Advanced Data Analysis: Blind Source Separation

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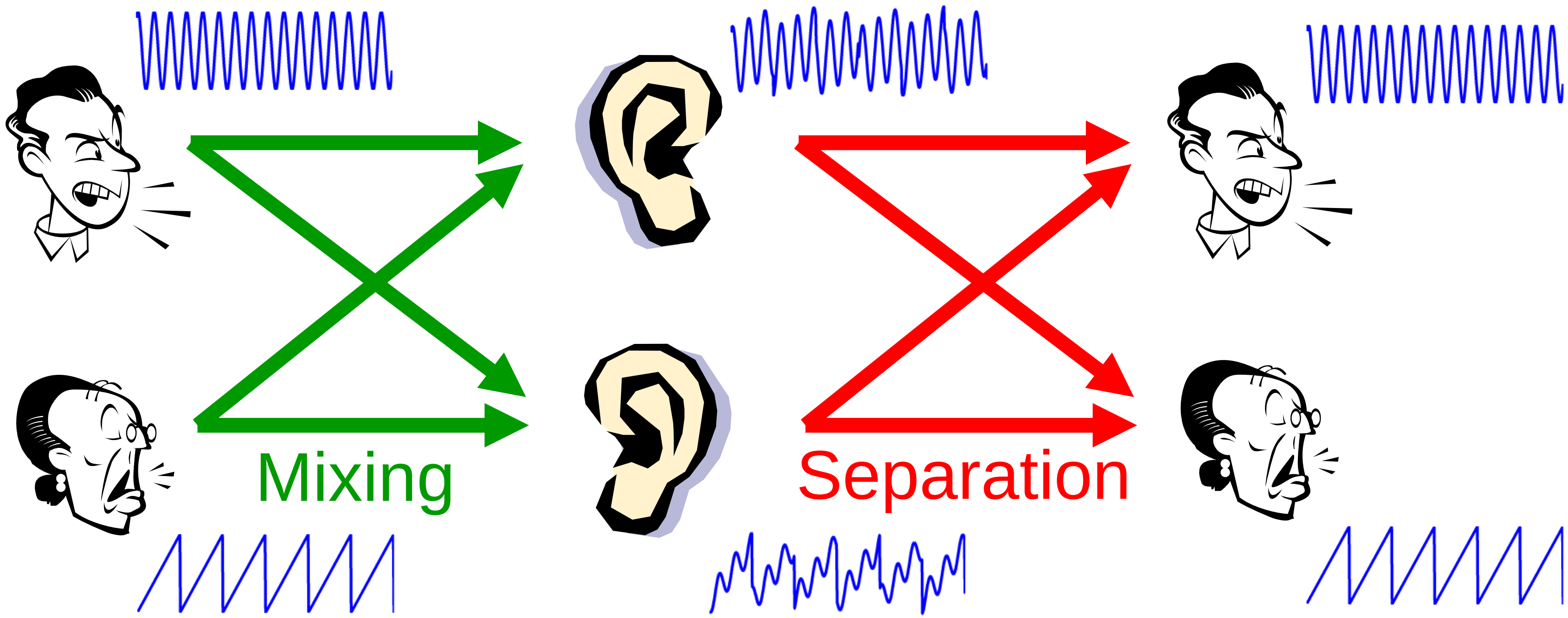
Blind Source Separation

- Cocktail-party problem:

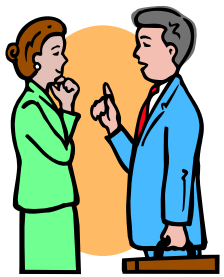


- We want to separate mixed signals into original ones.

