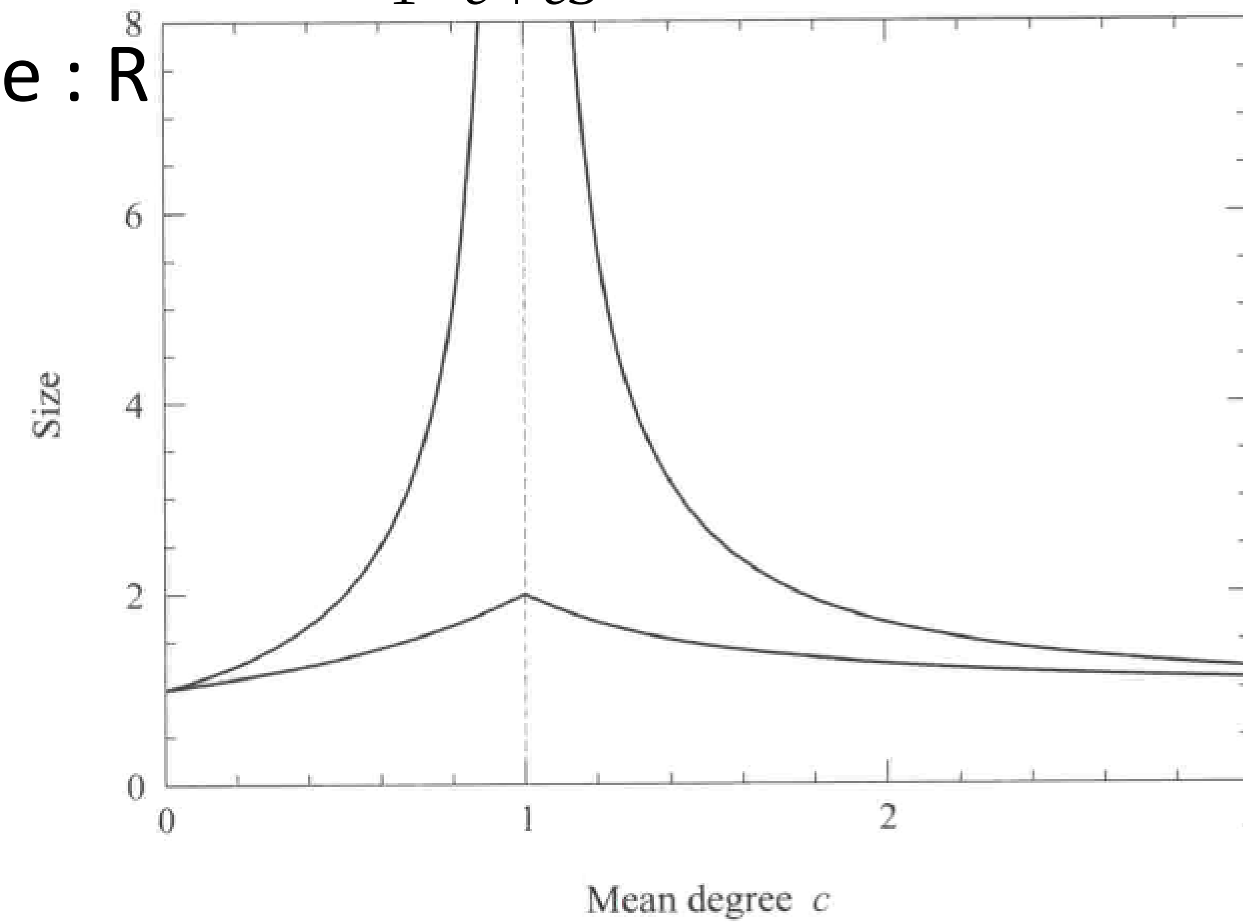
$$\langle s \rangle = \frac{1}{1-c+cS}$$

it doesn't grow with the number of vertices n

when c<1 and there is no giant component, <s>=1/(1-c)

divergence of the average size <s>

- upper curve : <s> $\langle s \rangle = \dfrac{1}{1-c+cS}$ diverges when c=1
- lower curve : R



average size of a small component

- <s>: the average size of the component to which a randomly chosen vertex belongs

