Probabilistic Concepts in Engineering Design

Class 3

Nov. 9th 2009

2.4 Useful Probability Distributions

(5) The geometric distribution

In a Bernoulli sequence, the number of trials until a specified event occurs for the first time is governed by the geometric distribution.

If the first occurrence of event is realized on the t-th trial, then there must be no occurrence of this event in any of the prior (t-1) trials. Therefore, if T is the appropriate random variable,

$$P(T=t) = p\, q^{t-1} \qquad t=1,2, \ldots$$

2.4 Useful Probability Distributions

(6) The return period

In a time (or space) problem, that can be modeled as a Bernoulli sequence, the number of time (or space) intervals until the first occurrence of an event is called the first occurrence time.

If the individual trials (or intervals) in the sequence are statically independent, the first occurrence time must also be the time between any two consecutive occurrences of the same event; that is, the recurrence time is equal to the first occurrence time.

2.4 Useful Probability Distributions

(6) The return period

The recurrence time, therefore, in a Bernoulli sequence also has a geometric distribution; the mean recurrence time, which is popularly known in engineering as the average return period is

$$\overline{T} = E(T) = \sum_{t=1}^{\infty} t \cdot pq^{t-1}$$

$$= p\left(\underline{1 + 2q + 3q^2 + \cdots}\right)$$

$$for\ q < 1.0 \quad \frac{1}{(1-q)^2} = \frac{1}{p^2}$$

Hence, $\overline{T} = \dfrac{1}{p}$

On the average, the time between two consecutive occurrences of events is equal to the reciprocal of the probability of the event.

- The return period is only an average duration between events
- should not be constructed as the actual time between occurrences.
- The actual time is T, which is a random variable

Example - 1

A radio transmission tower is designed for a "50-year wind"; that is, a wind velocity having a return period of 50 years.

(a) What is the probability that the design wind velocity will be exceeded for the first time on the fifth year after completion of the structure?

In this case $p = \frac{1}{50} = 0.02$

Then $P(T = 5) = (0.02)(0.98)^4 = 0.018$

(b) What is the probability that the first such wind velocity will occur within 5 years after completion of structure?

$$P(T \le 5) = \sum_{t=1}^{5} (0.02)(0.98)^{t-1}$$
$$= 0.02 + 0.196 + 0.192 + 0.188 + 0.184$$
$$= 0.096$$

This is the same as the event of at least 50 year wind in 5 years; thus, the desired probability may be obtained as

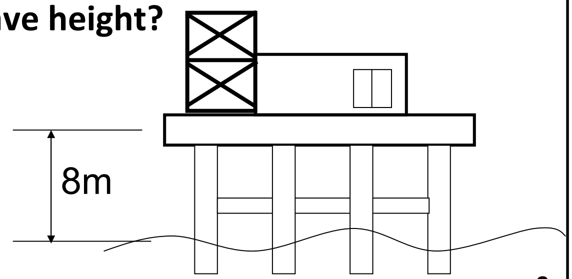
$$1 - (0.98)^5 = 0.096$$

However, this is different from the event of experiencing exactly one 50-year wind in 5 years:
The probability in this case would be

$$\binom{5}{1}(0.02)(0.98)^4 = 0.092$$

Example - 2
An offshore structure is designed for a height of 8m above the mean sea level. This height corresponds to 10% probability of being exceeded by sea waves in a year.

What is the probability that the structure will be subjected to waves exceeding 8m within the return period of the design wave height?



The return period of the wave is

$$\overline{T} = \frac{1}{0.1} = 10\,years$$

Therefore

$$P(H > 8m\ in\ 10\,years) = 1 - (0.9)^{10} = 0.6513$$

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If it is assumed that, when subjected to waves exceeding the design height, there is a probability of 20% that the structure may be damaged, what is the probability of damage to the structures within 3 years?

This probability should be taken into consideration that there may be 0,1,2, or 3 exceedances in 3 years, assuming the likelihood of more than one such wave in a year is negligible, assume that structure damages from more than one exceedance are statically independent. Then, according to the total probability theorem

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$$
\begin{aligned}
P(no\ damage\ in\ 3\ years) &= 1.00(0.90)^3 \\
&+ 0.80\left(3(0.10)(0.90)^2\right) \\
&+ (0.80)^2\left[3(0.10)^2(0.90)\right] \\
&+ (0.80)^3\left(0.10^3\right) \\
&= 0.9412
\end{aligned}
$$

Therefore

$$P(proceedings\ in\ 3\ years) = 0.0588$$

Assume the CDF of event $Fx(x)$

$$q = P_r[X \leq x] = F_x(x)$$

Then, the probability of X≦x within the continuing N years

$$Q = q^N \{F_x(x)\}^N$$

Here, the average return period is

$$\overline{T} = \frac{1}{p} = \frac{1}{1-q} = \frac{1}{1-Fx(x)}$$

Resulting

$$\overline{T} = \frac{1}{1-Q^{\frac{1}{N}}}$$

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Here, we assume N is the design life of structure, the probability that the event does not exceed the value of the average return period Q can be obtained

$$\overline{Q} = \left(1 - \frac{1}{\overline{T}}\right)^N$$

In the case N is large,

$$Q \fallingdotseq \exp\left(-\frac{N}{\overline{T}}\right)$$

In the case of large $\overline{T}\ (N = \overline{T})$

$$Q = e^{-1} = 0.364$$

The probability of occurrence within the average return period is

$$1-0.364=0.636$$

With probability