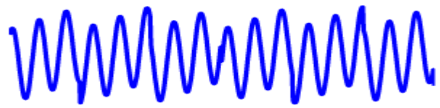
Demonstration

	Mixed signal	Separated signal 1	Separated signal 2
Conversation + Conversation			
Conversation + Instrument			

From http://www.brain.kyutech.ac.jp/~shiro/research/blindsep.html



Formulation

- **Source signals**:
  - Speaker 1: $s_1^{(1)}, s_2^{(1)}, \ldots, s_n^{(1)}$ 
  - Speaker 2: $s_1^{(2)}, s_2^{(2)}, \ldots, s_n^{(2)}$ 
- **Mixed signals**:
  - Left ear: $x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}$ 
  - Right ear: $x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}$ 

$$x_i^{(1)} = m_{11} s_i^{(1)} + m_{12} s_i^{(2)}$$
$$x_i^{(2)} = m_{21} s_i^{(1)} + m_{22} s_i^{(2)}$$

Formulation (cont.)

- In matrix form:

$$\boldsymbol{x}_i = \boldsymbol{M}\boldsymbol{s}_i$$

$$\boldsymbol{x}_i = \begin{pmatrix} x_i^{(1)} \\ x_i^{(2)} \end{pmatrix}$$

$$\boldsymbol{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \qquad \boldsymbol{s}_i = \begin{pmatrix} s_i^{(1)} \\ s_i^{(2)} \end{pmatrix}$$

- More generally
  - $\boldsymbol{x}_i, \boldsymbol{s}_i$: d -dimensional vectors
  - $\boldsymbol{M}$: d -dimensional matrix.

Problem

$$\boldsymbol{x}_i = \boldsymbol{M}\boldsymbol{s}_i$$

- We want to estimate $\{\boldsymbol{s}_i\}_{i=1}^n$ from $\{\boldsymbol{x}_i\}_{i=1}^n$.
- Approach: Estimate $\boldsymbol{M}$, and use its inverse for obtaining $\{\widehat{\boldsymbol{s}}_i\}_{i=1}^n$.

$$\widehat{\boldsymbol{s}}_i = \widehat{\boldsymbol{M}}^{-1}\boldsymbol{x}_i$$

- In BSS, the followings may not be important:
  - Permutation of separated signals
  - Scaling of separated signals
- Therefore, we estimate $\widehat{\boldsymbol{M}}^{-1}$ up to permutation and scaling of rows.

Assumptions

- $\{\boldsymbol{s}_i\}_{i=1}^n$ are i.i.d. random variables with mean zero and covariance identity:

$$\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{s}_i = \boldsymbol{0} \qquad \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{s}_i\boldsymbol{s}_i^\top = \boldsymbol{I}_d$$

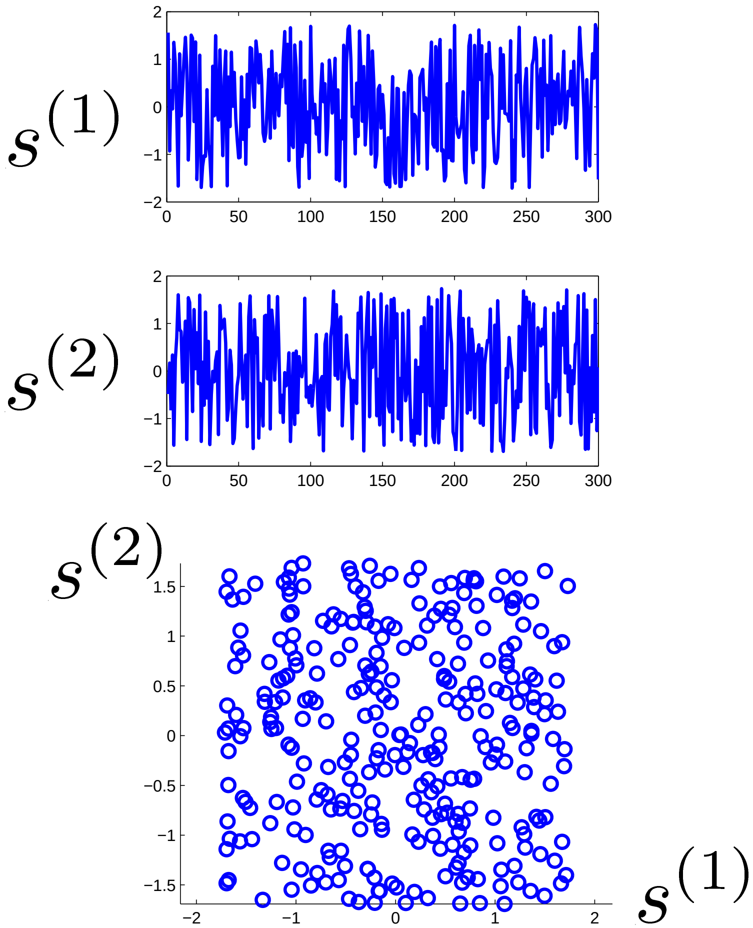
- $\{s^{(j)}\}_{j=1}^d$ are mutually independent:

