Basics of Structural Dynamics

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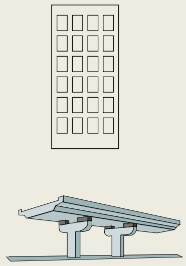
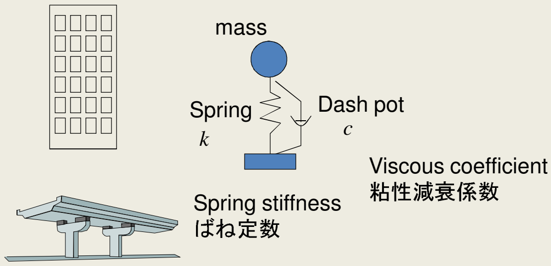
Preliminary Overview of Structural Dynamics



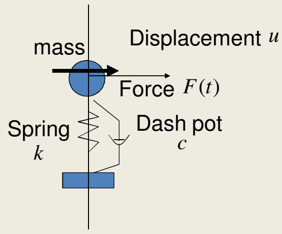
- What is the natural period? What is the definition of natural period?
- What is the damping ratio? What is the definition of damping ratio?
- How long natural period does a structure have?
- How large damping ratio does a structure have?

Single-degree-of-freedom Idealization (SDOF) 1自由度系

Idealization of a structure in terms of a SDOF system



Equation of Motion of a SDOF System



From the Newton's Law, the equilibrium of the system subjected to an external force F(t) becomes as

$$m\ddot{u} = F(t) - ku - c\dot{u}$$

$$m\ddot{u} + c\dot{u} + ku = F(t)$$

Dividing the equation by , we obtain

$$\ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = \frac{F(t)}{m}$$

Defining ω_n and h as

$$\omega_n = \sqrt{\frac{k}{m}}$$ Undamped angular frequency 非減衰円固有振動数

$$2h\omega_n = \frac{c}{m} \quad \therefore \ h = \frac{c}{2\sqrt{mk}} = \frac{c}{c_{cr}}$$ Damping ratio減衰定数

c_{cr}: Critical damping coefficient 臨界減衰係数

The equation of motion can be written as

$$\ddot{u} + 2h\omega_n\dot{u} + \omega_n^2 u = \frac{F(t)}{m}$$

The solution of the above equation can be obtained as a sum of the general solution (u_c) and a particular solution (up)

Free Oscillation of no damped, critically damped and over-damped SDOF system 非減衰，臨界減衰及び過減衰時の自由振動（一般解）

a) Free Oscillation of no damped system非減衰自由振動

$$\ddot{u} + \omega_n^2 u = 0$$

$$\therefore \ u_c = U_c \cos(\omega_n t - \varphi)$$

where

$$U_c = \sqrt{{u_0}^2 + \left(\frac{\dot{u}_0}{\omega_n}\right)^2}$$

$$\varphi = \tan^{-1}\left(\frac{\dot{u}_0}{\omega_n u_0}\right)$$

$$T_n = \frac{2\pi}{\omega_n}$$

Natural period (s)
固有周期

$$f_n = \frac{1}{T}$$

Natural frequency (Hz)
固有振動数

b) Free Oscillation of damped system減衰自由振動

If $0 < h < 1$

$$\ddot{u} + 2h\omega_n \dot{u} + \omega_n{}^2 u = 0$$

$$u_c = U_c e^{-h\omega_n t} \cos(\omega_d t - \varphi_c)$$

where

$$U_c = \sqrt{u_0{}^2 + \left(\frac{\dot{u}_0}{\omega_d} + \frac{h\omega_n u_0}{\omega_d}\right)^2}$$

$$\varphi_c = \tan^{-1}\left(\frac{\dot{u}_0 + h\omega_n u_0}{\omega_d u_0}\right)$$

