Probabilistic Concepts in Engineering Design

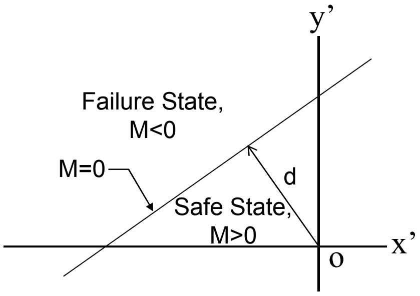
Class 7

Jan. 14th 2010

Second-Moment Method : Generalization

D: measure of reliability

d=ß



The reliability of an engineering system may involve multiple variables.

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For a function of several variables,

We define a performance function or state function

$$q(\mathbf{X}) = (X_1, X_2, ..., X_n)$$

Where $\mathbf{X} = (X_1, X_2, ..., X_n)$ is a vector of basic state or design variables of the system

The limiting performance requirement may be defined as

$$q(\mathbf{X}) = 0$$

which is the limit state of the system

$[q(\mathbf{X}) > 0]$ ≒ safe state

$[q(\mathbf{X}) < 0]$ = failure state

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Geometrically, the limit-state equation,

$q(\mathbf{X}) = 0$ is a n-dimensional surface that may be called "failure surface"

the one side of the failure surface is the safe state $q(\mathbf{X}) > 0$

the other side of surface is the failure state $q(\mathbf{X}) < 0$

Here, if the joint PDF of the design variables

$X_1, X_2, ..., X_n$ is $f_{X1,X2,,,Xn}(x_1, x_2,,, x_n)$

The probability of the safe state is

$$p_s = \int \cdots_{\{g(\mathbf{x})>0\}} \int f_{X1,\ldots,Xn}(x_1,\ldots,x_n)dx_1 \ldots dx_n$$

which may be written, as

$$p_s = \int_{g(\mathbf{x})>0} f_{\mathbf{x}}(\mathbf{x})d\mathbf{x}$$

This equation is simply the volume integral of over the safe region

conversely the probability of failure state is

$$p_F = \int_{g(\mathbf{x})<0} f_{\mathbf{x}}(\mathbf{x})d\mathbf{x}$$

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Uncorrelated variates

In general, the basic variables $(X_1, X_2, \ldots, X_n)$ may be correlated.

However, we consider first the case of uncorrelated variates.

Introduce the set of uncorrelated reduced variates,

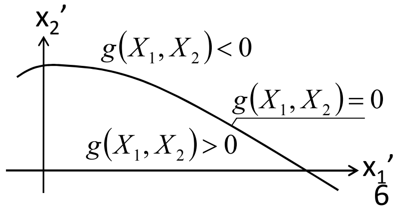
$$X_i' = \frac{X_i - \mu_{Xi}}{\sigma_{Xi}} \qquad i = 1,2,\ldots n$$

In terms of the reduced variates, the limit state equation would be

$$g(\sigma_{X1}X_1' + \mu_{X1}, \ldots, \sigma_{Xn}X_n' + \mu_{Xn}) = 0$$

The limit-state surface,

$$g(\mathbf{X}) = 0$$



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The position of the failure surface relative to the origin of the reduced variates should determine the safety or reliability of system

The position of the failure surface may be represented by the minimum distance from the surface $g(\mathbf{X}) = 0$ to the origin of reduced variates

The point on the failure surface with minimum distance to the origin is the most probable failure point

This minimum distance may be used as a measure of reliability

The required minimum distance may be determined as follows,

The distance from a point $\mathbf{X}' = (X_1', X_2', \dots, X_n')$

on the failure surface $q(\mathbf{X}) = 0$

to the origin of $\mathbf{X}'$ is

$$D = \sqrt{X_1'^2 + X_2'^2 + \dots + X_n'^2} = \left(\mathbf{X}'^t \, \mathbf{X}'\right)^{1/2}$$

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the point on the failure surface $\left(x_1'^*, x_2'^*, \ldots, x_n'^*\right)$

having the minimum distance to the origin may be determined by minimizing the function D , subject to the constraint $q(\mathbf{X}) = 0$; that is,

Minimize D
Subject to $q(\mathbf{X}) = 0$.

for this purpose, the method of Lagrange's multiplier may be used,

$$L = D + \lambda g(\mathbf{X})$$

or

$$L = \left(\mathbf{X}'^t\,\mathbf{X}'\right)^{1/2} + \lambda g(\mathbf{X})$$

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In scalar notation,

$$L = \sqrt{X_1'^2 + X_2'^2 + \ldots + X_n'^2} + \lambda g\left(X_1, X_2, \ldots, X_n\right)$$

in which $X_i = X_i' + \mu_{Xi}$

Minimizing L, we obtain the following set of n+1 equations with n+1 unknowns

$$\frac{\partial L}{\partial X_i} = \frac{X_i'}{\sqrt{X'^2_1 + X'^2_2 + \ldots + X'^2_n}} + \lambda \frac{\partial g}{\partial X'_i} = 0$$

and

$$\frac{\partial L}{\partial \lambda} = g\left(X_1, X_2, \ldots X_n\right) = 0$$

The solution of above set of equations should yield the most probable failure point