$$P(s^{(1)}, s^{(2)}, \ldots, s^{(d)}) = P(s^{(1)})P(s^{(2)})\cdots P(s^{(d)})$$

- $\{s^{(j)}\}_{j=1}^d$ are non-Gaussian.
- $\boldsymbol{M}$ is invertible.
- BSS under source independence is called independent component analysis.

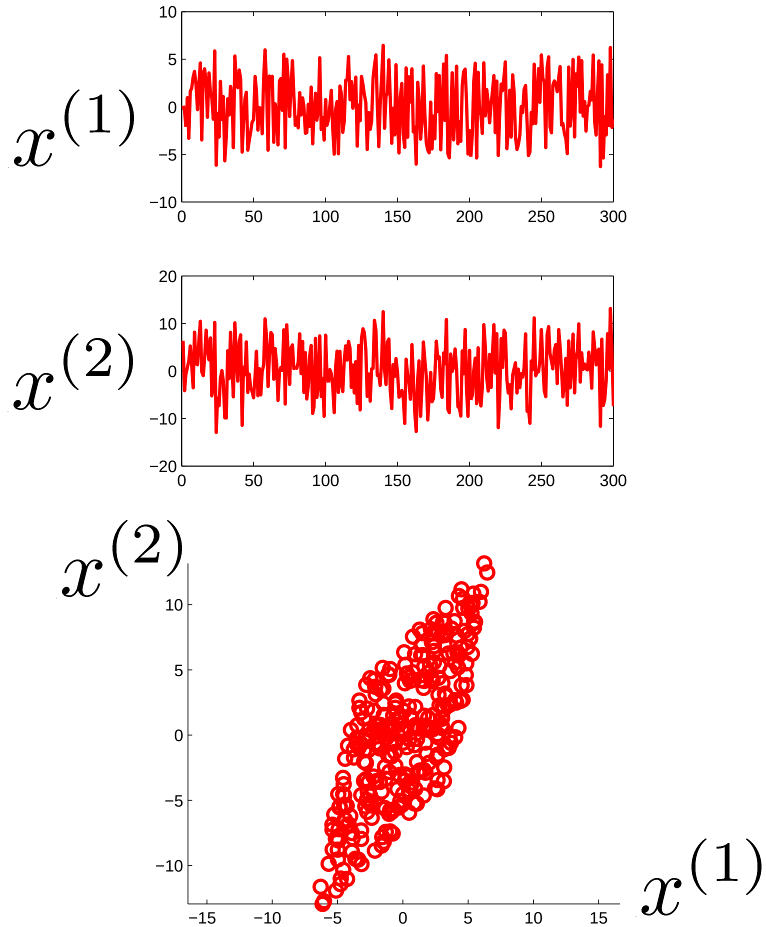
Example

Source signals
(uniform)



Mixed signals

$$\boldsymbol{M} = \begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix}$$



