Classification of CFD

Fluid material is definitely compressible.

Incompressible Fluid Model:

Negligible compressibility:

Pressure changes before density changes.

Fluid phenomena of daily life.

Typical time scale << typical size / sound speed

Compressible Fluid Model:

Density changes.

Typical velocity ~ sound velocity

Shock wave, Explosion, . . .

Compressible Fluid Equation

Euler Equation:

without viscosity and thermal conduction

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} = 0 , \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} , \quad \frac{\partial e}{\partial t} + u \frac{\partial e}{\partial x} = -\frac{p}{\rho}\frac{\partial u}{\partial x}$$

Conservative Form:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \qquad U = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} , \quad F = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ (E+p)u \end{bmatrix}$$

Convection + Acoustic Wave

Continuity Equation: $\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = 0$ **+** $\frac{\partial \rho}{\partial t} = -\rho \frac{\partial u}{\partial x}$

Momentum Equation: $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$ **+** $\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$

Linerization: $\rho = \rho_0 + \rho'$ $\quad u = u_0 + u' \quad (u_0 = 0)$

$$\frac{\partial \rho'}{\partial t} = -\rho_0 \frac{\partial u'}{\partial x}$$

$$\frac{\partial u'}{\partial t} = -\frac{1}{\rho_0}\left(\frac{\partial P}{\partial \rho}\right)\frac{\partial \rho'}{\partial x}$$

$$c_s^2 = \left(\frac{\partial P}{\partial \rho}\right)_s$$

$$\frac{\partial^2 u'}{\partial t^2} - \left(\frac{\partial P}{\partial \rho}\right)\frac{\partial^2 u'}{\partial x^2} = 0$$

Incompressible Fluid Equation

Constant density:

$$\frac{\partial \rho}{\partial t} = 0 \qquad \frac{\partial \rho}{\partial x} = 0 \quad \Rightarrow \quad \nabla \cdot \boldsymbol{u} = 0$$

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} = 0 ,$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla P + \nu \Delta \boldsymbol{u}$$

Semi-Implicit Method

The characteristics of the fluid equation consists of acoustic (sound) wave and flow velocity. When the sound velocity is much greater than flow velocity, flow phenomena is able to be described as incompressible fluid. The CFL condition is constrained by the sound speed in the explicit time integration, so we want to dump the acoustic mode by using semi-implicit time integration.

$$\nabla \cdot \boldsymbol{u} = 0$$

By solving Poisson equation and satisfying the divergence free condition, we dump the acoustic mode and obtain the pressure and velocity profiles to satisfy the incompressible flow.

MAC (Marker and Cell) (1/3)

2-dimensional incompressible fluid equation:

$$D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad (2)$$

$$\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) \Rightarrow \frac{\partial D}{\partial t} + \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 D}{\partial x^2} + \frac{\partial^2 D}{\partial y^2}\right)$$

MAC (Marker and Cell) (2/3)

$$\frac{\partial D}{\partial t} = \frac{D^{n+1} - D^n}{\Delta t} \rightarrow -\frac{D^n}{\Delta t} \quad \left(\text{satisfying } D^{n+1} = 0\right)$$

$$\frac{\partial^2 P^n}{\partial x^2} + \frac{\partial^2 P^n}{\partial y^2} = -\frac{\partial}{\partial x}\left(u^n\frac{\partial u^n}{\partial x} + v\frac{\partial u^n}{\partial y}\right) - \frac{\partial}{\partial y}\left(u^n\frac{\partial v^n}{\partial x} + v\frac{\partial v^n}{\partial y}\right) + \frac{D^n}{\Delta t} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 D^n}{\partial x^2} + \frac{\partial^2 D^n}{\partial y^2}\right)$$

MAC (Marker and Cell) (3/3)

By using the pressure, the velocity is corrected:

$$u^{n+1} = u^n + \Delta t\left\{-u^n\frac{\partial u^n}{\partial x} - v^n\frac{\partial u^n}{\partial y} - \frac{\partial P^n}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u^n}{\partial x^2} + \frac{\partial^2 u^n}{\partial y^2}\right)\right\}$$

$$v^{n+1} = v^n + \Delta t\left\{-u^n\frac{\partial v^n}{\partial x} - v^n\frac{\partial v^n}{\partial y} - \frac{\partial P^n}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v^n}{\partial x^2} + \frac{\partial^2 v^n}{\partial y^2}\right)\right\}$$

SMAC (1/2)

The intermediate $\overline{\boldsymbol{u}}$ is obtained from the n-step $\boldsymbol{u}^n,\ P^n$.

$$\overline{\boldsymbol{u}} = \boldsymbol{u}^n + \Delta t\left(-\boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^n - \nabla P^n + \frac{1}{\mathrm{Re}}\Delta \boldsymbol{u}^n\right)$$

assuming $\boldsymbol{u}^{n+1} = \overline{\boldsymbol{u}} + \boldsymbol{u}' \ ,\quad P^{n+1} = P^n + P'$

