# 7. Optical Fiber

7.1 Electromagnetic wave guided in an optical fiber

Cylindrical coordinate system:

$$\mathbf{E} = \mathbf{E}(r,\theta)e^{j(\omega t-\beta z)}, \mathbf{H} = \mathbf{H}(r,\theta)e^{j(\omega t-\beta z)}$$

$$\frac{1}{r}\frac{\partial E_z}{\partial \theta} + j\beta E_\theta = -j\omega\mu_0 H_r$$

$$-j\beta E_r - \frac{\partial E_z}{\partial r} = -j\omega\mu_0 H_\theta$$

$$\frac{1}{r}\frac{\partial (rE_\theta)}{\partial r} - \frac{1}{r}\frac{\partial E_r}{\partial \theta} = -j\omega\mu_0 H_z$$

$$\frac{1}{r}\frac{\partial H_z}{\partial \theta} + j\beta H_\theta = j\omega\varepsilon_0 n^2 E_r$$

$$-j\beta H_r - \frac{\partial H_z}{\partial r} = j\omega\varepsilon_0 n^2 E_\theta$$

$$\frac{1}{r}\frac{\partial (rH_\theta)}{\partial r} - \frac{1}{r}\frac{\partial H_r}{\partial \theta} = j\omega\varepsilon_0 n^2 E_z$$

$$\frac{1}{r}\frac{\partial E_z}{\partial \theta} + \frac{\beta}{\omega\varepsilon_0 n^2}\left(-j\beta H_r - \frac{\partial H_z}{\partial r}\right) = -j\omega\mu_0 H_r$$

$$H_r = \frac{j}{\beta_t^2}\left(\frac{\omega\varepsilon_0 n^2}{r}\frac{\partial E_z}{\partial \theta} - \beta\frac{\partial H_z}{\partial r}\right)$$

$$\beta_t^2 = n^2 k_0^2 - \beta^2$$

$$E_\theta = -\frac{j}{\beta_t^2}\left(\frac{\beta}{r}\frac{\partial E_z}{\partial \theta} - \omega\mu_0\frac{\partial H_z}{\partial r}\right)$$

# EMW in optical fiber

$$E_r = -\frac{j}{\beta_t^{\,2}}\left(\beta\frac{\partial E_z}{\partial r} + \omega\mu_0\frac{1}{r}\frac{\partial H_z}{\partial \theta}\right)$$

$$H_\theta = -\frac{j}{\beta_t^{\,2}}\left(\omega\varepsilon_0 n^2\frac{\partial E_z}{\partial r} + \frac{\beta}{r}\frac{\partial H_z}{\partial \theta}\right)$$

wave equations:

$$\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r}\frac{\partial H_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H_z}{\partial \theta^2} + \beta_t^{\,2} H_z = 0$$

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \theta^2} + \beta_t^{\,2} E_z = 0$$

variable separation $H_z, E_z = R(r)\Theta(\theta)$

$$\frac{r^2}{R}\left(\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + \beta_t^{\,2} r^2 + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = 0$$

$$\begin{cases} \dfrac{1}{\Theta}\dfrac{d^2\Theta}{d\theta^2} = -n^2 \\ \dfrac{r^2}{R}\left(\dfrac{d^2 R}{dr^2} + \dfrac{1}{r}\dfrac{dR}{dr}\right) + \beta_t^{\,2} r^2 = n^2 \end{cases}$$

# EMW in optical fiber

Bessel differential equations:

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(\beta_t^{\,2}-\frac{n^2}{r^2}\right)R=0$$

$$R(r)=\begin{cases} AJ_n(\beta_t r)+BN_n(\beta_t r) & :\beta_t^{\,2}>0 \\ CK_n(|\beta_t|r)+DI_n(|\beta_t|r) & :\beta_t^{\,2}<0 \end{cases}$$

core region: $\beta_t^2=n_1^2k_0^2-\beta^2>0$

$$N_n(\beta_t r)\rightarrow\infty \quad (r\rightarrow 0) \qquad \therefore B=0$$

clad region: $\beta_t^2=n_2^2k_0^2-\beta^2<0$

$$I_n(\beta_t r)\rightarrow\infty \quad (r\rightarrow\infty) \qquad \therefore D=0$$

$$\Theta=\begin{cases}\cos n\theta \\ \sin n\theta\end{cases}$$

# EMW in optical fiber

$$E_z = \begin{cases} AJ_n\left(\frac{u}{a}r\right)\sin n\theta & : r \le a \\ CK_n\left(\frac{w}{a}r\right)\sin n\theta & : r \ge a \end{cases}$$

$$H_z = \begin{cases} BJ_n\left(\frac{u}{a}r\right)\cos n\theta & : r \le a \\ DK_n\left(\frac{w}{a}r\right)\cos n\theta & : r \ge a \end{cases}$$

$$u = \beta_{t1}a = \sqrt{n_1^2k_0^2 - \beta^2}\,a$$

$$w = \beta_{t2}a = \sqrt{\beta^2 - n_2^2k_0^2}\,a$$

$n = 0$ : TE or TM → axial symmetry

$n \neq 0$ : hybrid → $E_z \neq 0$ and $H_z \neq 0$

# 7.2 Eigen value equation

boundary conditions: $E_z, \; H_z, \; E_\vartheta, \; H_\vartheta$ are to be continuous at r=a.

$$AJ_n(u) = CK_n(w)$$

$$BJ_n(u) = DK_n(w)$$

$$-A\frac{j\beta}{(u/a)^2}\frac{n}{a}J_n(u) + B\frac{j\omega\mu_0}{u/a}J_n'(u) = C\frac{j\beta}{(w/a)^2}\frac{n}{a}K_n(w) - D\frac{j\omega\mu_0}{w/a}K_n'(w)$$

$$-A\frac{j\omega\varepsilon_0 n_1^2}{u/a}J_n'(u) + B\frac{j\beta}{(u/a)^2}\frac{n}{a}J_n(u) = C\frac{j\omega\varepsilon_0 n_2^2}{w/a}K_n'(w) - D\frac{j\beta}{(w/a)^2}\frac{n}{a}K_n(w)$$

$$[M]\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0 \;-\; \det[\mathbf{M}] = 0$$

$$\left[\frac{J_n'(u)}{uJ_n(u)} + \frac{K_n'(w)}{wK_n(w)}\right]\left[\frac{n_1^2}{n_2^2}\frac{J_n'(u)}{uJ_n(u)} + \frac{K_n'(w)}{wK_n(w)}\right] = n^2\left(\frac{1}{u^2} + \frac{1}{w^2}\right)\left(\frac{n_1^2}{n_2^2}\frac{1}{u^2} + \frac{1}{w^2}\right)$$

# eigen mode in an optical fiber: TE and TM mode

(1) $n = 0$ $\Theta(\theta) = const$

| $r \geq a$ | $a \geq r$ |
|---|---|
| $E_z = CK_0\left(\frac{w}{a}r\right)$ | $E_z = AJ_0\left(\frac{u}{a}r\right)$ |
| $E_r = C\frac{j\beta}{w/a}K_0'\left(\frac{w}{a}r\right)$ | $E_r = -A\frac{j\beta}{u/a}J_0'\left(\frac{u}{a}r\right)$ |
| $E_\theta = -D\frac{j\omega\mu_0}{w/a}K_0'\left(\frac{w}{a}r\right)$ | $E_\theta = B\frac{j\omega\mu_0}{u/a}J_0'\left(\frac{u}{a}r\right)$ |
| $H_z = DK_0\left(\frac{w}{a}r\right)$ | $H_z = BJ_0\left(\frac{u}{a}r\right)$ |
| $H_r = D\frac{j\beta}{w/a}K_0'\left(\frac{w}{a}r\right)$ | $H_r = -B\frac{j\beta}{u/a}J_0'\left(\frac{u}{a}r\right)$ |
| $H_\theta = C\frac{j\omega\varepsilon_0 n_2^2}{w/a}K_0'\left(\frac{w}{a}r\right)$ | $H_\theta = -A\frac{j\omega\varepsilon_0 n_1^2}{u/a}J_0'\left(\frac{u}{a}r\right)$ |