Data Sphering

- Sphering (or pre-whitening):

$$\widetilde{\boldsymbol{x}}_i = \boldsymbol{C}^{-\frac{1}{2}} \boldsymbol{x}_i \qquad \boldsymbol{C} = \frac{1}{n} \sum_{j=1}^{n} \boldsymbol{x}_j \boldsymbol{x}_j^\top$$

- Then

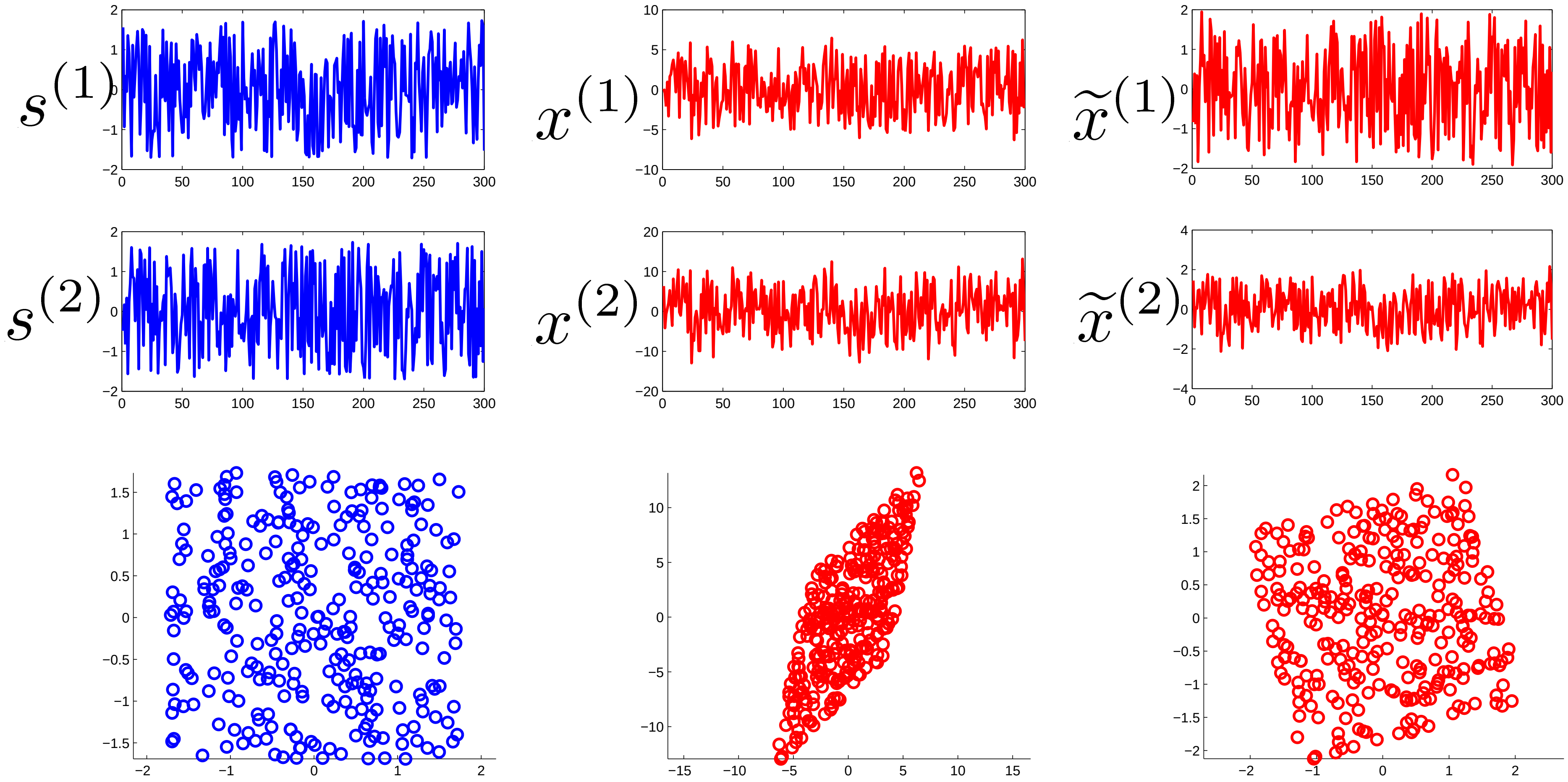
$$\widetilde{\boldsymbol{x}}_i = \widetilde{\boldsymbol{M}} \boldsymbol{s}_i \qquad \widetilde{\boldsymbol{M}} = \boldsymbol{C}^{-\frac{1}{2}} \boldsymbol{M}$$

