2-D Poisson Equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \rho$$

In the case of
incompressible flow computation : $\rho = \left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} \right) / \Delta t$

Discretization by the center finite difference method :

$$\frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{\Delta x^2} + \frac{p_{i,j+1} - 2p_{i,j} + p_{i,j-1}}{\Delta y^2} = \rho_{i,j}$$

2-D Poisson Equation

SOR method:

$$p_{i,j}^{(n+1)} = (1-\omega)\, p_{i,j}^{(n)} + \omega \left(\frac{p_{i+1,j}^{(n)} + p_{i-1,j}^{(n+1)}}{\Delta x^2} + \frac{p_{i,j+1}^{(n)} + p_{i,j-1}^{(n+1)}}{\Delta y^2} - \rho_{i,j} \right) \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}$$

n : iteration step

ω : relaxation factor ($0 < \omega < 2$)

Example: $\rho(x,y) = \sin(k_x x)\sin(k_y y)$
$(0 \le x \le 1,\ \ 0 \le y \le 1)$
periodic boundary condition

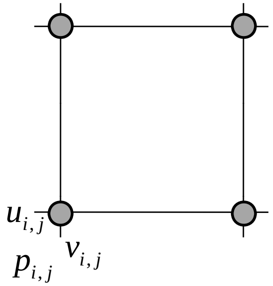
Source Code

Coupling between Velocity and Pressure

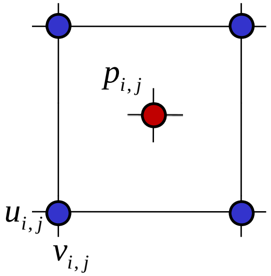
Poisson Equation : $\Delta p = \frac{\nabla \cdot \boldsymbol{u}}{\Delta t}$

Velocity Correction : $\boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \nabla p \cdot \Delta t$

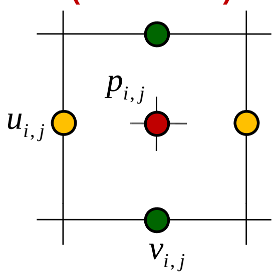
Collocated grid (Arakawa-A)



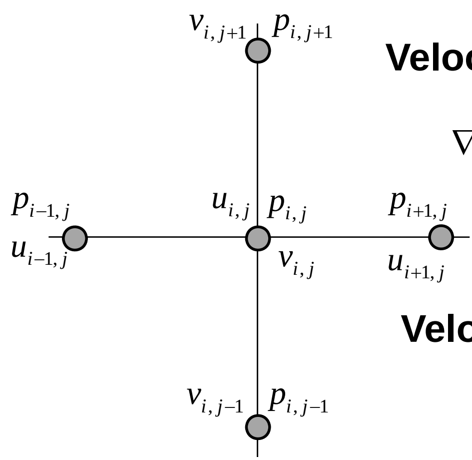
Staggered grid (Arakawa-B)



Staggered grid (Arakawa-B)



Collocated Grid



Velocity Divergence :

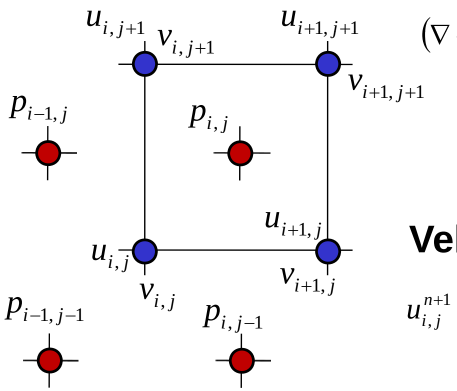
$$\nabla \cdot \boldsymbol{u} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}$$

Velocity Correction :

$$u_{i,j}^{n+1} = u_{i,j}^* - \frac{p_{i+1.j} - p_{i-1,j}}{2\Delta x}\Delta t$$

$$v_{i,j}^{n+1} = v_{i,j}^* - \frac{p_{i.j+1} - p_{i,j-1}}{2\Delta y}\Delta t$$

Staggered Grid
(Arakawa-B)



Velocity Divergence :