$$\omega_d = \omega_n \sqrt{1-h^2}$$

$$T_d = \frac{2\pi}{\omega_d}$$

Damped natural period (s)
減衰固有周期

$$f_d = \frac{1}{T}$$

Damped natural frequency (Hz)
減衰固有振動数

c) Critically damped free oscillation 臨界減衰自由振動

$h = 1$

d) Over damped free oscillation 過減衰自由振動

$h > 1$

Logarithmic damping ratio

Free oscillation of a damped SDOF system is

$$u_c = U_c e^{-h\omega_n t} \cos(\omega_d t - \varphi)$$

The amplitude at the m-th and (m+1)th oscillations is

$$u_{cm} = U_c e^{-h\omega_n t_m} \cos(\omega_d t_m - \varphi)$$

$$u_{c(m+1)} = U_c e^{-h\omega_n t_{m+1}} \cos(\omega_d t_{m+1} - \varphi)$$

$$\therefore \quad \frac{u_{cm}}{u_{cm+1}} = e^{h\omega_n T_d} = e^{2\pi h/\sqrt{1-h^2}}$$

$$\lambda \equiv \log_e \left|\frac{u_{cm}}{u_{cm+1}}\right| = \frac{2\pi h}{\sqrt{1-h^2}} \approx 2\pi h$$

Forced Oscillation of SDOF System (Particular solution)

$$\ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = \frac{F(t)}{m}$$

$$u_p = A\sin\omega t + B\cos\omega t$$

one can obtain up as

$$u_p = \frac{F_0}{k} \cdot \frac{1}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left\{2h\frac{\omega}{\omega_n}\right\}^2}} \sin(\omega t - \varphi_p)$$

where

$$\varphi_p = \tan^{-1}\left(\frac{2h(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right)$$

Solution of SDOF system subjected to an external force F(t)

$$\ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = \frac{F(t)}{m}$$

Transient response

Steady state response

$$u = U_c e^{-h\omega_n t} \cos(\omega_d t - \varphi)$$

$$+ \frac{F_0}{k} \cdot \frac{1}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left\{2h\frac{\omega}{\omega_n}\right\}^2}} \sin(\omega t - \varphi)$$

where

$$U_c = \sqrt{u_0{}^2 + \left(\frac{\dot{u}_0}{\omega_d} + \frac{h\omega_n u_0}{\omega_d}\right)^2}$$

$$\varphi_c = \tan^{-1}\left(\frac{\dot{u}_0 + h\omega_n u_0}{\omega_d u_0}\right) \qquad \varphi_p = \tan^{-1}\left(\frac{2h(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right)$$

Exercise 1 Compute and plot response u under the following conditions

$u_0 = 1m$

$\dot{u}_0 = 0$

$T_n = 0.5s$ and $T_n = 1.0s$

$h = 0.01$, $h = 0.05$ and $h = 0.10$

$T = 0.3s$

$F_0 / k = 0.5m$

Dynamic Amplification Factor

Because the transient response soon decays, the SDOF system continues to oscillate with an amplitude of A_D

$$A_D = \frac{F_0}{k} \cdot \frac{1}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left\{2h\frac{\omega}{\omega_n}\right\}^2}}$$

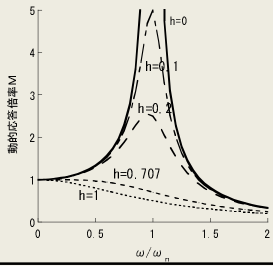
Because F_0/k represents the static displacement, denoting F_0/k as A_S,

$$A_D = A_S \cdot \frac{1}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left\{2h\frac{\omega}{\omega_n}\right\}^2}}$$

The dynamic amplification factor M can be written as

$$M = \frac{A_D}{A_S} = \frac{1}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left\{2h\frac{\omega}{\omega_n}\right\}^2}}$$

$$= \frac{1}{\sqrt{\left\{1-\left(\frac{T_n}{T}\right)^2\right\}^2 + \left\{2h\frac{T_n}{T}\right\}^2}}$$



From

$$\frac{dM}{d(\omega/\omega_n)} = \frac{-4(\frac{\omega}{\omega_n}) + 4(\frac{\omega}{\omega_n})^3 + 8h^2(\frac{\omega}{\omega_n})}{-2\left\{1-2(\frac{\omega}{\omega_n})^2 + (\frac{\omega}{\omega_n})^4 + 4h^2(\frac{\omega}{\omega_n})^2\right\}^{3/2}} = 0$$