- Now we want to estimate $\widetilde{\boldsymbol{M}}$ from $\{\widetilde{\boldsymbol{x}}_i\}_{i=1}^{n}$, and obtain $\{\widehat{\boldsymbol{s}}_i\}_{i=1}^{n}$ by

$$\widehat{\boldsymbol{s}}_i = \boldsymbol{W} \widetilde{\boldsymbol{x}}_i \qquad \boldsymbol{W} = \widetilde{\boldsymbol{M}}^{-1}$$

Example

Source signals (uniform)	Mixed signals	Sphered signals
	$\boldsymbol{M} = \begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix}$	$\widetilde{\boldsymbol{x}}_i = \boldsymbol{C}^{-\frac{1}{2}} \boldsymbol{x}_i$



Orthogonal Matrix

- $\widetilde{M}$ is an orthogonal matrix since

$$\widetilde{C} = \frac{1}{n}\sum_{i=1}^{n}\widetilde{\boldsymbol{x}}_i\widetilde{\boldsymbol{x}}_i^\top = \boldsymbol{I}_d$$

$$\widetilde{C} = \widetilde{\boldsymbol{M}}\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{s}_i\boldsymbol{s}_i^\top\right)\widetilde{\boldsymbol{M}}^\top = \widetilde{\boldsymbol{M}}\widetilde{\boldsymbol{M}}^\top$$

- Therefore,

$$\widehat{\boldsymbol{s}}_i = \boldsymbol{W}\widetilde{\boldsymbol{x}}$$

$$\boldsymbol{W} = \widetilde{\boldsymbol{M}}^{-1} = \widetilde{\boldsymbol{M}}^\top \equiv (\boldsymbol{w}^{(1)}|\boldsymbol{w}^{(2)}|\cdots|\boldsymbol{w}^{(d)})^\top$$