$E_z$, $E_r$, $H_\theta$ → $TM_{0l}$

$$\frac{n_1^2}{n_2^2}\frac{J_0'(u)}{uJ_0(u)} + \frac{K_0'(w)}{wK_0(w)} = 0$$

$E_\theta$, $H_z$, $H_r$ → $TE_{0l}$

$$\frac{J_0'(u)}{uJ_0(u)} + \frac{K_0'(w)}{wK_0(w)} = 0$$

# eigen mode in an optical fiber: hybrid mode

(2) $n \neq 0$ : hybrid → $E_z \neq 0$ and $H_z \neq 0$

$n_1 \approx n_2$ weakly guided approximation

$$\left[\frac{J_n'(u)}{uJ_n(u)} + \frac{K_n'(w)}{wK_n(w)}\right]^2 = n^2\left(\frac{1}{u^2} + \frac{1}{w^2}\right)^2$$

$$\therefore \frac{J_n'(u)}{uJ_n(u)} + \frac{K_n'(w)}{wK_n(w)} = \pm n\left(\frac{1}{u^2} + \frac{1}{w^2}\right)$$

With the help of Bessel function's equalities, all the modes can be included in the following characteristic equation.

$$\frac{uJ_{m-1}(u)}{J_m(u)} = -\frac{wK_{m-1}(w)}{K_m(w)}$$

$$m = \begin{cases} 1 & : TE, TM \ \text{mode} \\ n+1 & : EH \ \text{mode} \\ n-1 & : HE \ \text{mode} \end{cases} \qquad LP_{ml}$$

# 7.3 Guided mode

Normalized frequency:

$$V^2 = u^2 + w^2 = k_0^2 (n_1^2 - n_2^2) a^2 \approx k_0^2 n_1^2 a^2 2 \frac{n_1 - n_2}{n_1} = k_0^2 n_1^2 a^2 2\Delta \qquad V = k_0 n_1 a \sqrt{2\Delta}$$

Characteristic equation:

$$\frac{uJ_{m-1}(u)}{J_m(u)} + \frac{wK_{m-1}(w)}{K_m(w)} = 0$$

$$V^2 = u^2 + w^2$$

Cut-off : w=0

$J_{m-1}(u) = 0$ the -th zero of order Bessel function

$$\downarrow$$

$V_c = j_{m-1,l}$ $\quad (m \ge 1,\ l \ge 1)$ $\longrightarrow$ $LP_{ml}$ mode( $EH_{m-1,l}$ $HE_{m+1,l}$ )

$LP_{1l}$ mode ( $TM_{0l}$ $TE_{0l}$ $HE_{2l}$ )

$LP_{01}$ ( $HE_{11}$ n=1, m=0) mode $\longrightarrow$ $V_c = 0$

$LP_{11}$ mode $\longrightarrow$ $J_0(V_c) = 0 \rightarrow Vc = 2.405$

single mode

$V_c \le 2.405$

# 7.4 Group velocity and wavelength dispersion

group velocity:

$$v_g = \frac{1}{\frac{\partial \beta}{\partial \omega}} = v_p - \lambda \frac{dv_p}{d\lambda}$$