M_{max} is developed at

$$\frac{\omega}{\omega_n} = \begin{cases} \sqrt{1-2h^2} \cdots\cdots\cdots h < \frac{1}{\sqrt{2}} \\ 0 \cdots\cdots\cdots\cdots\cdots\cdots h \geq \frac{1}{\sqrt{2}} \end{cases}$$

Consequently, M_{max} becomes as

$$M_{\max} = \begin{cases} \frac{1}{2h\sqrt{1-h^2}} \cdots\cdots\cdots h < \frac{1}{\sqrt{2}} \\ 1 \cdots\cdots\cdots\cdots\cdots\cdots h \geq \frac{1}{\sqrt{2}} \end{cases}$$

- This means that a peak value of M does not exist and M_{max} becomes 1.0 at $\omega/\omega_n = 0$ if $h > 1/\sqrt{2}$
- M_{max} becomes infinitively large at $\omega/\omega_n = 1.0$ if h=0
- If $h < 1/\sqrt{2}$, M_{max} becomes

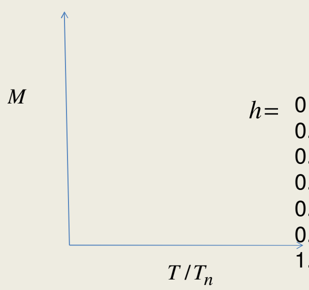
$$M_{\max} = \frac{1}{2h\sqrt{1-h^2}} \approx \frac{1}{2h}$$

Therefore, the maximum amplitude A_{Dmax} becomes

$$A_{D\max} \approx \frac{A_S}{2h}$$

Consequently, if h=0.05, M_{max}=1/(2x0.05)=10, and A_{Dmax}=10A_S.

Exercise 2 Compute the dynamic amplification factor M for various damping ratio h



$h=$ 0
0.02
0.05
0.1
0.5
0.707
1.0

Equivalent Damping Ratio

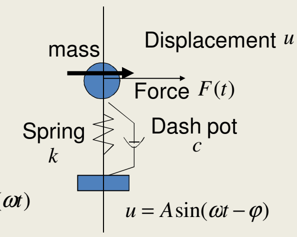
- Energy dissipated by a damper per cycle W_d can be written as

$$c\dot{u} \cdot du = c\dot{u}\frac{du}{dt}dt$$

$$W_d = \int_0^{2\pi/\omega} c\dot{u} \cdot \dot{u}dt$$

$$= \frac{c}{\omega}\int_0^{2\pi/\omega} \omega^2 A^2 \cos^2(\omega t - \varphi)d(\omega t)$$

$$= c\omega A^2 \pi$$



If we can evaluate the energy dissipation per cycle W_d based on an experiment, and if the same amount of energy is dissipated by a damper, the viscous damping coefficient c_{eq} should be as

$$c_{eq} = \frac{W_d}{\pi\omega A^2}$$

- c_{eq} is called as the equivalent damping coefficient
- The damping ratio corresponding to c_{eq} (equivalent damping ratio) is

$$h_{eq} = \frac{c_{eq}}{c_{cr}} = \frac{W_d}{\pi\omega A^2} \Big/ 2\sqrt{mk}$$

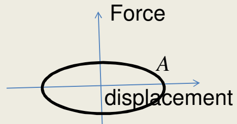
$$= \frac{1}{2\pi}\frac{W_d}{kA^2}$$

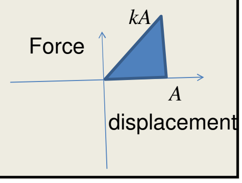
Because

$$W_e = \frac{1}{2}kA^2$$

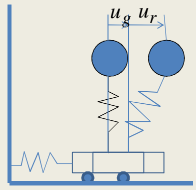
represents the elastic energy,

$$h_{eq} = \frac{1}{4\pi}\frac{W_d}{W_e}$$





SDOF System subjected to Earthquake Ground Motion



u_r : relative displacement (= displacement of a SDOF system relative to the ground)

u_g : ground displacement due to earthquake

From the Newton's law, the equilibrium of the SDOF system subjected to ground motion u_g is

$$m(\ddot{u}_r + \ddot{u}_g) = -ku_r - c\dot{u}_r$$

$$m\ddot{u}_r + c\dot{u}_r + ku_r = -m\ddot{u}_g$$

How can we compute response of structures subjected to an earthquake ground motion?