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Linear Performance Functions

A linear performance function may be represented as

$$g(\mathbf{X}) = a_0 + \sum_i a_i X_i$$

where

a_0 and a_i are constants

the corresponding limit-state equation is

$$a_0 + \sum_i a_i X_i = 0$$

in terms of the reduced variates, the limit-state equation becomes

$$a_0 + \sum_i a_i \left(\sigma_{Xi} X'_i + \mu_{Xi}\right) = 0$$

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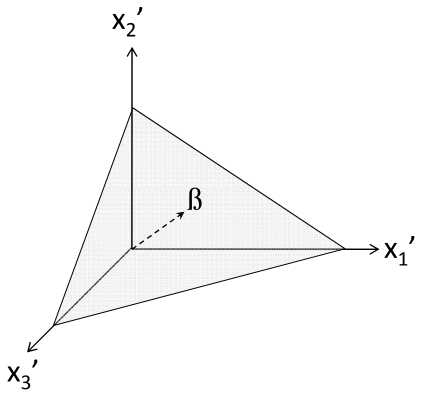
in three dimensions

$$a_0 + a_1\left(\sigma_{X1} X'_1 + \mu_{X1}\right) + a_2\left(\sigma_{X2} X'_2 + \mu_{X2}\right) + a_3\left(\sigma_{X3} X'_3 + \mu_{X3}\right) = 0$$

the distance of the failure plane to the origin of the reduced variates

$$\beta = \frac{a_0 + \sum_i a_i \mu_{Xi}}{\sqrt{\sum_i \left(a_i \sigma_{Xi}\right)^2}}$$



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Example of linear performance function : structural elements

The safety of a structural element may be evaluated on the basis of linear performance function, for example

$$g(\mathbf{X}) = R - Q$$

R: The resistance of the element
Q: The total load effect on the element: e.g. Q=D+L
D: Dead load effect
L: Live load effect

LRFD design --- ACI

$$0.9Rn \geq 1.4Dn + 1.7Ln$$

Rn, Dn, Ln are respective nominal values of resistance and loads

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Structural engineers often use “nominal” values of loads and resistances, which are generally different from the corresponding mean values,

For example, the ratios of the mean loads to the respective specified nominal loads for office buildings in U.S.A. are as follows

$$\frac{\overline{D}}{D_n} = 1.05, \quad \frac{\overline{L}}{L_n} = 1.15$$

Whereas in the case of flexural capacity of RC beams, the corresponding ratio is

$$\frac{\overline{R}}{R_n} = 1.05$$

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The corresponding c.o.v.'s associated with these variables are

$$\Omega_R = 0.11 \quad \Omega_D = 0.10 \quad \Omega_L = 0.25$$

The nominal dead load effect Dn is due to the weight of structure.
The nominal live load is usually specified by codes or standards.

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For buildings, the live load intensity L0 in U.S. is specified by the American National Standards Institute(ANSI).

A load-reduction is also permitted yielding a normal live load as follows

$$L_n = L_o\left\{1 - \min\left[0.0008A_T; 0.23\left(1 + \frac{\mathrm{D_n}}{L_o}\right); 0.6\right]\right\}$$

in which

$A_T =$ The tributary floor area

$\dfrac{L_o}{D_n} =$ The live load-dead load ratio

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For a live load-dead load ration of $\frac{L_o}{D_n} = 1.0$
and a tributary areas $A_T = 400 ft^2$

The above load-reduction factor would be

$$\frac{L_n}{L_o} = 1 - 0.32 = 0.68$$

in summary, the reliability underlying a reinforced concrete beam in an office building designed in accordance with the ACI and ANSI load specification is as follows

$$\frac{\overline{L}/1.15}{\overline{D}/1.05} = 0.68 \quad or \quad \overline{L} = 0.745\overline{D}$$

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whereas the ACI requirement becomes

$$0.9\frac{\overline{R}}{1.05} = 1.4\frac{\overline{D}}{1.05} + 1.7\frac{\overline{L}}{1.15}$$

from which the required mean resistance is

$$\overline{R} = 2.831\overline{D}$$

Therefore, according to E q., the safety index of the beam is

$$\beta = \frac{\overline{R} - \overline{D} - \overline{L}}{\sqrt{\sigma_R^2 + \sigma_D^2 + \sigma_L^2}} = \frac{2.831\overline{D} - \overline{D} - 0.745\overline{D}}{\sqrt{\left(0.11 \times 2.831\overline{D}\right)^2 + \left(0.10\overline{D}\right)^2 + \left(0.25 \times 0.745\overline{D}\right)^2}}$$

$$= 2.885$$

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If the variables can be assumed to be normal, the underlying probability of failure would be

$$p_F = 1 - \Phi(2.885) = 1.96 \times 10^{-3}$$

END

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