Group Velocity Dispersion (GVD)

$$\frac{dv_g}{d\lambda} = \frac{d}{d\lambda}\left(\frac{1}{t_g}\right) = -\frac{1}{t_g{}^2}\frac{dt_g}{d\lambda} = -v_g{}^2 \frac{dt_g}{d\lambda}$$

material dispersion

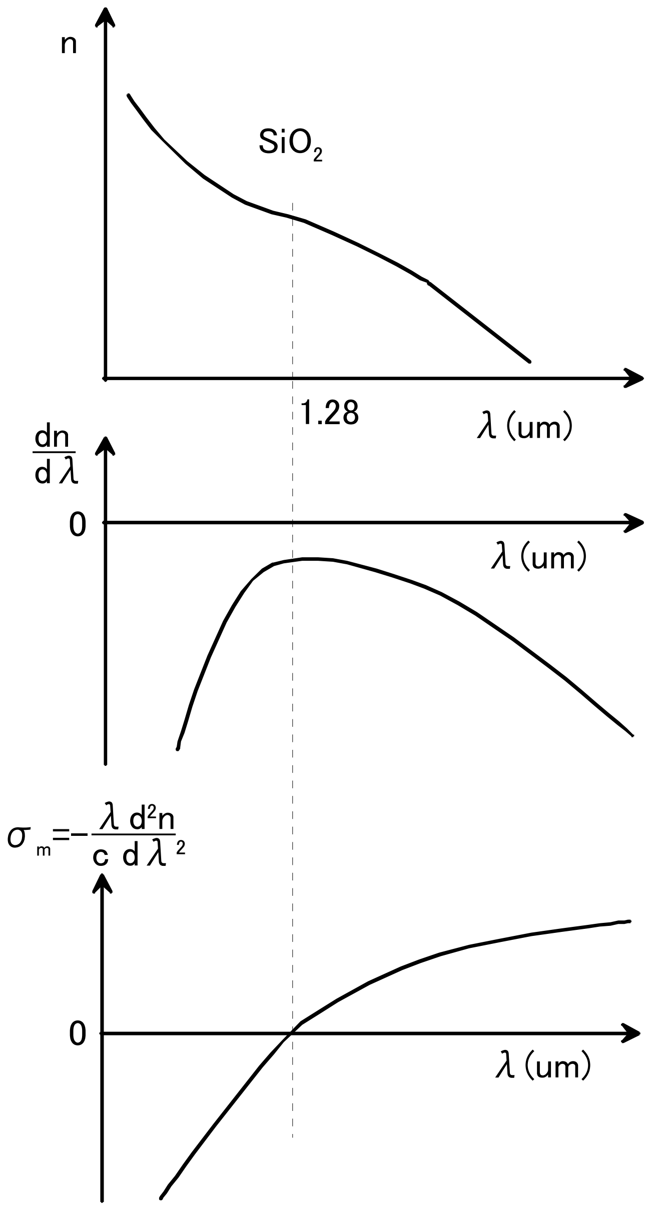
$$v_g = \frac{c}{n} - \lambda \frac{d}{d\lambda}\left(\frac{c}{n}\right) = \frac{c}{n}\left(1 + \frac{\lambda}{n}\frac{dn}{d\lambda}\right)$$

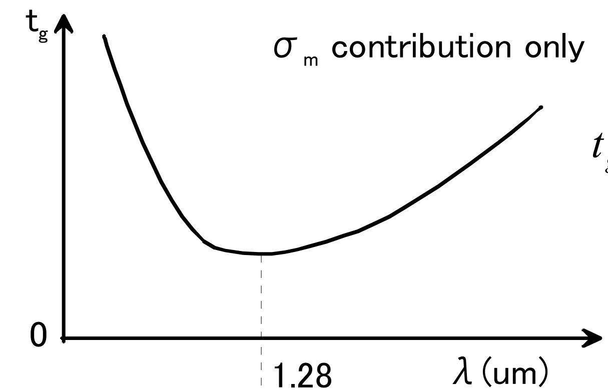
$$t_g = v_g{}^{-1} \approx \frac{n}{c}\left(1 - \frac{\lambda}{n}\frac{dn}{d\lambda}\right)$$

$$\therefore \sigma_m = \frac{dt_g}{d\lambda} = -\frac{\lambda}{c}\frac{d^2 n}{d\lambda^2}$$