- The seismic response of a structure subjected to a combination of sinusoidal motion can be solved by evaluating a general solution and a particular solution.
- The seismic response of a structure under an earthquake ground acceleration can be solved by one of the two methods
  - ✓Mode superposition method (only for linear elastic structures)
  - ✓Direct integration method (for both linear and nonlinear method)

Nonlinear Dynamic Response Analysis

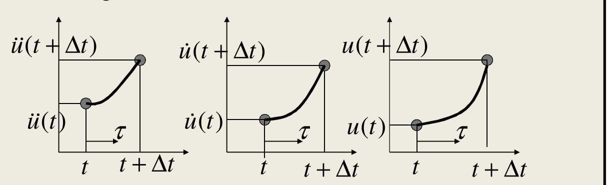
Equations of motion in the incremental form

$$M\Delta\ddot{u}(t) + C_t\Delta\dot{u}(t) + K_t\Delta u(t) = \Delta R(t) + MB\Delta\ddot{u}_g(t) \qquad (2.47)$$

$$\Delta\ddot{u}(t) = \ddot{u}(t+\Delta t) - \ddot{u}(t)$$
$$\Delta\dot{u}(t) = \dot{u}(t+\Delta t) - \dot{u}(t)$$
$$\Delta u(t) = u(t+\Delta t) - u(t)$$
$$\Delta R(t) = R(T+\Delta t) - R(t)$$
$$\Delta\ddot{u}_g(t) = \ddot{u}_g(t+\Delta t) - \ddot{u}_g(t)$$

Nonlinear Dynamic Response Analysis (continued)

Newmark's generalized acceleration method



$$\dot{u}(\tau) = \int_0^\tau \ddot{u}(\tau)d\tau + C_1$$

$$u(\tau) = \int\int_0^\tau \ddot{u}(\tau)d\tau + C_1\tau + C_2$$

$$\dot{u}_t = \dot{u}(0) = C_1$$

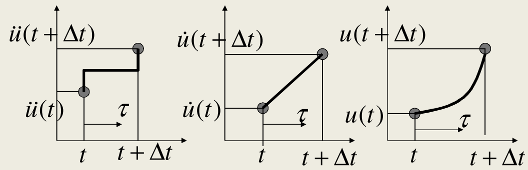
$$\dot{u}_{t+\Delta t} = \dot{u}(\Delta t) = \int_0^{\Delta t}\ddot{u}(\tau)d\tau + C_1$$

$$u_t = u(0) = C_2$$

$$u_{t+\Delta t} = u(\Delta t) = \int\int_0^{\Delta t}\ddot{u}(\tau)d\tau + C_1\Delta t + C_2$$

Newmark's generalized acceleration method (continued)

Constant Acceleration Method



$$\ddot{u}(\tau) = \frac{\ddot{u}(t) + \ddot{u}(t+\Delta t)}{2}$$

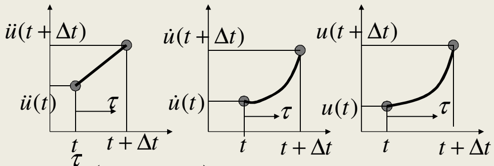
$$\dot{u}(\tau) = \dot{u}_t + \frac{\tau}{2}(\ddot{u}_t + \ddot{u}_{t+\Delta t}) \qquad \therefore\ \dot{u}_{t+\Delta t} = \dot{u}_t + \frac{\Delta t}{2}(\ddot{u}_t + \ddot{u}_{t+\Delta t})$$

$$u(\tau) = u_t + \dot{u}_t\tau + \frac{\tau^2}{4}(\ddot{u}_t + \ddot{u}_{t+\Delta t})$$

$$\therefore\ u_{t+\Delta t} = u_t + \dot{u}_t\Delta t + \frac{\Delta t^2}{4}(\ddot{u}_t + \ddot{u}_{t+\Delta t})$$