$\{\boldsymbol{w}^{(j)}\}_{j=1}^{d}$:Orthonormal basis

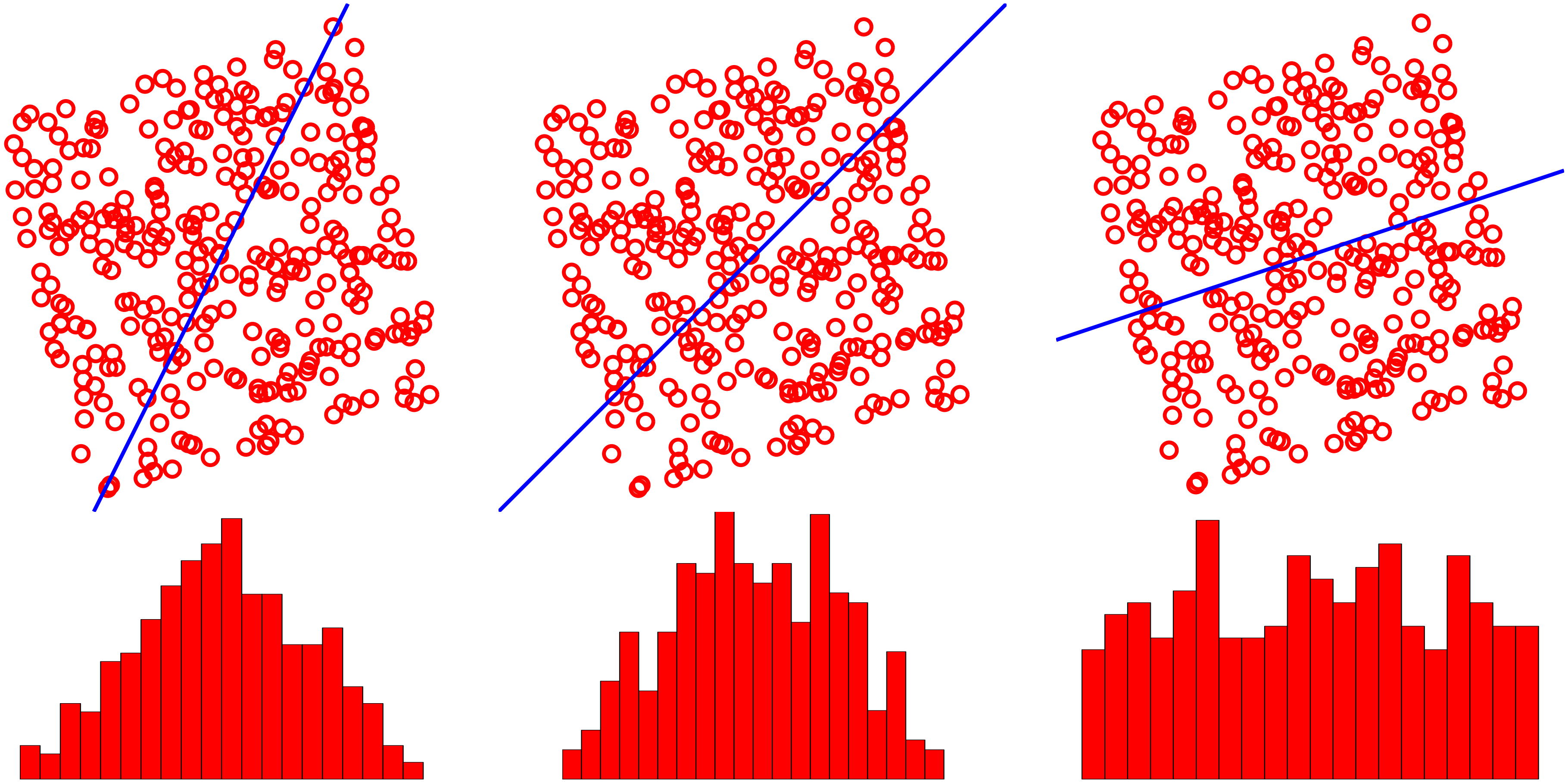
$$\widehat{s}_i^{(j)} = \langle \boldsymbol{w}^{(j)}, \widetilde{\boldsymbol{x}}_i \rangle$$

Non-Gaussian Is Independent

- Now we want to find an ONB $\{\boldsymbol{w}^{(j)}\}_{j=1}^{d}$ such that $\{\widehat{s}^{(j)}\}_{j=1}^{d}$ are independent.
- Central limit theorem: Sum of independent variables tends to be Gaussian.
- Conversely, non-Gaussian variables are independent.
- We find non-Gaussian directions in $\{\widetilde{\boldsymbol{x}}_i\}_{i=1}^{n}$.

Example (cont.)