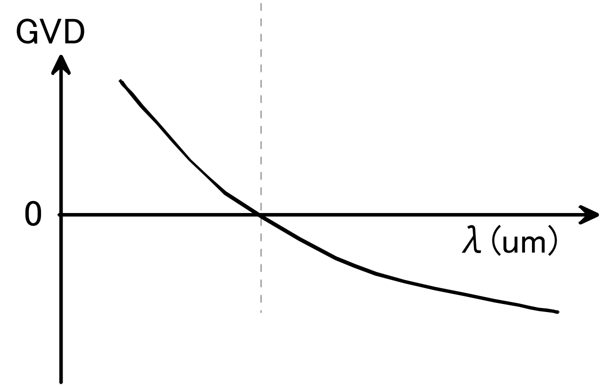
GVD $$\frac{dv_g}{d\lambda} = \left(\frac{c}{n}\right)^2 \left(1 + \frac{\lambda}{n}\frac{dn}{d\lambda}\right)^2 \frac{\lambda}{c}\frac{d^2 n}{d\lambda^2} \approx \frac{c\lambda}{n^2}\frac{d^2 n}{d\lambda^2} = -\left(\frac{c}{n}\right)^2 \sigma_m$$

# material dispersion




$$t_g = v_g^{-1} \approx \frac{n}{c}\left(1 - \frac{\lambda}{n}\frac{dn}{d\lambda}\right)$$



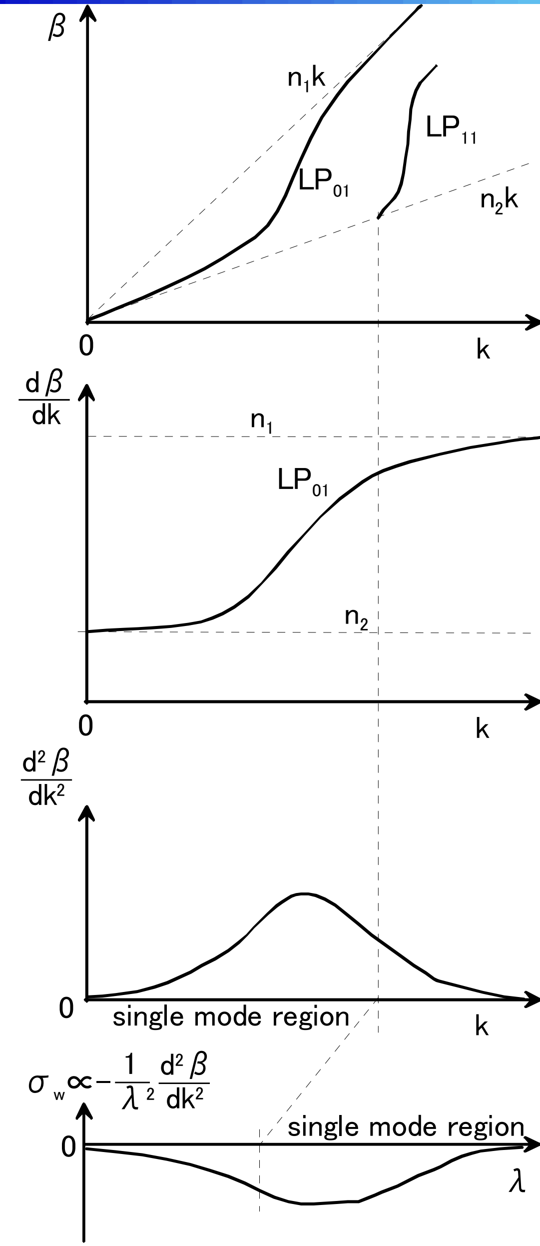

$$\frac{dv_g}{d\lambda} = -\left(\frac{c}{n}\right)^2 \sigma_m$$

# waveguide dispersion

β changes as wavelength changes due to confinement variation into waveguide core.

$$t_g = v_g^{-1} = \frac{d\beta}{d\omega} = \frac{1}{c}\frac{d\beta}{dk}$$

$$\therefore \sigma_w = \frac{dt_g}{d\lambda} = \frac{1}{c}\frac{d}{d\lambda}\left(\frac{d\beta}{dk}\right) = -\frac{2\pi}{c\lambda^2}\frac{d^2\beta}{dk^2}$$



# total dispersion

total dispersion: $\sigma = \sigma_m + \sigma_w$

(1)normal dispersion (GVD>0, σ<0):
$v_g$ increases as a wavelength becomes longer.