Newmark's generalized acceleration method (continued)

Linear Acceleration Method



$$\ddot{u}(\tau)=\ddot{u}_t+\frac{\tau}{\Delta t}(\ddot{u}_{t+\Delta t}-\ddot{u}_t)$$

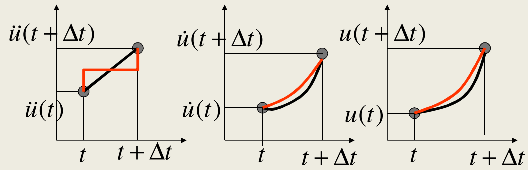
$$\dot{u}(\tau)=\dot{u}_t+\ddot{u}_t\tau+\frac{\tau^2}{2\Delta t}(\ddot{u}_{t+\Delta t}-\ddot{u}_t)$$

$$\therefore\ \dot{u}_{t+\Delta t}=\dot{u}_t+\ddot{u}_t\Delta t+\frac{\Delta t^2}{2}(\ddot{u}_{t+\Delta t}-\ddot{u}_t)$$

$$u(\tau)=u_t+\dot{u}_t\tau+\frac{\ddot{u}_t\tau^2}{2}+\frac{\tau^3}{6}(\ddot{u}_{t+\Delta t}-\ddot{u}_t)$$

$$\therefore\ u_{t+\Delta t}=u_t+\dot{u}_t\Delta t+\frac{\ddot{u}_t\Delta t^2}{2}+\frac{\Delta t^3}{6}(\ddot{u}_{t+\Delta t}-\ddot{u}_t)$$

Newmark's generalized acceleration method (continued)



$$\dot{u}(t+\Delta t)=\dot{u}(t)+[(1-\delta)\ddot{u}(t)+\delta\ddot{u}(t+\Delta t)] \qquad (2.50)$$
$$u(t+\Delta t)=u(t)+\dot{u}(t)\Delta t+[(1/2-\sigma)\ddot{u}(t)+\sigma\ddot{u}(t+\Delta t)]$$

$\delta=\frac{1}{2}$ $\quad\sigma=\frac{1}{4}$ constant acceleration method

$\delta=\frac{1}{2}$ $\quad\sigma=\frac{1}{6}$ linear acceleration method

Newmark's generalized acceleration method (continued)

$$\begin{aligned}\Delta\ddot{u}(t)&=C_1\Delta u(t)-C_3\dot{u}(t)-C_4\ddot{u}(t)\\ \Delta\dot{u}(t)&=C_2\Delta u(t)-C_4\dot{u}(t)-C_5\ddot{u}(t)\end{aligned} \qquad (2.51)$$

constant acceleration method	linear acceleration method
$C_1=4/\Delta t^2$	$C_1=6/\Delta t^2$
$C_2=2/\Delta t$	$C_2=3/\Delta t$
$C_3=4/\Delta t$	$C_3=6/\Delta t$
$C_4=2$	$C_4=3$
$C_5=0$	$C_5=\Delta t/2$

Newmark's generalized acceleration method (continued)

$$M\Delta\ddot{u}(t)+C_t\Delta\dot{u}(t)+K_t\Delta u(t)=\Delta R(t)+MB\Delta\ddot{u}_g(t) \qquad (2.47)$$

$$\begin{aligned}\Delta\ddot{u}(t)&=C_1\Delta u(t)-C_3\dot{u}(t)-C_4\ddot{u}(t)\\ \Delta\dot{u}(t)&=C_2\Delta u(t)-C_4\dot{u}(t)-C_5\ddot{u}(t)\end{aligned} \qquad (2.51)$$