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^n + \Delta t\left(-\boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^n - \nabla P^{n+1} + \frac{1}{\mathrm{Re}}\Delta \boldsymbol{u}^n\right) \qquad (3)$$

$$= \overline{\boldsymbol{u}} - \Delta t \cdot \nabla P'$$

SMAC (2/2)

Taking divergence of the equation (3),

$$\nabla \cdot \boldsymbol{u}^{n+1} = \nabla \cdot \overline{\boldsymbol{u}} - \Delta t \cdot \Delta P'$$
$$= 0$$

Poisson Equation:

$$\Delta P' = \frac{\nabla \cdot \overline{\boldsymbol{u}}}{\Delta t}$$

$$P^{n+1} = P^n + P'$$

2-D Burgers Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \kappa\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Advection term: 3rd-order upwind scheme:

Diffusion term: 2nd-order central difference:

Use the Runge-Kutta 3-stage or 4-stage Time Integration

Divergence-Free Initial Condition

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

When we assume

$$u_x(x,y) = \sin(k_x x)\sin(k_y y)$$
$$v_y(x,y) = -\sin(k_x x)\sin(k_y y)$$

we have the following divergence-free velocity profile.

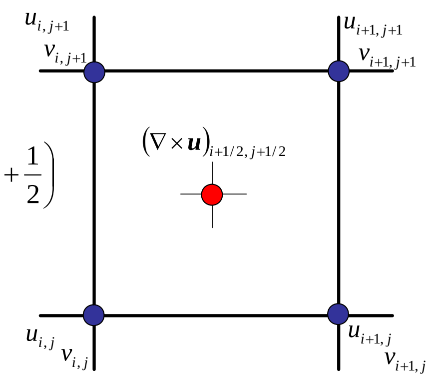
$$u(x,y) = -\frac{1}{k_x}\cos(k_x x)\sin(k_y y)$$
$$v(x,y) = \frac{1}{k_y}\sin(k_x x)\cos(k_y y)$$

Divergence-Free Initial Condition

```
int      jx,    jy;
double   x,     y,    kx = 2.0*M_PI,    ky = 2.0*M_PI;
for(jy=0 ; jy < ny; jy++) {
    for(jx=0 ; jx < nx; jx++) {
        x = dx*(double)(jx - 1);
        y = dy*(double)(jy - 1);
        u[jy][jx] = - cos(kx*x)*sin(ky*y)/kx;
        v[jy][jx] = sin(kx*x)*cos(ky*y)/ky;
        x += 0.3;   y += 0.7;
        u[jy][jx] += - 0.6*cos(2.0*kx*x)*sin(2.0*ky*y)/kx;
        v[jy][jx] += 0.6*sin(2.0*kx*x)*cos(2.0*ky*y)/ky;
    }
}
```


Velocity Rotation (Vorticity)

$$\omega = \nabla \times \boldsymbol{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

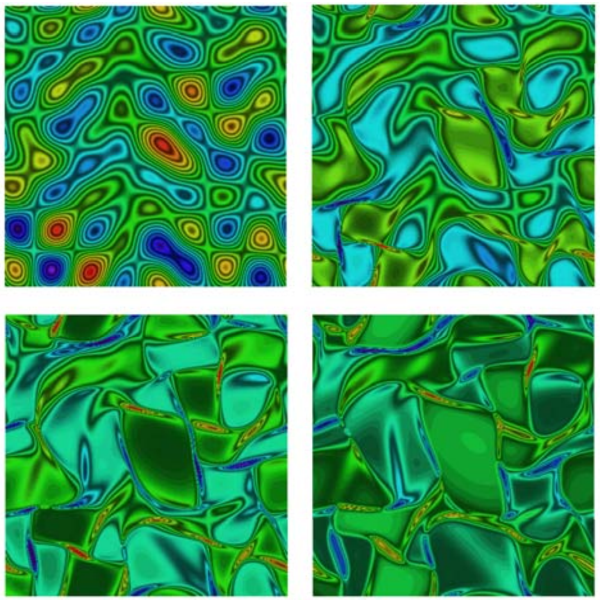


Central Finite Difference at $\left(i+\frac{1}{2},\ j+\frac{1}{2}\right)$

Compact and 2nd-order accuracy

$$\omega_{i+1/2,j+1/2} = \frac{1}{2}\left(\frac{v_{i+1,j+1} - v_{i,j+1}}{\Delta x} + \frac{v_{i+1,j} - v_{i,j}}{\Delta x}\right) - \frac{1}{2}\left(\frac{u_{i+1,j+1} - u_{i+1,j}}{\Delta y} + \frac{u_{i,j+1} - u_{i,j}}{\Delta y}\right)$$

Computational Results of 2-D Burgers Equation



Source Code