(2)anomalous dispersion (GVD<0, σ>0):
$v_g$ decreases as a wavelength becomes longer.

pulse broadening: $\Delta T = L|\sigma|\delta\lambda$

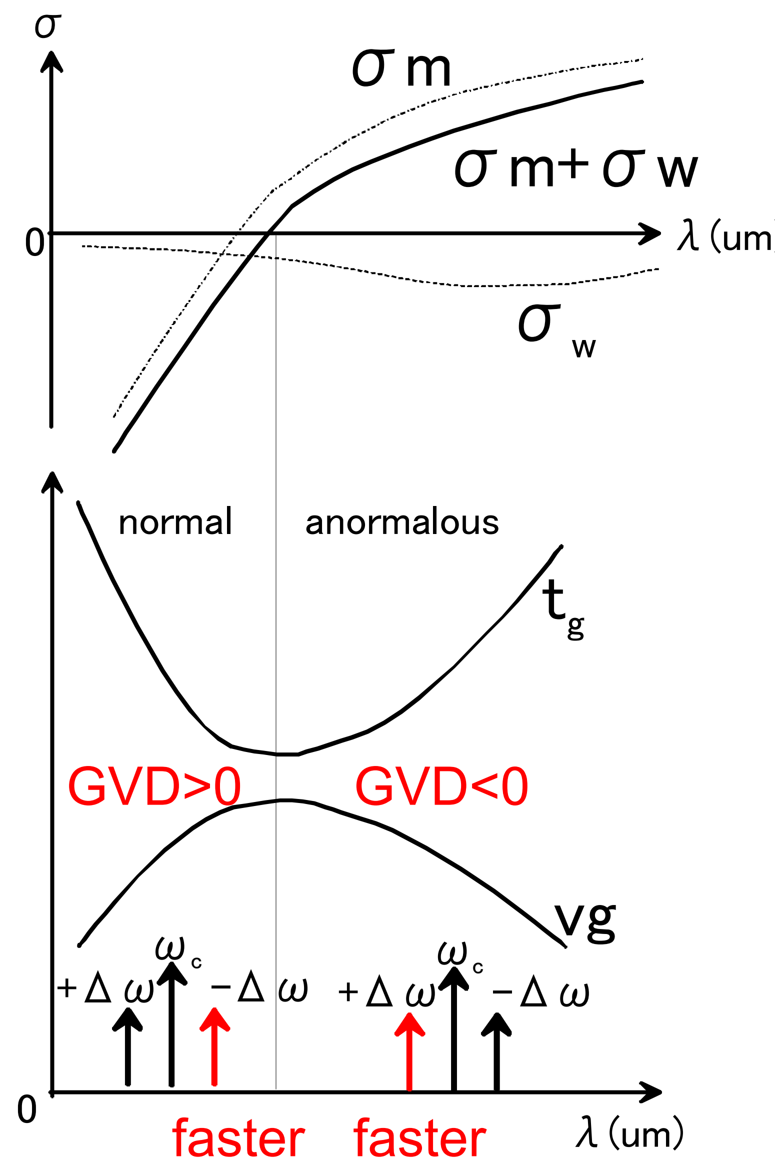
W: bit rate →limit $\Delta T < T = \frac{1}{W}$