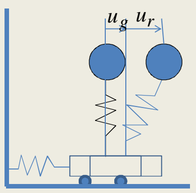
$$\tilde{K}_t\Delta u(t)=\Delta\tilde{R}(t) \qquad (2.52)$$

where,

$$\tilde{K}_t=C_1M+C_2C_t+K_t \qquad (2.53)$$

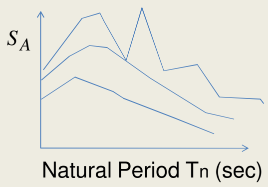
$$\Delta\tilde{R}(t)=\Delta R(t)+MB\Delta\ddot{u}_g(t)+\{C_3M+C_4C_t\}\dot{u}(t)+\{C_4M+C_5C_t\}\ddot{u}(t) \qquad (2.54)$$

Earthquake Response Spectrum



- Compute maximum response acceleration $\left|\ddot{u}_r+\ddot{u}_g\right|_{\max}$ for a given natural period Tn and a damping ratio h under a ground motion
- Response acceleration response spectrum is defined as

$$S_A(T,h)\equiv\left|\ddot{u}_r+\ddot{u}_g\right|_{\max}$$



How does response acceleration vary depending on damping ratio?

: A most simple evaluation for the effect of enhancing the energy dissipation capability for reducing structural response

A modification factor of response acceleration spectrum depending on damping ratio

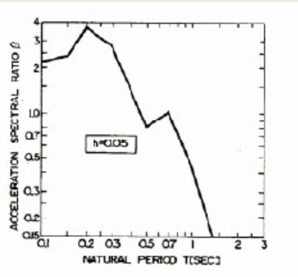
$$\xi_{SA}(T,h)=\frac{S_A(T,h)}{S_A(T,0.05)}$$

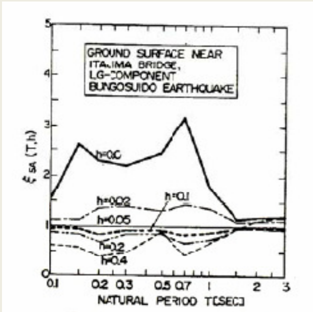
Response spectrum shape (Ratio of response acceleration and the peak ground acceleration, or response acceleration ratio)

$$\beta(T,h)=\frac{S_A(T,h)}{a_{\max}}$$

How does response acceleration vary depending on damping ratio?

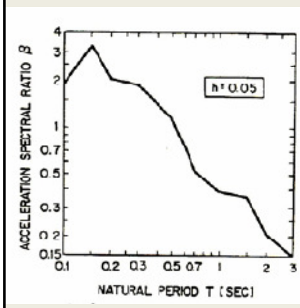
A ground motion recorded at Itajima Bridge site

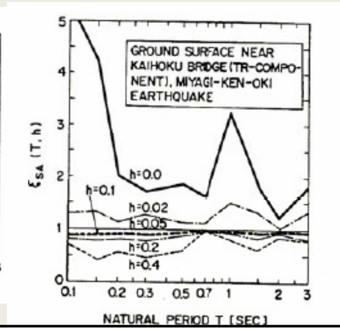




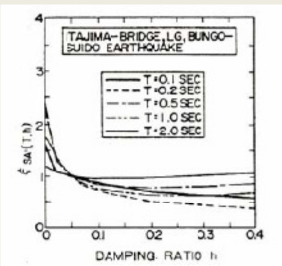
How does response acceleration vary depending on damping ratio?

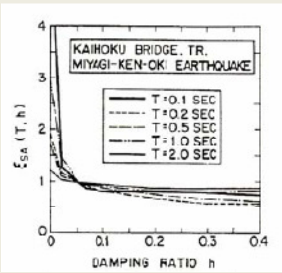
A ground motion recorded at Kaihoku Bridge site



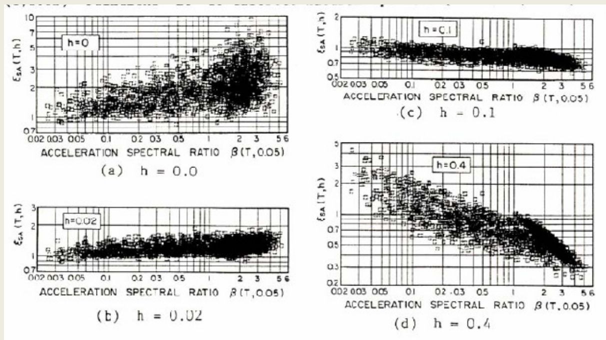


How does $\xi_{SA}(T,h)$ vary in accordance with damping ratio?