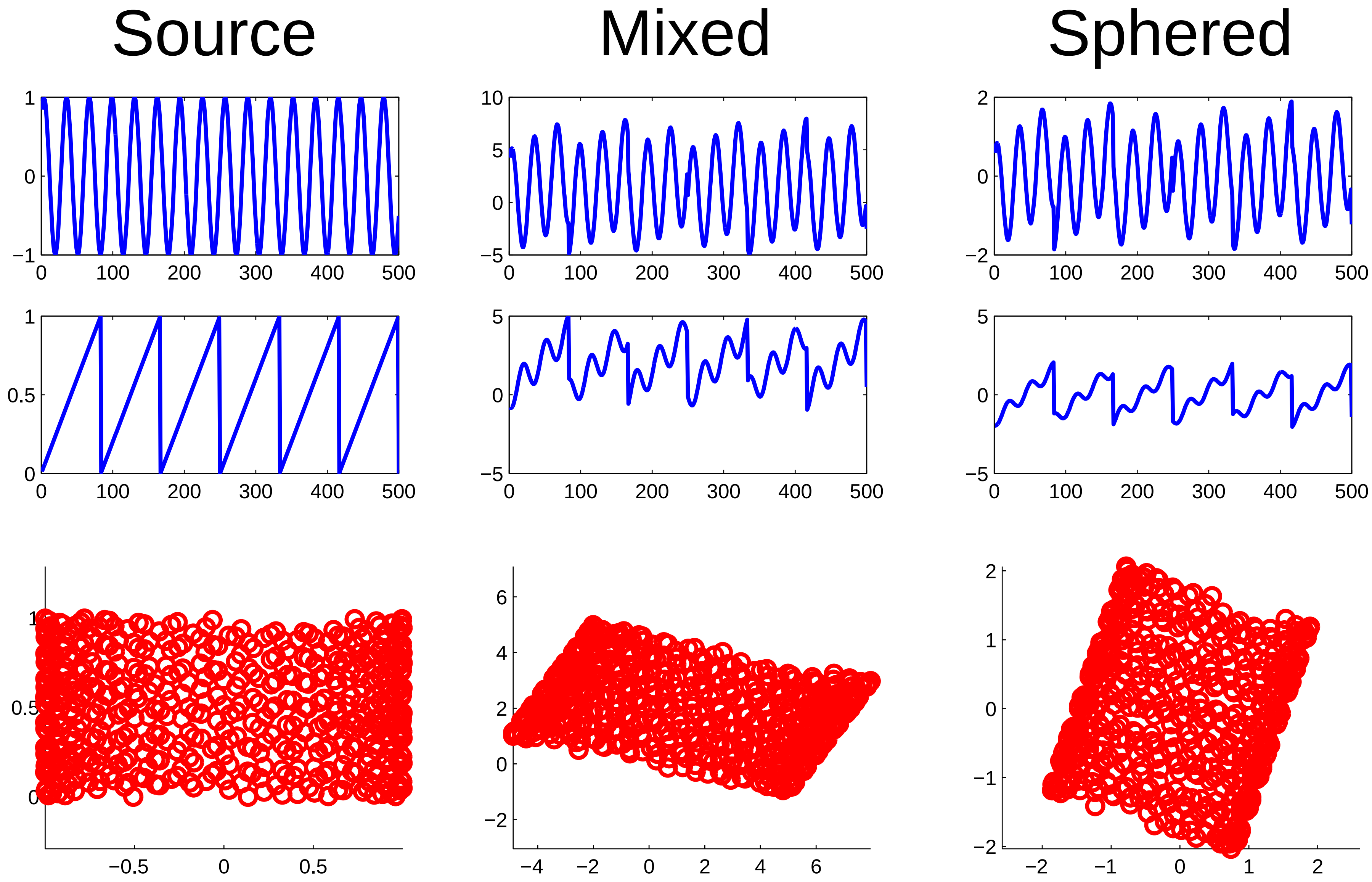
- Non-Gaussian direction is independent.