$$WL|\sigma|\delta\lambda < 1$$

modulation→ $\delta\lambda = \frac{\lambda^2}{c} W$

$$\therefore W^2 L|\sigma| < \frac{c}{\lambda^2}$$

σ=16ps/km/nm (λ=1550nm)
W=2.5Gbps → L < 1248km
W=10Gbps → L < 78km



# 7.5.1 Nonlinear effect (SPM) and soliton

optical Kerr effect: $n = n_0 + n_2(\omega)|E|^2$

$\Delta\phi = k_0 n_2(\omega)|E|^2 L$ $\quad \Delta\omega = -\dfrac{d\Delta\phi}{dt} = -\dfrac{2\pi}{\lambda} n_2 L \dfrac{d|E|^2}{dt}$

rising edge: $\Delta\omega < 0$

falling edge: $\Delta\omega > 0$

(1)normal dispersion region (GVD>0):

$t_g$ is larger for shorter wavelength (at higher frequency)

rising edge : faster

trailing edge : slowly

pulse broadening

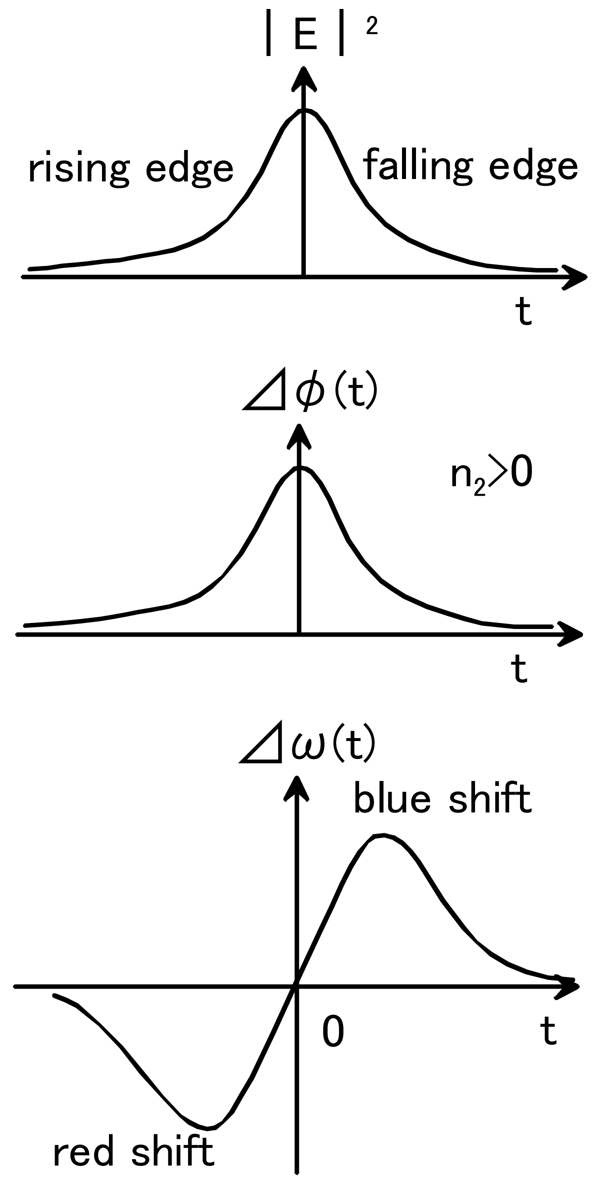
(2)anomalous region (GVD<0):

$t_g$ is larger for longer wavelength (at lower frequency)

rising edge : slowly

trailing edge : faster

pulse compression

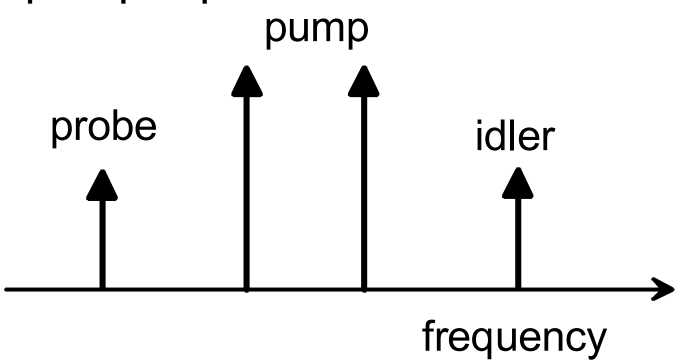


# 7.5.2 Nonlinear effect (FWM: Four Wave Mixing)

caused by an optical Kerr effect (third-order nonlinearity)