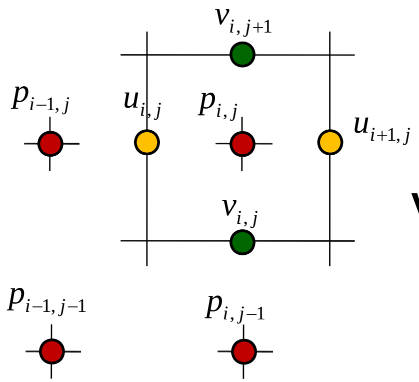
$$(\nabla \cdot \boldsymbol{u})_{i,j} = \frac{1}{2}\left(\frac{u_{i+1,j+1} - u_{i,j+1}}{\Delta x} + \frac{u_{i+1,j} - u_{i,j}}{\Delta x}\right) + \frac{1}{2}\left(\frac{v_{i+1,j+1} - v_{i+1,j}}{\Delta y} + \frac{v_{i,j+1} - v_{i,j}}{\Delta y}\right)$$

Velocity Correction :

$$u_{i,j}^{n+1} = u_{i,j}^{*} - \frac{1}{2}\left(\frac{p_{i.j} - p_{i-1,j}}{\Delta x} + \frac{p_{i.j-1} - p_{i-1,j-1}}{\Delta x}\right)\Delta t$$

$$v_{i,j}^{n+1} = v_{i,j}^{*} - \frac{1}{2}\left(\frac{p_{i.j} - p_{i,j-1}}{\Delta y} + \frac{p_{i-1.j} - p_{i-1,j-1}}{\Delta y}\right)\Delta t$$

Staggered Grid
(Arakawa-C)



Velocity Divergence :

$$(\nabla \cdot \boldsymbol{u})_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + \frac{v_{i,j+1} - v_{i,j}}{\Delta y}$$

Velocity Correction :

$$u_{i,j}^{n+1} = u_{i,j}^{*} - \frac{p_{i.j} - p_{i-1,j}}{\Delta x}\Delta t$$

$$v_{i,j}^{n+1} = v_{i,j}^{*} - \frac{p_{i.j} - p_{i,j-1}}{\Delta y}\Delta t$$

2-D Incompressible Flow

Navior-Stokes Equation (normalized)

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla P + \frac{1}{\mathrm{Re}} \Delta \boldsymbol{u}$$

Incompressible flow condition

$$\nabla \cdot \boldsymbol{u} = 0$$

STEP I : Solving Burgers Equation - - - - - - - - - - - $\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \frac{1}{\mathrm{Re}} \Delta \boldsymbol{u}$

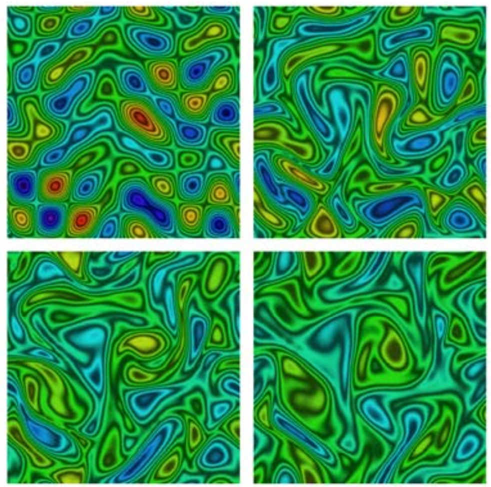
STEP II : Getting Velocity Divergence - - - - - - - - - - $\rho = \left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right) / \Delta t$

STEP III : Solving Pressure Poisson Equation - - - - $\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \rho$

STEP IV : Correcting Velocity - - - - - - - - - - - - - - - - - $\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla P$

Homogeneous Turbulent Flow

Initial velocity profile satisfying the divergence-free condition,



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u_x(x,y) = \sin(k_x x)\sin(k_y y)$$

$$v_y(x,y) = -\sin(k_x x)\sin(k_y y)$$

$$u(x,y) = -\frac{1}{k_x}\cos(k_x x)\sin(k_y y)$$

$$v(x,y) = \frac{1}{k_y}\sin(k_x x)\cos(k_y y)$$

$$(0 \le x \le 1, \quad 0 \le y \le 1)$$

Periodic boundary condition

Source Code