ICA by Projection Pursuit

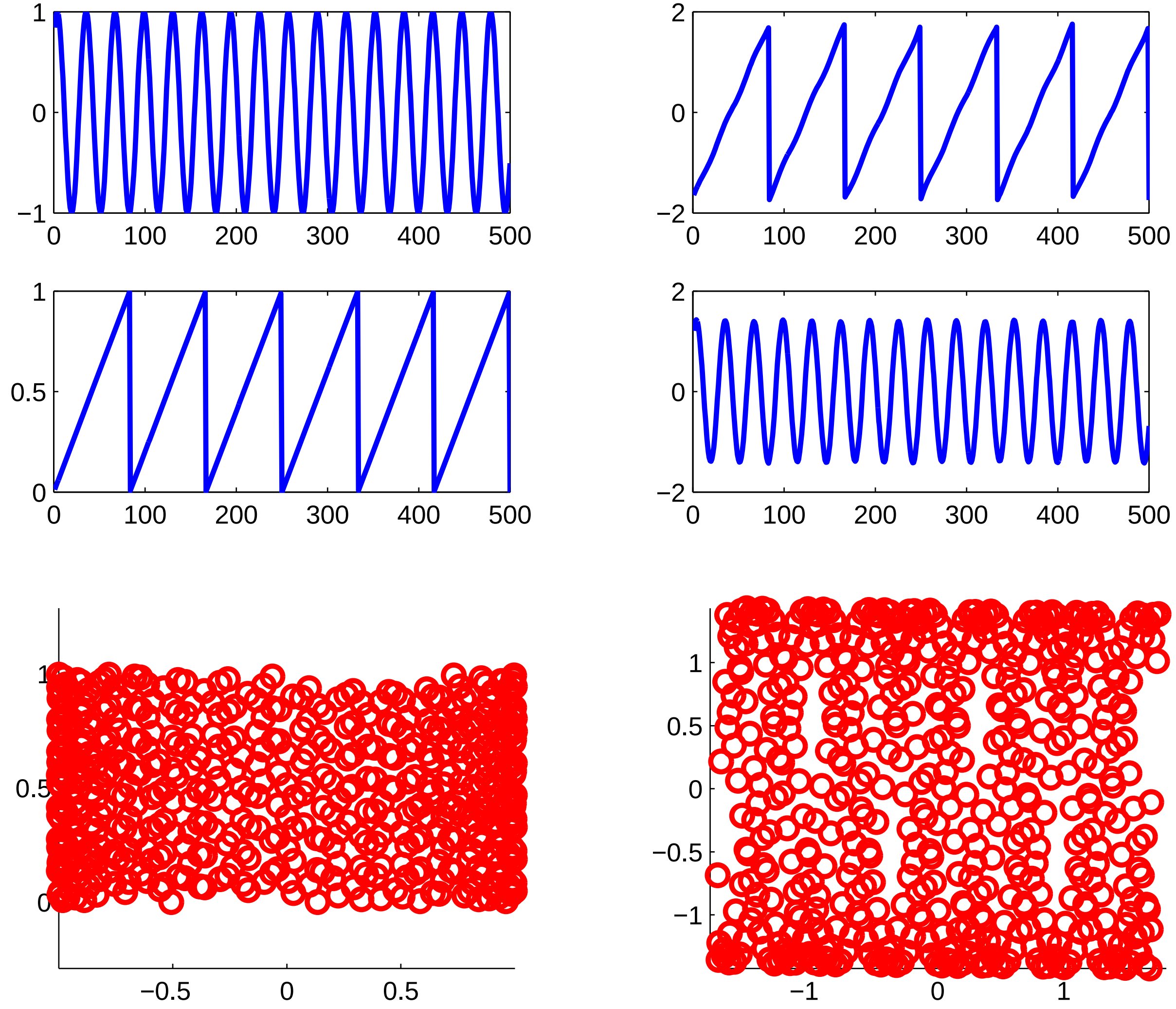
- Finding non-Gaussian directions can be achieved by projection pursuit algorithms!
  - Center and sphere the data.
  - Find non-Gaussian directions by PP.
- We may use an approximate Newton-based PP method, which is called FastICA.

Example 2



Example 2 (cont.)

Source | Separated



- Original signals are recovered up to permutation and scaling.