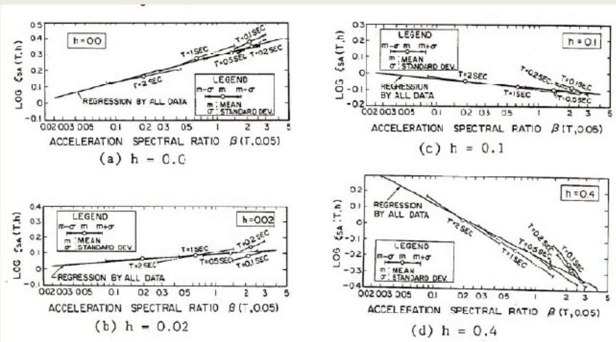


How does $\xi_{SA}(T,h)$ vary in accordance with response spectrum shape?

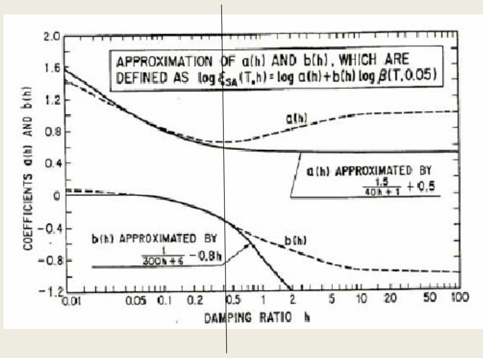


How does $\xi_{SA}(T,h)$ vary in accordance with response spectrum shape?

$$\log \xi_{SA}(T,h) = \log a(h) + b(h) \log \beta(h, 0.05)$$



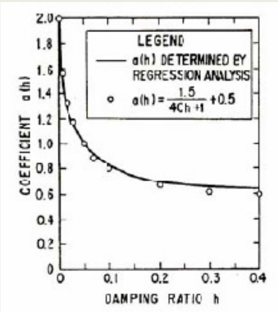
Damping Ratio Dependence of a(h) and b(h)



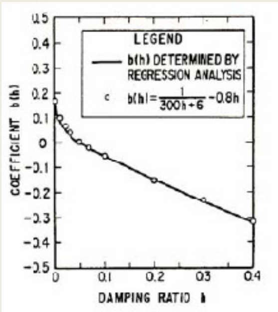
If we are interested in structural response under damping ratio less than 0.4, a(h) and b(h) may be approximated as

$$\log \xi_{SA}(T,h) = \log a(h) + b(h) \log \beta(h,0.05)$$

$a(h)$



$b(h)$



Least square fitting for a(h) and b(h)

$$a(h) = \frac{1.5}{40h+1} + 0.5$$

$$b(h) = \frac{1}{300h+6} - 0.8h$$

Therefore, $\xi_{SA}(T,h)$ may be evaluated as,

$$\xi_{SA}(T,h) = a(h) \times \beta(T,0.05)^{\left(\frac{1}{300h+6} - 0.8h\right)}$$

Practical Evaluation of Damping Ratio Dependence of Acceleration Response Spectra S_A(T,h)

$$\xi_{SA}(T,h) = \frac{S_A(T,h)}{S_A(T,0.05)}$$

$$= a(h) \times \beta(T,0.05)^{\left(\frac{1}{300h+6} - 0.8h\right)}$$

$$\approx \frac{1.5}{40h+1} + 0.5$$

This equation is widely used to estimate the response acceleration with an arbitrary damping ratio based on the response acceleration with 5% damping ratio

$$S_A(T,h) \approx \left(\frac{1.5}{40h+1} + 0.5\right) \times S_A(T,0.05)$$

Exercise 3 Evaluate how much can we reduce response acceleration spectrum by increasing damping ratio?

- From h=0.02 to h=0.05
- From h=0.04 to h=0.12