  ≠ average size of a component
- n_s : the actual number of components of size s
- sn_s: the number of vertices that belong to components of size s
- the probability that a randomly chosen vertex belongs to a component of size s is $\pi_s = \frac{sn_s}{n}$

average size of a small component

- R: average size of a component

$$R=\frac{\sum_s sn_s}{\sum_s n_s}=\frac{n\sum_s \pi_s}{n\sum_s \pi_s/s}=\frac{1-S}{\sum_s \pi_s/s}$$

$$\int_0^1\frac{h(z)}{z}dz=\sum_{s=1}^{\infty}\pi_s\int_0^1 z^{s-1}dz=\sum_{s=1}^{\infty}\frac{\pi_s}{s}$$

$$h(z)=\sum_{s=1}^{\infty}\pi_s z^s$$

$$\frac{h(z)}{z}=[1-ch(z)]\frac{dh}{dz} \qquad \because h'(z)=\frac{h(z)}{z[1-ch(z)]}$$

$$\sum_{s=1}^{\infty}\frac{\pi_s}{s}=\int_0^1[1-ch(z)]\frac{dh}{dz}dz=\int_0^{1-S}(1-ch)dh=1-S-\frac{1}{2}c(1-S)^2$$

$$\because h(1)=\sum_s \pi_s=1-S \qquad \therefore R=\frac{2}{2-c+cS}$$

it does not diverge at c=1

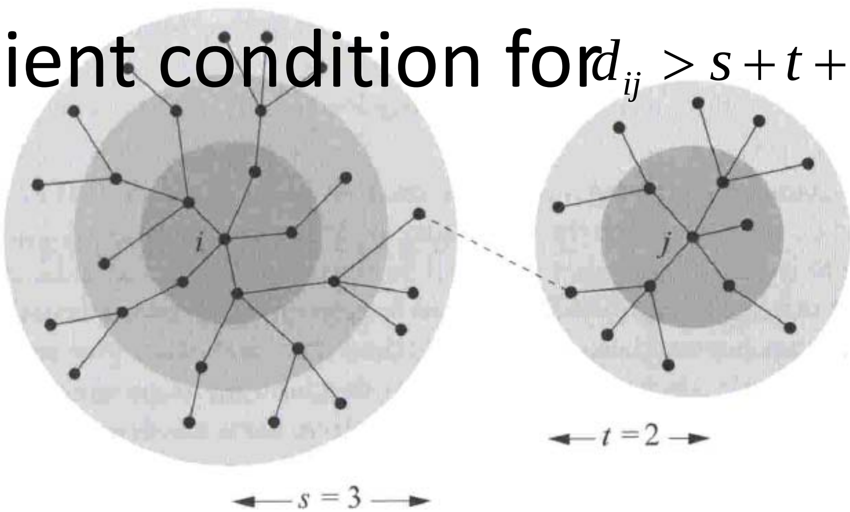
path lengths (1)

- small world effect : typical length of paths between vertices in network tend to be short
- the diameter of a random graph varies with the number n of vertices as $\ln n$
  - the average number of vertices s steps away from a randomly chosen vertex in a random graph is c^s
  - it grows exponentially with s $c^s \cong n$
  - diameter of the network is approximately $s \cong \ln n / \ln c$
- this argument is true when c^s is much less than n

path lengths (2)

- two different starting vertices (i and j)
- if there is a dashed line between the surfaces, the shortest path between i and j is s+t+1
- the absence of an edge between the surfaces is a necessary and sufficient condition for $d_{ij} > s+t+1$
- $c^s \times c^t$ pairs of vertices

$$P(d_{ij} > s+t+1) = (1-p)^{c^{s+t}}$$
$$l = s+t+1$$
$$P(d_{ij} > l) = (1-p)^{c^{l-1}} = \left(1-\frac{c}{n}\right)^{c^{l-1}}$$
$$\ln P(d_{ij} > l) = c^{l-1}\ln\left(1-\frac{c}{n}\right) \cong -\frac{c^l}{n}$$



$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n \qquad for all |x| < 1$$

path length (3)

$$P(d_{ij} > l) = \exp\left(-\frac{c^l}{n}\right)$$

tend to zero only if c^l grows faster than n

$c^l = an^{1+\varepsilon}$
$\varepsilon \to 0$

- diameter : the smallest value of l s.t. $P(d_{ij} > l) = 0$

$$l = \frac{\ln a}{\ln c} + \lim_{\varepsilon \to 0} \frac{(1+\varepsilon)\ln n}{\ln c} = A + \frac{\ln n}{\ln c}$$

diameter increases slowly with n

- logarithmic dependence of the diameter on n
  - acquaintance network of the entire world (7 billion people)

$$l = \frac{\ln n}{\ln c} = \frac{\ln(7 \times 10^9)}{\ln 1000} = 3.3..$$

small enough to account for the results of the small-world experiments of Milgram

problems with the random graph (1)

- no transitivity or clustering
  - $C = \dfrac{c}{n-1}$ tens to zero in the limit of large n
  - the acquaintance network of the human population in the world
    - $n \cong 7{,}000{,}000{,}000$
    - $C \cong \dfrac{1000}{7{,}000{,}000{,}000} \cong 10^{-7}$

clustering coefficient of real acquaintance network is much bigger (0.01 or 0.5)

- no correlation between the degrees of adjacent vertices (no communities)

problems with the random graph (2)

- the shape of degree distribution is different
  - real network : right-skewed