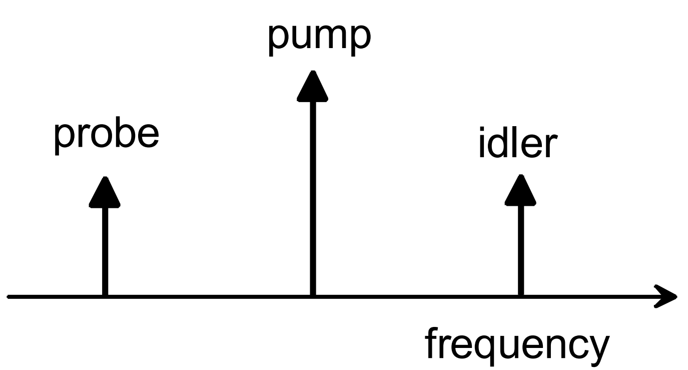
FWM: Four Wave Mixing

2 pump + probe → idler




DFWM: Degenerate Four Wave Mixing

2 pump (with same $\lambda$) + probe → idler




<u>bad news</u>

FWM produces a spurious light that could has the same wavelength as a signal to be transmitted in WDM.

<u>good news</u>

FWM can be used as a wavelength converter.

# 7.5.3 Nonlinear effect (Raman effect)

Frequency shifted light is generated due to Raman effect:

$\lambda(\omega) \rightarrow \lambda + \Delta\lambda\ (\omega - \Delta\omega)$

$-\Delta\omega$ : molecular vibration, optical phonon energy

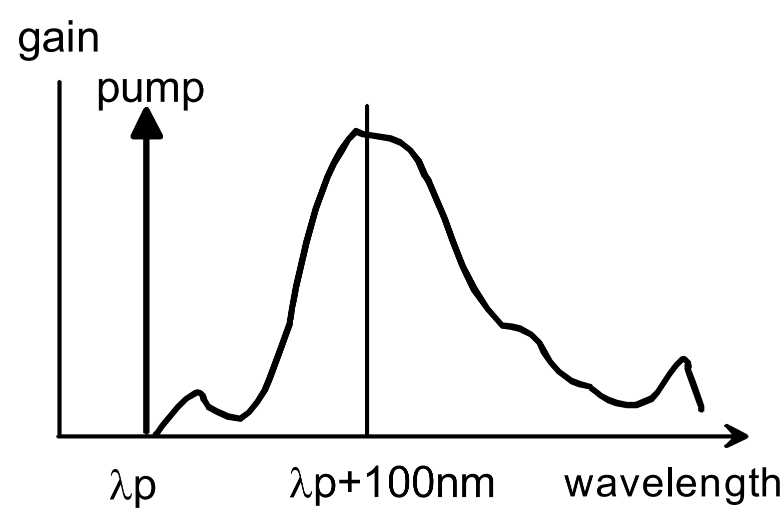
Stimulated Raman effect in optical fibers:

optical gain at a wavelength region approximately 100nm-longer than a pump wavelength

utilized as an optical amplifier

can be tuned, in principle, at an arbitrary wavelength

fiber itself is a gain medium



# 7.5.4 Nonlinear effect (Brillouin scattering)

Wavelength-shifted light is generated:
shifted by the frequency corresponding to acoustic phonon energy

acoustic phonon → compressional wave
→ periodic variation in refractive index
→ red-shifted scatted light
(propagating in backward)

coherent and intense incident light
→ Stimulated Brillouin scattering (SBS)

power transfer from incident light
to scattered light
→ suppresses transmission power